FOURIER COEFFICIENTS OF GENERALIZED EISENSTEIN SERIES OF DEGREE TWO II

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§1. Statement of Results.

This is a continuation of Part I ([13]). We treat a generalization of the previous result and investigate a non-vanishing property of Fourier coefficients. We follow the notation of Part I throughout this paper.

Let $f \in M_k(I_1)$ be a normalized elliptic eigen modular form of weight k , and let $[f] \in M_k(I_2)$ be the generalized Eisenstein series of degree two attached to f in the sense of Langlands [8] and Klingen [4] (see (1.4) of Part I). Then $[f]$ is an eigen modular form of weight k satisfying $\Phi([f]) = f$ where Φ is the Siegel operator (Proposition in §2 of Part I); conversely, by Kurokawa [5], $[f]$ is characterized by these properties. Let

$$[f](Z) = \sum_{T \geq 0} a(T, [f]) e(\text{tr}(TZ)) \quad (1.1)$$

be the Fourier expansion of $[f]$, where $e(z) = \exp(2\pi iz)$, tr denotes the trace of matrices, and T runs over all symmetric positive semi-definite semi-integral matrices of size 2. We investigate these $a(T, [f])$.

Let

$$f(z) = \sum_{n=0}^{\infty} a(n) e(nz) \quad (1.2)$$

be the Fourier expansion of f , hence $T(n)f = a(n)f$ for each $n \geq 1$. For $|T| = 0$ we have $a(T, [f]) = a(n)$ if T is unimodularly equivalent to $\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}$, since $\Phi([f]) = f$. For $T > 0$, from $T(p)[f] = (1 + p^{k-2})a(p)[f]$ (p a prime number) we have:

$$\begin{aligned} a(pT, [f]) &= (1 + p^{k-2})a(p)a(T, [f]) - p^{2k-3}a(p^{-1}T, [f]) \\ &\quad - p^{k-2}a\left(p^{-1}T \begin{bmatrix} 0 & p \\ 1 & 0 \end{bmatrix}, [f]\right) - p^{k-2} \sum_{j=0}^{p-1} a\left(p^{-1}T \begin{bmatrix} 1 & 0 \\ j & p \end{bmatrix}, [f]\right) \end{aligned} \quad (1.3)$$

as in Maaß [11]. Here we understand that $a(*, [f]) = 0$ unless $*$ is semi-integral. The relation (1.3) enables us to write each $a(T, [f])$ as an explicit $\mathbf{Z}(f)$ -linear combination of $a(T, [f])$ for primitive $T > 0$, where $\mathbf{Z}(f)$ is the integer ring of the totally real number field $\mathbf{Q}(f) = \mathbf{Q}(a(n) | n \geq 1)$ and we say that $T = \begin{pmatrix} t_1 & t/2 \\ t/2 & t_2 \end{pmatrix}$

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is primitive if g. c. d. $(t_1, t_2, t) = 1$. Thus we may limit our attention to the case where $T > 0$ is primitive. We put $|2T| = \Delta(T)\mathfrak{f}^2$ with a positive integer $\mathfrak{f}$, where $-\Delta(T)$ is the discriminant of the quadratic field $\mathbf{Q}(\sqrt{-|2T|})$. We denote by $\chi = \left(\frac{-\Delta(T)}{*}\right)$ the Dirichlet character associated with $\mathbf{Q}(\sqrt{-|2T|})$.

Now we state our main result.

THEOREM 1. *Let f be a normalized eigen modular form of weight k with respect to $\Gamma_1 = \text{SL}(2, \mathbf{Z})$ with a Fourier expansion (1.2). Suppose $T > 0$ is primitive. Then we have:*

$$a(T, [f]) = (-1)^{k/2} \frac{(k-1)!}{(2k-2)!} (2\pi)^{k-1} \Delta(T)^{k-(3/2)} \frac{L(k-1, \chi)}{L_2(2k-2, f)} \cdot \sum_{\substack{s|\mathfrak{f} \\ s>0}} M(\mathfrak{f}s^{-1}) \sum_{\substack{t|\mathfrak{f} \\ t>0}} \mu(t) D(k-1, f, \mathcal{G}_T^{(s/t)}). \quad (1.4)$$

Here

$$L_2(s, f) = \prod_p (1 - \alpha_p^2 p^{-s})^{-1} (1 - \alpha_p \beta_p p^{-s})^{-1} (1 - \beta_p^2 p^{-s})^{-1}$$

is the "second" L -function attached to f , α_p and β_p being complex numbers such that $\alpha_p + \beta_p = a(p)$ and $\alpha_p \beta_p = p^{k-1}$,

$$\mathcal{G}_T(z) = \sum_{(m,n) \in \mathbf{Z}^2} e(z(t_1 m^2 + t m n + t_2 n^2)) \quad \text{for } T = \begin{pmatrix} t_1 & t/2 \\ t/2 & t_2 \end{pmatrix},$$

$$D(s, f, \mathcal{G}_T^{(p)}) = \sum_{n=1}^{\infty} a(n) b_T(v^2 n) n^{-s} \quad \text{if we write } \mathcal{G}_T(z) = \sum_{n=0}^{\infty} b_T(n) e(nz),$$

$L(s, \chi)$ is the Dirichlet L -function, μ is the Möbius function, and

$$M(a) = \sum_{\substack{d|a \\ d>0}} \mu(d) \chi(d) d^{k-2} \sigma_{2k-3}(a d^{-1}) \quad (1.5)$$

where $\sigma_s(d) = \sum_{\substack{g|d \\ g>0}} g^s$. We understand that each L -function is analytically continued to the whole s -plane.

We note that a direct computation shows that (1.4) is equivalent to:

$$a(T, [f]) = (-1)^{k/2} \frac{(k-1)!}{(2k-2)!} (2\pi)^{k-1} \Delta(T)^{k-(3/2)} \frac{L(k-1, \chi)}{L_2(2k-2, f)} \cdot \sum_{\substack{v|\mathfrak{f} \\ v>0}} D(k-1, f, \varphi_{T,v}). \quad (1.4a)$$

where

$$\varphi_{T,v} = (\mathfrak{f}v^{-1})^{2k-3} \prod_{\substack{p|\mathfrak{f} \\ p: \text{prime}}} (1 - \chi(p) p^{1-k}) \mathcal{G}_T^{(p)}.$$

In case $\Phi f \neq 0$, i. e. $f(z) = G_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) e(nz)$ (k is an even integer ≥ 4), then, as we shall see in § 3, (1.4) may also be written:

$$a(T, [G_k]) = (-1)^{k/2} \frac{(k-1)!}{(2k-2)!} 2(2\pi)^{k-1} A(T)^{k-(3/2)} \frac{L(k-1, \chi)}{\zeta(2k-2)} M(\mathfrak{f}) \quad (1.6)$$

which coincides with the formula of Maaß [11] for $a(T, [G_k])$.

Here we note that *all* $a(T, [f])$ have unified expression for both cases $\Phi f = 0$ and $\Phi f \neq 0$, by Theorem 1 and by the relation (1.3). This suggests that the higher degree cases are in similar situations.

Exactly as in Part I, we have the following algebraicity result from Theorem 1 and (1.3) combined with the algebraicity concerning $D(k-1, f, \mathfrak{g}_f^{(p)})$ and $L_2(2k-2, f)$ shown by Shimura [15], Sturm [17], and Zagier [18]:

THEOREM 2. *Let f be as in Theorem 1. Then, for all $T \geq 0$ we have:*

- (1) $a(T, [f])^\sigma = a(T, [f^\sigma])$ for all $\sigma \in \text{Aut}(C)$.
- (2) $a(T, [f])$ belong to $\mathbf{Q}(f)$.

Here $\mathbf{Q}(f) = \mathbf{Q}(a(n) | n \geq 1)$. Theorem 2 is a special case of the result of Kurokawa [5].

In §4 we treat a non-vanishing problem suggested by Professor Maaß.

THEOREM 3. *Let f be as in Theorem 1. Suppose that $T \geq 0$ (as a quadratic form) represents 1 over $\mathbf{Z}$. Then: $a(T, [f]) \neq 0$.*

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Notes. (1) A generalization to the Hilbert-Siegel modular case is treated in [14]. (2) Theorem 1 for the case $\Phi f = 0$ is also obtained by S. Böcherer "Über das Verhalten der Fourierentwicklung bei Liftung von Modulformen" (thesis, Albert-Ludwigs-Universität, Freiburg, 1981) using a different method; Professor Böcherer communicated his result to the author in October 1981.

Notation.

- 1°. The letter p always denotes a prime number, and $\prod_p$ means the product taken over all prime numbers. For an integer a , the fact " $p^t | a$ and $p^{t+1} \nmid a$ " is denoted by $p^t || a$.
- 2°. For a set X , its cardinality is denoted by $\#X$.
- 3°. For a real number x , $[x]$ is the maximal integer $\leq x$.
- 4°. We understand that summation (resp. product) over the empty set is equal to 0 (resp. 1).
- 5°. If A is a ring with 1, then $A^\times$ denotes the group of units.

§2. Proof of Theorem 1: The Case $\Phi f = 0$.

Throughout this section we suppose that f is a normalized eigen cusp form with the Fourier expansion (1.2) with $a(0) = 0$. By (3.5) in Part I, for each $T > 0$

we have:

$$a(T, [f]) = \xi |T|^{k-(3/2)} \sum_{(r,s) \in \mathcal{P}} \sum_{d^2 | Q(r,s)} \sum_{d > 0} \{a(Q(r,s)d^{-2})Q(r,s)^{1-k} \cdot \sum_{g=1}^{\infty} g^{-k} \sum_{\sigma \bmod g} \sum_{\substack{\rho \bmod d \\ (\rho,d)=1}} e(R_{r,s,\sigma} \rho / gd) K_g(Q(r,s)d^{-2} \rho^2, Q(u_0 + \sigma r, v_0 + \sigma s))\}, \quad (2.1)$$

where

$$\mathcal{P} = \{(x, y) \in \mathbf{Z}^2 \mid \begin{array}{l} x \text{ and } y \text{ are relatively prime;} \\ x > 0 \text{ or } (x, y) = (0, 1) \end{array}\};$$

$$\begin{pmatrix} Q(r, s) & R_{r,s,\sigma}/2 \\ R_{r,s,\sigma}/2 & Q(u_0 + \sigma r, v_0 + \sigma s) \end{pmatrix} = T \begin{bmatrix} r & u_0 + \sigma r \\ s & v_0 + \sigma s \end{bmatrix}$$

with integers u_0, v_0 satisfying $rv_0 - su_0 = 1$; for $a, b \in \mathbf{Z}$ we put

$$K_g(a, b) = \sum_{\substack{x \bmod g \\ (x,g)=1}} e((ax + bx^{-1})/g), \quad (2.2)$$

(x^{-1} denoting an integer such that $xx^{-1} \equiv 1 \pmod{g}$); and

$$\xi = (-1)^{k/2} 2^{k-1} \pi^{k-(1/2)} \Gamma(k-(1/2))^{-1}. \quad (2.3)$$

We investigate the exponential sums involving K_g . For a while we fix $(r, s) \in \mathcal{P}$ and put

$$T \begin{bmatrix} r & u_0 \\ s & v_0 \end{bmatrix} = \begin{pmatrix} l & m/2 \\ m/2 & n \end{pmatrix}, \quad (2.4)$$

so $T \begin{bmatrix} r & u_0 + \sigma r \\ s & v_0 + \sigma s \end{bmatrix} = \begin{pmatrix} l & m/2 \\ m/2 & n \end{pmatrix} \begin{bmatrix} 1 & \sigma \\ 0 & 1 \end{bmatrix}$. Note that: $Q(r, s) = l$, $R_{r,s,\sigma} = 2l\sigma + m$, and $Q(u_0 + \sigma r, v_0 + \sigma s) = l\sigma^2 + m\sigma + n$. The sum in question is:

$$\begin{aligned} A(d, g) &= A(l, m, n; d, g) \\ &= \sum_{\sigma \bmod g} \sum_{\substack{\rho \bmod d \\ (\rho,d)=1}} e((2l\sigma + m)\rho / gd) K_g(d^{-2}l\rho^2, l\sigma^2 + m\sigma + n), \end{aligned} \quad (2.5)$$

where $d^2 | l$.

LEMMA 1. $A(d, g)$ is simultaneously multiplicative in the following sense: if $d = \prod_i p_i^{\alpha_i}$, $g = \prod_i p_i^{\beta_i}$ with distinct primes p_i and non-negative integers α_i, β_i , then $A(d, g) = \prod_i A(p_i^{\alpha_i}, p_i^{\beta_i})$.

Proof. By (2.2) we have

$$K_g(a, b) = \varphi(gd)^{-1} \varphi(g) \sum_{\substack{x \bmod gd \\ (x, gd)=1}} e((ax + bx^{-1})/g)$$

for each integer $d > 0$, where φ is the Euler's function. Thus

$$\begin{aligned} \Lambda(d, g) &= \varphi(gd)^{-1} \varphi(g) \sum_{\sigma \bmod g} \sum_{\substack{x \bmod gd \\ (x, gd)=1}} \sum_{\substack{\rho \bmod d \\ (\rho, d)=1}} e((gd)^{-1}(2l\sigma + m)\rho) \\ &\quad + g^{-1} \{x^{-1}d^{-2}l\rho^2 + x(\sigma^2l + \sigma m + n)\}. \end{aligned}$$

Here we may replace ρ by ρx , since each x satisfies $(x, d)=1$. Hence

$$\begin{aligned} \Lambda(d, g) &= \varphi(gd)^{-1} \varphi(g) \sum_{\sigma \bmod g} \sum_{\substack{x \bmod gd \\ (x, gd)=1}} \sum_{\substack{\rho \bmod d \\ (\rho, d)=1}} \\ &\quad \cdot e((gd)^{-1}x \{(2l\sigma + m)\rho + d^{-1}l\rho^2 + d(\sigma^2l + \sigma m + n)\}) \\ &= \varphi(gd)^{-1} \varphi(g) \sum_{\sigma \bmod g} \sum_{\substack{x \bmod gd \\ (x, gd)=1}} \sum_{\substack{\rho \bmod d \\ (\rho, d)=1}} \\ &\quad \cdot e((gd)^{-1}x \{d^{-1}l(\rho + d\sigma)^2 + m(\rho + d\sigma) + dn\}) \\ &= \varphi(gd)^{-1} \varphi(g) \sum_{\substack{x \bmod gd \\ (x, gd)=1}} \sum_{\substack{\rho \bmod gd \\ (\rho, d)=1}} e((gd)^{-1}x(d^{-1}l\rho^2 + m\rho + dn)). \end{aligned} \quad (2.6)$$

Now suppose that $g = g_1g_2$ and $d = d_1d_2$ with positive integers g_1, g_2, d_1, d_2 such that $(g_1d_1, g_2d_2)=1$. Then, the elements of the set

$$\{x \bmod gd \mid (x, gd)=1\} \quad (\text{resp. } \{\rho \bmod gd \mid (\rho, d)=1\})$$

correspond bijectively to those of

$$\prod_{j=1,2} \{x_j \bmod g_jd_j \mid (x_j, g_jd_j)=1\} \quad (\text{resp. } \prod_{j=1,2} \{\rho_j \bmod g_jd_j \mid (\rho_j, d_j)=1\})$$

through the relation $x = x_1g_2d_2 + x_2g_1d_1$ (resp. $\rho = \rho_1g_2d_2 + \rho_2g_1d_1$). Therefore, noting that $(d_1d_2)^2 \mid l$, we have:

$$\begin{aligned} \Lambda(d, g) &= \varphi(g_1)\varphi(g_1d_1)^{-1} \sum_{\substack{x_1 \bmod g_1d_1 \\ (x_1, g_1d_1)=1}} \sum_{\substack{\rho_1 \bmod g_1d_1 \\ (\rho_1, d_1)=1}} \\ &\quad \cdot e((g_1d_1)^{-1}d_2x_1\{d_1^{-1}l(\rho_1g_2)^2 + m(\rho_1g_2) + d_1n\}) \\ &\quad \cdot \varphi(g_2)\varphi(g_2d_2)^{-1} \sum_{\substack{x_2 \bmod g_2d_2 \\ (x_2, g_2d_2)=1}} \sum_{\substack{\rho_2 \bmod g_2d_2 \\ (\rho_2, d_2)=1}} \\ &\quad \cdot e((g_2d_2)^{-1}d_1x_2\{d_2^{-1}l(\rho_2g_1)^2 + m(\rho_2g_1) + d_2n\}) \\ &= \Lambda(d_1, g_1)\Lambda(d_2, g_2) \end{aligned}$$

by (2.6), which shows that $\Lambda(d, g)$ is simultaneously multiplicative.

Remark 1. If $-|2T|$ is a fundamental discriminant, we do not need Lemma 1 to compute the values of $\Lambda(d, g)$ (cf. Part I, Lemma 3).

Let p be a prime number and r, δ be non-negative integers such that $p^{2r} \mid l$. We calculate the value of $\Lambda(p^r, p^\delta) = \Lambda(l, m, n; p^r, p^\delta)$. If $\delta=0$, then by (2.5),

$$A(p^r, 1) = \sum_{\substack{\rho \bmod p^r \\ (\rho, p^r)=1}} e(p^{-r}m\rho) = \begin{cases} \varphi(p^r) & \text{if } p^{2r} \mid |2T|, \\ -p^{r-1} & \text{if } p^{2r-2} \parallel |2T|, \\ 0 & \text{if } p^{2r-2} \nmid |2T|. \end{cases} \quad (2.7)$$

If $\delta > 0$, then by (2.6) we have:

$$\begin{aligned} A(p^r, p^\delta) &= p^{-r} \sum_{\substack{x \bmod p^{r+\delta} \\ (x, p)=1}} \sum_{\substack{\rho \bmod p^{r+\delta} \\ (\rho, p^r)=1}} e(p^{-r-\delta}x(p^{-r}l\rho^2 + m\rho + p^r n)) \\ &= p^{-r} p^{r+\delta} (A(p; r, \delta) - A(p; r, \delta-1)), \end{aligned} \quad (2.8)$$

where

$$A(p; r, u) = \# \left\{ \rho \bmod p^{r+u} \mid \begin{array}{l} p^{-r}l\rho^2 + m\rho + p^r n \equiv 0 \pmod{p^{r+u}}, \\ (\rho, p^r)=1 \end{array} \right\}.$$

Note that $A(p; r, u) = 0$ if $p^r \nmid m$, since $p^{2r} \nmid l$. Hence

$$A(p^r, p^\delta) = 0 \quad \text{if } \delta > 0, \quad p^{2r} \nmid |2T|. \quad (2.9)$$

Next suppose $p^{2r} \mid l$, $p^r \mid m$. Then

$$A(p; r, u) = \begin{cases} \varphi(p^r) & \text{if } u=0 \\ p^r B(p; r, u) & \text{if } u > 0, \end{cases}$$

where

$$B(p; r, u) = \# \left\{ \rho \bmod p^u \mid \begin{array}{l} p^{-2r}l\rho^2 + p^{-r}m\rho + n \equiv 0 \pmod{p^u}, \\ (\rho, p^r)=1 \end{array} \right\}. \quad (2.10)$$

Hence,

$$A(p^r, p^\delta) = \begin{cases} p(p^r B(p; r, 1) - \varphi(p^r)) & \text{if } \delta=1, \quad p^{2r} \mid |2T|, \\ p^{r+\delta} (B(p; r, \delta) - B(p; r, \delta-1)) & \text{if } \delta \geq 2, \quad p^{2r} \mid |2T|. \end{cases} \quad (2.11)$$

LEMMA 2. *Let the notation be as above. Take the non-negative integer b such that $p^b \parallel |2T|$. Suppose that $T > 0$ is primitive and that $p^{2r} \mid l$, $p^{2r} \mid |2T|$. Then, $B(p; r, u)$ ($u \geq 1$) takes the following values:*

	p : odd prime		$p=2$	
$p \nmid p^{-2r}l$	$1 \leq u \leq b-2r$	$p^{\lfloor u/2 \rfloor}$	$1 \leq u \leq b-2r-2$	$2^{\lfloor u/2 \rfloor}$
			$u = b-2r-1$	$\delta(b/2)2^{(b/2)-r-1}$
			$u = b-2r$	$\delta(b/2)\chi(2)^2 2^{(b/2)-r}$
	$u \geq b-2r+1$	$\delta(b/2)\chi(p)(1+\chi(p))p^{(b/2)-r}$		
$p \mid p^{-2r}l$	$r < b/2$	0		
	$r = b/2$	$\chi(p)(1+\chi(p))/2$		

where $\delta(x)=1$ or 0 according as x is an integer or not.

Proof. First, observe the following :

(1°) By the above assumptions, $(l, m, n)=1$; if $\gamma>0$, then $p \nmid n$.

(2°) If $\gamma>0$, by (2.10) we have

$$B(p; \gamma, u) = \# \left\{ \rho \bmod p^u \mid \begin{matrix} p^{-2\gamma}l + p^{-\gamma}m\rho + n\rho^2 \equiv 0 \pmod{p^u}, \\ (\rho, p) = 1 \end{matrix} \right\}.$$

(3°) If $\gamma>0$ and $p \nmid p^{-2\gamma}l$, then

$$B(p; \gamma, u) = \# \{ \rho \bmod p^u \mid p^{-2\gamma}l + p^{-\gamma}m\rho + n\rho^2 \equiv 0 \pmod{p^u} \}.$$

(4°) Let $F(\rho) = p^{-2\gamma}l + p^{-\gamma}m\rho + n\rho^2$. Then its discriminant is equal to $-p^{-2\gamma}|2T|$. Namely, $4(p^{-2\gamma}l)n - (p^{-\gamma}m)^2 = p^{b-2\gamma}v$, where $|2T| = p^b v$, $p \nmid v$.

(5°) By (4°) we have: $4nF(\rho) - p^{-2\gamma}|2T| = (2n\rho + p^{-\gamma}m)^2$.

Now we examine each case in the table.

(i) $p \geq 3$, $p \nmid p^{-2\gamma}l$.

If $\gamma>0$, then by (1°) (3°) (5°) we have

$$B(p; \gamma, u) = \# \{ \rho \bmod p^u \mid (2n\rho + p^{-\gamma}m)^2 \equiv -p^{b-2\gamma}v \pmod{p^u} \}$$

and $p \nmid 2n$. If $\gamma=0$, then by (2.10) we have

$$\begin{aligned} B(p; \gamma, u) &= \# \{ \rho \bmod p^u \mid l\rho^2 + m\rho + n \equiv 0 \pmod{p^u} \} \\ &= \# \{ \rho \bmod p^u \mid (2l\rho + m)^2 \equiv -p^{b-2\gamma}v \pmod{p^u} \}, \end{aligned}$$

and $p \nmid 2l$. Thus for each $\gamma \geq 0$, $B(p; \gamma, u)$ is equal to the number of the solutions of the congruence

$$x^2 \equiv -p^{b-2\gamma}v \pmod{p^u}, \quad (2.12)$$

where $p \nmid v$.

(i-a) $u > b - 2\gamma$.

If (2.12) is solvable, then it is necessary that $b \equiv 0 \pmod{2}$ and $\chi(p)=1$; here we note that $\left(\frac{-v}{p}\right) = \chi(p)$. Conversely, if these conditions are satisfied, then (2.12) is solvable and has $2p^{(b-2\gamma)/2}$ solutions.

(i-b) $1 \leq u \leq b - 2\gamma$.

$$B(p; \gamma, u) = \# \{ x \bmod p^u \mid x^2 \equiv 0 \pmod{p^u} \} = p^{\lceil u/2 \rceil}.$$

(ii) $p \geq 3$, $p \mid p^{-2\gamma}l$.

Suppose first that $\gamma>0$, hence $p \nmid n$ by (1°). Then, from (5°) we have

$$\begin{aligned}
B(p; \gamma, u) &= \# \left\{ \rho \bmod p^u \mid \begin{array}{l} (2n\rho + p^{-\gamma}m)^2 \equiv -p^{b-2\gamma}v \pmod{p^u}, \\ (\rho, p) = 1 \end{array} \right\} \\
&= \# \left\{ x \bmod p^u \mid \begin{array}{l} x^2 \equiv -p^{b-2\gamma}v \pmod{p^u}, \\ x \not\equiv p^{-\gamma}m \pmod{p} \end{array} \right\}.
\end{aligned} \tag{2.13}$$

Note that

$$(p^{-\gamma}m)^2 \equiv -p^{b-2\gamma}v \pmod{p} \tag{2.14}$$

in this case. If $b > 2\gamma$, then (2.14) implies $p^{-\gamma}m \equiv 0 \pmod{p}$, hence $B(p; \gamma, u) = 0$ by (2.13). If $b = 2\gamma$, then (2.14) implies that $p^{-\gamma}m$ is one of the two solutions of $x^2 \equiv -v \pmod{p}$, so we must have $\chi(p) = 1$, and $B(p; \gamma, u) = 1$ by (2.13).

Next suppose that $\gamma = 0$. If moreover $b > 0$, then by (4°) we have $p \mid m$, hence $p \nmid n$. Thus

$$B(p; 0, u) = \# \{ \rho \bmod p^u \mid l\rho^2 + m\rho + n \equiv 0 \pmod{p^u} \} = 0$$

since $p \mid l$, $p \mid m$, $p \nmid n$. If $\gamma = b = 0$, the proof of

$$B(p; 0, u) = \chi(p)(1 + \chi(p))/2$$

is similar to (1°) (iii) of the proof of Lemma 4 in Part I.

(iii) $p = 2$, $2 \nmid 2^{-2\gamma}l$.

If $\gamma > 0$, then by (1°) (3°) (5°),

$$B(2; \gamma, u) = \# \{ \rho \bmod 2^u \mid (2n\rho + 2^{-\gamma}m)^2 \equiv -2^{b-2\gamma}v \pmod{2^{u+2}} \}$$

and $2 \nmid nv$. If $\gamma = 0$, then (2.10) implies

$$B(2; \gamma, u) = \# \{ \rho \bmod 2^u \mid (2l\rho + m)^2 \equiv -2^b v \pmod{2^{u+2}} \}$$

and $2 \nmid lv$. Hence for each $\gamma \geq 0$ we have

$$B(2; \gamma, u) = \# \{ \rho \bmod 2^u \mid (2\rho + 2^{-\gamma}m)^2 \equiv -2^{b-2\gamma}v \pmod{2^{u+2}} \}. \tag{2.15}$$

Suppose $\gamma < b/2$, hence $b - 2\gamma \geq 2$. Then $2 \mid 2^{-\gamma}m$ and

$$B(2; \gamma, u) = \# \{ \rho \bmod 2^u \mid (\rho + 2^{-\gamma-1}m)^2 \equiv -2^{b-2\gamma-2}v \pmod{2^u} \}.$$

Remaining argument is similar to (i).

Next suppose $\gamma = b/2$. The congruence $x^2 \equiv -v \pmod{2^{u+2}}$ is solvable if and only if $\left(\frac{-v}{2}\right) = \chi(2) = 1$, and if it is so, there exist four solutions. For each solution x_0 of this congruence, ρ in the right hand side of (2.15) is determined by $2\rho + 2^{-\gamma}m \equiv x_0 \pmod{2^{u+1}}$; note that $2^{-\gamma}m \equiv x_0 \pmod{2}$. This yields two distinct values of $\rho \bmod 2^u$. Hence

$$B(2; b/2, u) = \delta(b/2)\chi(2)(1 + \chi(2)).$$

(iv) $p=2, 2|2^{-2\gamma}l$.

Suppose $\gamma < b/2$. Then $2|2^{-\gamma}m$, hence $2 \nmid n$. Thus by (2.10) we have

$$B(2; \gamma, u) = \left\{ \rho \bmod 2^u \left| \begin{array}{l} 2^{-2\gamma}l\rho^2 + 2^{-\gamma}m\rho + n \equiv 0 \pmod{2^u}, \\ (\rho, 2^\gamma) = 1 \end{array} \right. \right\} = 0.$$

Next suppose that $\gamma = b/2 > 0$. Then $2 \nmid n$ by (1°). As in (ii),

$$B(2; b/2, u) = \left\{ \rho \bmod 2^u \left| \begin{array}{l} (2n\rho + 2^{-\gamma}m)^2 \equiv -v \pmod{2^{u+2}}, \\ (\rho, 2) = 1 \end{array} \right. \right\}.$$

Then we proceed as in (iii), noting that ρ is odd. Thus

$$B(2; b/2, u) = \delta(b/2)\chi(2)(1 + \chi(2))/2.$$

Finally suppose $b=2\gamma=0$. In this case the proof is similar to (1°) (iv) of the proof of Lemma 4 in Part I.

Hereafter we suppose that $T > 0$ is primitive. Then it follows from (2.7), (2.9), (2.11) and Lemma 2 that the values of $A(l, m, n; d, g)$ depend only on l, d, g whenever (l, m, n) is defined by (2.4). So we write

$$A(l, m, n; d, g) = A(l; d, g)$$

and put

$$S_p(l, p^r) = \sum_{\delta=0}^{\infty} p^{-k\delta} A(l; p^r, p^\delta). \quad (2.16)$$

By Lemma 1,

$$\sum_{g=1}^{\infty} g^{-k} A(l; d, g) = \prod_p S_p(l, [d]_p), \quad (2.17)$$

where $[d]_p$ denotes the p -part of d . Note that $S_p(l, 1) \neq 0$ since $A(l; 1, 1) = 1$. Therefore (2.17) may also be written

$$\sum_{g=1}^{\infty} g^{-k} A(l; d, g) = \prod_p S_p(l, 1) \cdot \prod_{p \nmid d} S_p(l, 1)^{-1} S_p(l, [d]_p). \quad (2.18)$$

We compute the values of $S_p(l, p^r)$, $p^{2\gamma} | l$. By (2.7) and (2.9),

$$S_p(l, p^r) = \begin{cases} 0 & \text{if } r > 0, p^{2\gamma-2} \nmid |2T|, \\ -p^{r-1} & \text{if } r > 0, p^{2\gamma-2} \parallel |2T|. \end{cases} \quad (2.19)$$

So we next suppose that $p^{2\gamma} \nmid |2T|$, $\gamma > 0$. Then by (2.7) and (2.11),

$$\begin{aligned} S_p(l, p^r) &= \varphi(p^r) + p^{1-k} (B(p; \gamma, 1) - \varphi(p^r)) \\ &\quad + p^r \sum_{\delta=2}^{\infty} p^{(1-k)\delta} (B(p; \gamma, \delta) - B(p; \gamma, \delta-1)), \end{aligned}$$

and by Lemma 2 we obtain the following values (after some computation). If $p \nmid p^{-2\gamma}l$, then:

$$S_p(l, p^r) = -\varepsilon(r)p^{r-1}(1-p^{1-k}) + p^r(1-\chi(p)p^{1-k})^{-1}(1-p^{2-2k}) \\ \cdot \left\{ (1-\chi(p)p^{1-k}) \sum_{a=0}^{j-\gamma} p^{a(3-2k)} + \chi(p)^2 p^{(j-\gamma+1)(3-2k)} \right\}, \quad (2.20)$$

where $\varepsilon(r)=1$ or 0 according as $r \geq 1$ or $r=0$, and $j = [(b-1)/2]$ or $[(b-2)/2]$ according as $p \geq 3$ or $p=2$. If $p \mid p^{-2\gamma}l$, then:

$$S_p(l, p^r) = \begin{cases} (1-p^{1-k})\varphi(p^r) & \text{for } \gamma < b/2, \\ (1-p^{1-k})\varphi(p^r) + 2^{-1}p^{\gamma+1-k}\chi(p)(1+\chi(p)) & \text{for } \gamma = b/2. \end{cases} \quad (2.21)$$

In the above computation we use the following identity appearing also in Maaß [11]:

$$\sum_{u=0}^a p^{(1-k)u} p^{[u/2]} = (1+p^{1-k}) \sum_{n=0}^{[(a-1)/2]} p^{u(3-2k)} + \delta(a/2) p^{a(3-2k)/2}.$$

Now we return to (2.1) and derive a "product expression for $a(T, [f])$ ". As in Part I, let $H(-|2T|)$ be the group of I_1 -equivalence classes of primitive binary positive definite quadratic forms (which we identify with semi-integral matrices of size 2) with discriminant $-|2T|$. Let $-A(T)$ be the discriminant of the quadratic field $K = \mathbf{Q}(\sqrt{-|2T|})$. Put $|2T| = A(T)\mathfrak{f}^2$ with a positive integer $\mathfrak{f}$. Let $T_1, \dots, T_{h(\mathfrak{f})}$ be representatives picked up from each class in $H(-|2T|)$, where $h(\mathfrak{f})$ denotes the cardinality of $H(-|2T|)$. We denote by $\mathcal{O}_K(\mathfrak{f})$ the order of K with discriminant $-A(T)\mathfrak{f}^2$, and by $w(\mathfrak{f})$ the number of roots of unity contained in $\mathcal{O}_K(\mathfrak{f})$. For $\mathfrak{f}=1$, we write $h(1)=h_K$ and $w(1)=w_K$. Let $\mathcal{M}_K(\mathfrak{f})$ be the set of all full modules in K with coefficient ring $\mathcal{O}_K(\mathfrak{f})$. Here, a full module means a $\mathbf{Z}$ -module of rank 2, and for a full module M contained in K , its coefficient ring is defined by

$$\mathcal{O}_M = \{\alpha \in K \mid \alpha M \subset M\}. \quad (2.22)$$

As is well known (see, for example, Borevich-Shafarevich [2]), $\mathcal{M}_K(\mathfrak{f})$ modulo scalar multiplication by $K^\times$ forms a finite abelian group $\mathcal{C}_K(\mathfrak{f})$, which is isomorphic to $H(-|2T|)$.

Let $\phi: H(-|2T|) \rightarrow \mathbf{C}^\times$ be an arbitrary character. By (2.1), (2.4), (2.5) and (2.18), we have:

$$\sum_{j=1}^{h(\mathfrak{f})} \phi(T_j) a(T_j, [f]) = (w(\mathfrak{f})/2) \xi |T|^{k-(3/2)} \prod_p A(p) \\ \cdot \sum_{\nu=1}^{\infty} \sum_{\substack{d^2 \mid \nu \\ d > 0}} \left\{ C(\phi, \nu) a(\nu d^{-2}) \nu^{1-k} \prod_{p \mid \nu} A(p)^{-1} B(p) \prod_{p \nmid d} B(p)^{-1} S_p(\nu, [d]_p) \right\}. \quad (2.23)$$

Here ξ is the constant (2.3), $A(p)$ (resp. $B(p)$) denotes the right hand side of (2.20) (resp. (2.21)) with $\gamma=0$, and

$$C(\phi, \nu) = 2w(\mathfrak{f})^{-1} \sum_{j=1}^{h(\mathfrak{f})} \phi(T_j) \# \left\{ (r, s) \in \mathcal{P} \mid T \begin{bmatrix} r \\ s \end{bmatrix} = \nu \right\}. \quad (2.24)$$

Note that

$$\prod_p S_p(\nu, 1) = \prod_p A(p) \prod_{p|\nu} A(p)^{-1} B(p).$$

LEMMA 3. $C(\phi, \nu)$ is multiplicative with respect to ν .

Proof. By (2.24), we have

$$C(\phi, \nu) = \sum_{a \in \mathcal{R}(\nu)} \phi((\nu, a, n(a))), \quad (2.25)$$

where

$$\mathcal{R}(\nu) = \left\{ a \bmod 2\nu \mid \begin{array}{l} (*) \\ \text{g. c. d.}(\nu, a, (a^2 + |2T|)/4\nu) = 1 \end{array} \right\},$$

and $(\nu, a, n(a))$ denotes the quadratic form $\nu x^2 + axy + n(a)y^2$; for each a , we put $n(a) = (a^2 + |2T|)/4\nu$ (cf. Landau [7]). It is easily verified that the property $(*)$ depends only on the residue of a modulo 2ν . Moreover, for relatively prime positive integers ν_1, ν_2 , the mapping:

$$\begin{aligned} \mathcal{R}(\nu_1 \nu_2) &\longrightarrow \mathcal{R}(\nu_1) \times \mathcal{R}(\nu_2) \\ a \bmod 2\nu_1 \nu_2 &\longmapsto (a \bmod 2\nu_1, a \bmod 2\nu_2) \end{aligned} \quad (2.26)$$

is bijective. Therefore it is sufficient to prove the following: if $a \longleftrightarrow (a_1, a_2)$ under (2.26), then

$$(\nu_1, a_1, n(a_1)) \circ (\nu_2, a_2, n(a_2)) = (\nu_1 \nu_2, a, n(a)), \quad (2.27)$$

where $\circ$ denotes the product in $H(-|2T|)$.

For $c_1, \dots, c_m \in \mathcal{C}$, we put $\{c_1, \dots, c_m\} = \mathbf{Z}c_1 + \dots + \mathbf{Z}c_m$ (a $\mathbf{Z}$ -module). Then, putting $-\mathcal{A}(T) = D$,

$$\begin{aligned} &\{\nu_1, (a_1 - \mathfrak{f}\sqrt{D})/2\} \{\nu_2, (a_2 - \mathfrak{f}\sqrt{D})/2\} \\ &= \{\nu_1, (a - \mathfrak{f}\sqrt{D})/2\} \{\nu_2, (a - \mathfrak{f}\sqrt{D})/2\} \quad (\because a \equiv a_j \pmod{2\nu_j}) \\ &= \{\nu_1 \nu_2, \nu_1(a - \mathfrak{f}\sqrt{D})/2, \nu_2(a - \mathfrak{f}\sqrt{D})/2, (a^2 + \mathfrak{f}^2 D - 2a\mathfrak{f}\sqrt{D})/4\} \\ &= \{\nu_1 \nu_2, (a - \mathfrak{f}\sqrt{D})/2, (a^2 + \mathfrak{f}^2 D - 2a\mathfrak{f}\sqrt{D})/4\} \quad (\because (\nu_1, \nu_2) = 1) \\ &= \{\nu_1 \nu_2, (a - \mathfrak{f}\sqrt{D})/2, (\mathfrak{f}^2 D - a^2)/4\}. \end{aligned}$$

Since $a^2 \equiv \mathfrak{f}^2 D \pmod{4\nu_1 \nu_2}$, the above module is equal to $\{\nu_1 \nu_2, (a - \mathfrak{f}\sqrt{D})/2\}$. This implies (2.27) through the isomorphism $\mathcal{C}_K(\mathfrak{f}) \cong H(-|2T|)$.

Since f is an eigen cusp form, (2.23) with Lemma 3 yields the following product expression:

$$\sum_{j=1}^{h(\mathfrak{f})} \phi(T_j) a(T_j, [f]) = (w(\mathfrak{f})/2) \xi |T|^{k-(3/2)} \prod_p A(p) G(p), \quad (2.28)$$

where

$$G(p) = \sum_{\delta=0}^{\infty} \sum_{\substack{d^2 | p^\delta \\ d > 0}} \left\{ C(\phi, p^\delta) a(p^\delta d^{-2}) p^{\delta(1-k)} \cdot \left(\prod_{p|d} A(p)^{-1} B(p) \right) \cdot \left(\prod_{p|d} B(p)^{-1} S_p(p^\delta, d) \right) \right\}, \quad (2.29)$$

and $A(p)$, $B(p)$ are the same as in (2.23).

First we compute $G(p)$ for $p \nmid \mathfrak{f}$. If $p \nmid \mathfrak{f}$, then for each $\delta \geq 1$ we have the following: if $\chi(p)=1$ and $(p)=\mathfrak{p}\bar{\mathfrak{p}}$ in $\mathcal{O}_K(\mathfrak{f})$, then $C(\phi, p^\delta) = \phi(\mathfrak{p})^\delta + \phi(\bar{\mathfrak{p}})^\delta$; if $\chi(p)=0$ and $(p)=\mathfrak{p}^2$ in $\mathcal{O}_K(\mathfrak{f})$, then $C(\phi, p) = \phi(\mathfrak{p})$ and $C(\phi, p^\delta) = 0$ for $\delta \geq 2$; if $\chi(p)=-1$, then $C(\phi, p^\delta) = 0$ for $\delta \geq 1$. (Here, $\mathfrak{p}$ denotes a prime ideal in $\mathcal{O}_K(\mathfrak{f})$ lying above p .) The proof is similar to that of Part I, Lemma 5(2), with ν running over all positive integers relatively prime to $\mathfrak{f}$. We note that the uniqueness of the prime decomposition holds in $\mathcal{O}_K(\mathfrak{f})$ for each $p \nmid \mathfrak{f}$. (Cf. Proposition in § 3 below where another proof is given.) Hence

$$G(p) = 1 \quad \text{if } p \nmid \mathfrak{f}, \chi(p) = -1. \quad (2.30)$$

Suppose $p \nmid \mathfrak{f}$, $\chi(p)=0$. Then

$$G(p) = 1 + \phi(\mathfrak{p}) a(p) p^{1-k} A(p)^{-1} B(p).$$

In this case we have $A(p) = 1 - p^{2-2k}$, $B(p) = 1 - p^{1-k}$, so

$$G(p) = (1 + p^{1-k})(1 + a(p)\phi(\mathfrak{p})p^{1-k} + \phi(\mathfrak{p})^2 p^{1-k}). \quad (2.31)$$

Next suppose $p \nmid \mathfrak{f}$, $\chi(p)=1$. Then $A(p) = 1 + p^{1-k}$, $B(p) = 1$ and

$$G(p) = 1 + \sum_{\delta=1}^{\infty} \left[\{ \phi(\mathfrak{p})^\delta + \phi(\bar{\mathfrak{p}})^\delta \} p^{\delta(1-k)} (1 + p^{1-k})^{-1} \sum_{\substack{d^2 | p^\delta \\ d > 0}} \left\{ a(p^\delta d^{-2}) \prod_{p|d} S_p(p^\delta, d) \right\} \right]. \quad (2.32)$$

Since $p \nmid |2T|$, we have $S_p(p^\delta, d) = \mu(d)$ if $p|d$, by (2.19). Thus, if $p \nmid \mathfrak{f}$, $\chi(p)=1$, then

$$G(p) = 1 + (1 + p^{1-k})^{-1} \left[\{ \phi(\mathfrak{p}) + \phi(\bar{\mathfrak{p}}) \} p^{1-k} a(p) + \sum_{\delta=2}^{\infty} \{ \phi(\mathfrak{p})^\delta + \phi(\bar{\mathfrak{p}})^\delta \} p^{\delta(1-k)} (a(p^\delta) - a(p^{\delta-2})) \right].$$

The above summation is equal to F_p for $\chi(p)=1$ defined in Part I, (3.21). Namely,

$$\begin{aligned} G(p) &= (1 + p^{1-k})^{-1} (1 - a(p)\phi(\mathfrak{p})p^{1-k} + \phi(\mathfrak{p})^2 p^{1-k})^{-1} \\ &\quad \cdot (1 - a(p)\phi(\bar{\mathfrak{p}})p^{1-k} + \phi(\bar{\mathfrak{p}})^2 p^{1-k})^{-1} \\ &\quad \cdot (1 - p^{1-k})(1 - a(p)p^{1-k} + p^{1-k})(1 + a(p)p^{1-k} + p^{1-k}). \end{aligned} \quad (2.33)$$

We put

$$g_\phi = w(\mathfrak{f})^{-1} \sum_{j=1}^{h(\mathfrak{f})} \phi(T_j) \mathcal{G}_{T_j}, \quad (2.34)$$

which is a modular form of weight 1 and character $\left(\frac{-|2T|}{*}\right)$ with respect to the congruence subgroup $\Gamma_0(|2T|)$ of Γ_1 . Let

$$g_\psi(z) = \sum_{n=0}^{\infty} t(\psi, n) e(nz) \quad (2.35)$$

be its Fourier expansion. By (2.24) we have

$$t(\psi, n) = \sum_{\substack{d^2 | n \\ d > 0}} C(\psi, nd^{-2}) \quad \text{for } n \geq 1, \quad (2.36)$$

hence $t(\psi, n)$ is also multiplicative.

As in Part I (3.22), (3.22a), we see from (2.30), (2.31), (2.33) that: if $p \nmid f$, then

$$A(p)G(p) = \text{the } p\text{-factor of } \frac{L(k-1, \chi)D(k-1, f, g_\psi)}{L_2(2k-2, f)}. \quad (2.37)$$

Now we compute the remaining p -factors. Suppose $p^\alpha \parallel f$, $\alpha \geq 1$.

LEMMA 4. $S_p(p^{\delta+2u}, p^u)$ ($\delta \geq 0, u \geq 1$) takes the following values:

	$1 \leq u \leq \alpha-1$	$u = \alpha$	$u = \alpha+1$	$u \geq \alpha+2$
$\delta \geq 1$	$\varphi(p^u)(1-p^{1-k})$	$-p^{\alpha-1}(1-p^{1-k})$ $+ p^\alpha \left[1 + \left\{ 2^{-1}\chi(p)(1+\chi(p)) - 1 \right\} p^{1-k} \right]$	$-\chi(p)^2 p^\alpha$	0
$\delta = 0$	$-p^{u-1}(1-p^{1-k})$ $+ p^u(1-p^{2-2k})(1-\chi(p)p^{1-k})^{-1} p^{(\alpha-u)(3-2k)} M(p^{\alpha-u})$			

Proof. The above table is a direct consequence of (2.19), (2.20), (2.21); verification is easy if p is an odd prime or if $p=2$ and $\chi(2) \neq 0$. For the case $p=2$ and $\chi(2)=0$, note that $2^{2\alpha+2} \nmid l$ in (2.4). In fact, if $l=2^{2\alpha+2}l'$ with $l' \in \mathbf{Z}$, then $m^2 - 2^{2\alpha+4}l'n = 2^{2\alpha+2}d$ with $d \equiv 2$ or $3 \pmod{4}$, so $m=2^{\alpha+1}m'$ for some $m' \in \mathbf{Z}$ and $(m')^2 - 4l'n = d$, $(m')^2 \equiv d \pmod{4}$, a contradiction.

Hereafter in this section, we fix ψ and put

$$c(n) = C(\psi, n), \quad t(n) = t(\psi, n),$$

$$\mathcal{S}(u) = \sum_{\delta=0}^{\infty} a(p^\delta) c(p^{\delta+2u}) p^{\delta(1-k)}, \quad (2.38)$$

and

$$H(p) = p^{\alpha(2k-3)} (1-\chi(p)p^{1-k})(1-p^{1-k})^{-1} A(p)G(p), \quad (2.39)$$

i. e.,

$$H(p) = (1+p^{1-k})M(p^\alpha)G(p).$$

Remark 2. By Proposition in § 3 below, the series $\mathcal{S}(u)$ is a rational function of p^{-k} with coefficients lying in $\mathbf{Q}(a(n), \phi(T_j) | n \geq 1, 1 \leq j \leq h(\mathfrak{f}))$. But we do not use this fact here.

By (2.29), (3.39) we have:

$$H(p) = p^{\alpha(2k-3)}(1-\chi(p)p^{1-k})(1-p^{1-k})^{-1} \left\{ A(p) + \sum_{\delta=1}^{\infty} a(p^{\delta})c(p^{\delta})p^{\delta(1-k)}(1-p^{1-k}) \right. \\ \left. + \sum_{u=1}^{\alpha+1} \sum_{\delta=0}^{\infty} a(p^{\delta})c(p^{\delta+2u})p^{(\delta+2u)(1-k)}S_p(p^{\delta+2u}, p^u) \right\}.$$

Lemma 4 shows that $S_p(p^{\delta+2u}, p^u) = S_p(p^{1+2u}, p^u)$ for all $\delta \geq 1$, so

$$H(p) = (1+p^{1-k})M(p^{\alpha}) + p^{\alpha(2k-3)}(1-\chi(p)p^{1-k})(\mathcal{S}(0)-1) \\ + p^{\alpha(2k-3)}(1-\chi(p)p^{1-k})(1-p^{1-k})^{-1} \sum_{u=1}^{\alpha} c(p^{2u})p^{2u(1-k)} \{S_p(p^{2u}, p^u) - S_p(p^{2u+1}, p^u)\} \\ + p^{\alpha(2k-3)}(1-\chi(p)p^{1-k})(1-p^{1-k})^{-1} \sum_{u=1}^{\alpha+1} p^{2u(1-k)}S_p(p^{2u+1}, p^u)\mathcal{S}(u). \quad (2.40)$$

Here we used $S_p(p^{2u}, p^u) - S_p(p^{2u+1}, p^u) = 0$ for $u = \alpha + 1$. If $\chi(p) \neq 1$, then it is easy to see that $c(p^{\delta}) = 0$ for all $\delta \geq 2\alpha + 2$ (cf. Proposition in § 3 below), hence $\mathcal{S}(\alpha+1) = 0$. Thus we have always

$$\chi(p)^2(1-\chi(p)p^{1-k})(1-p^{1-k})^{-1}\mathcal{S}(\alpha+1) = \mathcal{S}(\alpha+1). \quad (2.41)$$

By the same reason, $\mathcal{S}(\alpha) = c(p^{2\alpha})$ if $\chi(p) \neq 1$. So the terms in (2.40) corresponding to $u = \alpha$ have the form:

$$p^{\alpha(2k-3)}(1-\chi(p)p^{1-k})(1-p^{1-k})^{-1}c(p^{2\alpha})p^{2\alpha(1-k)} \{S_p(p^{2\alpha}, p^{\alpha}) - S_p(p^{2\alpha+1}, p^{\alpha})\} \\ + p^{\alpha(2k-3)}(1-\chi(p)p^{1-k})(1-p^{1-k})^{-1}p^{2\alpha(1-k)}S_p(p^{2\alpha+1}, p^{\alpha})\mathcal{S}(\alpha) \\ = p^{1-k}c(p^{2\alpha}) + (1-p^{-1}+\chi(p)p^{-k})\mathcal{S}(\alpha). \quad (2.42)$$

For $1 \leq u \leq \alpha - 1$, by Lemma 4 we have

$$p^{\alpha(2k-3)}p^{2u(1-k)}(1-\chi(p)p^{1-k})(1-p^{1-k})^{-1} \{S_p(p^{2u}, p^u) - S_p(p^{2u+1}, p^u)\} \\ = (1+p^{1-k})M(p^{\alpha-u}) - p^{(\alpha-u)(2k-3)}(1-\chi(p)p^{1-k}) \\ = p^{1-k}M(p^{\alpha-u}) + M(p^{\alpha-u-1}).$$

Here we used the following equality:

$$M(p^u) - M(p^{u-1}) = p^{u(2k-3)}(1-\chi(p)p^{1-k}). \quad (2.43)$$

Thus by (2.40) and Lemma 4 we see:

$$\begin{aligned}
H(p) &= p^{1-k}M(p^\alpha) + M(p^{\alpha-1}) + \sum_{u=1}^{\alpha-1} c(p^{2u})\{p^{1-k}M(p^{\alpha-u}) + M(p^{\alpha-u-1})\} \\
&\quad + p^{1-k}c(p^{2\alpha}) + p^{\alpha(2k-3)}(1-\chi(p)p^{1-k})S(0) \\
&\quad + \sum_{u=1}^{\alpha-1} p^{(\alpha-u)(2k-3)}(1-p^{-1})(1-\chi(p)p^{1-k})S(u) \\
&\quad + (1-p^{-1}+\chi(p)p^{-k})S(\alpha) - p^{2-2k}S(\alpha+1). \tag{2.44}
\end{aligned}$$

By (2.43), for each $u \geq 1$ we have

$$\begin{aligned}
M(p^u) - M(p^{u-1}) &+ p^{2-2k}(M(p^u) - M(p^{u+1})) \\
&= p^{u(2k-3)}(1-p^{-1})(1-\chi(p)p^{1-k}). \tag{2.45}
\end{aligned}$$

Putting $M(x)=0$ for $x \notin \mathbf{Z}$, the left hand side of (2.45) is equal to $1-p^{-1}+\chi(p)p^{-k}$ for $u=0$, and $-p^{2-2k}$ for $u=-1$. Comparing these values with (2.44), we obtain

$$\begin{aligned}
H(p) &= M(p^\alpha)\{p^{1-k}+S(0)-p^{2-2k}S(1)\} + \sum_{u=1}^{\alpha} M(p^{\alpha-u}) \\
&\quad \cdot \{c(p^{2u})p^{1-k}+c(p^{2u-2})-S(u-1)+(1+p^{2-2k})S(u)-p^{2-2k}S(u+1)\}. \tag{2.46}
\end{aligned}$$

By (2.36), $c(p^\delta)=t(p^\delta)-t(p^{\delta-2})$ for $\delta \geq 2$. Moreover $t(p^{2u-1})=c(p^{2u-1})=0$ ($1 \leq u \leq \alpha$), which follows easily from the definition of $t(n)$; cf. also Proposition in §3 below. Using

$$a(p^{\delta+2})p^{2-2k}+a(p^{\delta-2})=(a(p^2)a(p^\delta)-p^{k-1}a(p^\delta))p^{2-2k} \quad (\delta \geq 2),$$

we get

$$p^{1-k}+S(0)-p^{2-2k}S(1)=(1+p^{1-k}+p^{2-2k}-a(p^2)p^{2-2k})\sum_{\delta=0}^{\infty} a(p^\delta)t(p^\delta)p^{\delta(1-k)},$$

and

$$\begin{aligned}
&c(p^{2u})p^{1-k}+c(p^{2u-2})-S(u-1)+(1+p^{2-2k})S(u)-p^{2-2k}S(u+1) \\
&=(1+p^{1-k}+p^{2-2k}-a(p^2)p^{2-2k})\sum_{\delta=0}^{\infty} a(p^\delta)(t(p^{\delta+2u})-t(p^{\delta+2u-2}))p^{\delta(1-k)}
\end{aligned}$$

for $1 \leq u \leq \alpha$. Thus (2.46), with (2.39), implies

$$\begin{aligned}
A(p)G(p) &= p^{\alpha(3-2k)}(1-\chi(p)p^{1-k})^{-1}(1-p^{1-k})(1+p^{1-k}+p^{2-2k}-a(p^2)p^{2-2k}) \\
&\quad \cdot \left\{ M(p^\alpha) \sum_{\delta=0}^{\infty} a(p^\delta)t(p^\delta)p^{\delta(1-k)} \right. \\
&\quad \left. + \sum_{u=1}^{\alpha} M(p^{\alpha-u}) \sum_{\delta=0}^{\infty} a(p^\delta)(t(p^{\delta+2u})-t(p^{\delta+2u-2}))p^{\delta(1-k)} \right\}. \tag{2.47}
\end{aligned}$$

Noting that

$$L_2(2k-2, f)^{-1} = \prod_p (1-p^{1-k})(1+p^{1-k}+p^{2-2k}-a(p^2)p^{2-2k}),$$

by (2.28), (2.37), (2.47) we have :

$$\begin{aligned} \sum_{j=1}^{h(\mathfrak{f})} \phi(T_j) a(T_j, [f]) = & (-1)^{k/2} \frac{(k-1)!}{(2k-2)!} (2\pi)^{k-1} w(\mathfrak{f}) \mathcal{A}(T)^{k-(3/2)} \frac{L(k-1, \chi)}{L_2(2k-2, f)} \\ & \cdot \sum_{s|\mathfrak{f}} M(\mathfrak{f}s^{-1}) \sum_{t|s} \mu(t) D(k-1, f, g_{\phi}^{(s/t)}), \end{aligned} \quad (2.48)$$

where

$$g_{\phi}^{(v)}(z) = \sum_{n=0}^{\infty} t(\phi, nv^2) e(nz).$$

Thus we obtain (1.4) for a normalized eigen cusp form f . Observe that the right hand side of (2.48) gives an Euler-product expression.

§ 3. Proof of Theorem 1: The Case $\Phi f \neq 0$.

In this section we treat the case $f = G_k$. Since (1.4) is equivalent to (2.48), we shall prove :

LEMMA 5. *Let*

$$V(\phi) = \sum_{s|\mathfrak{f}} M(\mathfrak{f}s^{-1}) \sum_{t|s} \mu(t) D(k-1, G_k, g_{\phi}^{(s/t)}).$$

(1) *If $\phi = \phi_0$ is trivial, then :*

$$V(\phi_0) = M(\mathfrak{f}) \frac{\zeta_K(k-1) \zeta_K(0)}{L(k-1, \chi)} \mathfrak{f} \prod_{p|\mathfrak{f}} (1 - \chi(p) p^{-1}).$$

(2) *If ϕ is non-trivial, then : $V(\phi) = 0$.*

Here $K = \mathbf{Q}(\sqrt{-|2T|})$ and $\zeta_K(s)$ is the Dedekind zeta function of K .

Suppose Lemma 5 is valid. Theorem 1 for $f = G_k$ is equivalent to :

$$\begin{aligned} a(T, [G_k]) = & h(\mathfrak{f})^{-1} w(\mathfrak{f}) (-1)^{k/2} \frac{(k-1)!}{(2k-2)!} (2\pi)^{k-1} \mathcal{A}(T)^{k-(3/2)} \frac{L(k-1, \chi)}{L_2(2k-2, G_k)} \\ & \cdot \sum_{\phi} \bar{\phi}(T) V(\phi), \end{aligned} \quad (3.1)$$

where ϕ runs over all characters of $H(-|2T|)$. By Lemma 5 which we assume now, the right hand side of (3.1) is equal to :

$$\frac{w(\mathfrak{f})}{h(\mathfrak{f})} (-1)^{k/2} \frac{(k-1)!}{(2k-2)!} (2\pi)^{k-1} \mathcal{A}(T)^{k-(3/2)} M(\mathfrak{f}) \frac{\zeta_K(k-1) \zeta_K(0)}{L_2(2k-2, G_k)} \mathfrak{f} \prod_{p|\mathfrak{f}} (1 - \chi(p) p^{-1}). \quad (3.2)$$

Here,

$$\zeta_K(0) = -\frac{h_K}{w_K}, \quad L_2(2k-2, G_k) = \zeta(0) \zeta(k-1) \zeta(2k-2), \quad \text{and}$$

$$\frac{h(\mathfrak{f})}{w(\mathfrak{f})} = \frac{h_K}{w_K} \mathfrak{f} \prod_{p|\mathfrak{f}} (1 - \chi(p)p^{-1}), \quad (3.3)$$

so (3.2) is equal to

$$(-1)^{k/2} \frac{(k-1)!}{(2k-2)!} 2(2\pi)^{k-1} \mathcal{A}(T)^{k-(3/2)} \frac{L(k-1, \chi)}{\zeta(2k-2)} M(\mathfrak{f})$$

which is exactly the formula of Maaß [11] for $a(T, [G_k])$. Thus Theorem 1 for $\phi \neq 0$ follows from Lemma 5.

The rest of §3 is devoted to the proof of Lemma 5. Let $T > 0$ be primitive and $K = \mathbf{Q}(\sqrt{-|2T|})$. Let $\mathcal{A}(T)$, $\mathfrak{f}$, $H(-|2T|)$, $\mathcal{O}_K(\mathfrak{f})$, $h(\mathfrak{f})$, $w(\mathfrak{f})$, h_K and w_K be as in §2 (see before and after (2.22)). We denote by $\mathcal{M}_K(\mathfrak{f})$ the set of all full modules in K with coefficient ring $\mathcal{O}_K(\mathfrak{f})$. For an order $\mathcal{O}$ in K and a positive integer ν , we put

$$\mathcal{M}_K(\mathfrak{f}, \mathcal{O}) = \{M \in \mathcal{M}_K(\mathfrak{f}) \mid M \subset \mathcal{O}\} \quad (3.4)$$

and

$$\mathcal{M}_K(\mathfrak{f}, \mathcal{O}; \nu) = \{M \in \mathcal{M}_K(\mathfrak{f}, \mathcal{O}) \mid N(M) = \nu\}. \quad (3.5)$$

Here $N(M)$ is the norm of M , i.e., $M\bar{M} = N(M)\mathcal{O}_M$ (cf. (2.22)). As in §2, we put

$$\mathcal{C}_K(\mathfrak{f}) = \mathcal{M}_K(\mathfrak{f}) \text{ modulo scalar multiplication by } K^\times. \quad (3.6)$$

If $\mathfrak{f}' \mid \mathfrak{f}$, the group homomorphism $\mathcal{C}_K(\mathfrak{f}) \rightarrow \mathcal{C}_K(\mathfrak{f}')$ induced by $M \mapsto M\mathcal{O}_K(\mathfrak{f}')$ is denoted by $\nu(\mathfrak{f}, \mathfrak{f}')$. It is well-known that $\nu(\mathfrak{f}, \mathfrak{f}')$ is surjective. Through $\mathcal{C}_K(\mathfrak{f}) \cong H(-\mathcal{A}(T)\mathfrak{f}^2)$, we see also $\nu(\mathfrak{f}, \mathfrak{f}')$ to be

$$\nu(\mathfrak{f}, \mathfrak{f}') : H(-\mathcal{A}(T)\mathfrak{f}^2) \longrightarrow H(-\mathcal{A}(T)(\mathfrak{f}')^2). \quad (3.7)$$

To prove Lemma 5, we investigate the values of $t(\phi, n)$, which is defined by (2.35) for an arbitrary character $\phi : H(-|2T|) \rightarrow \mathbf{C}^\times$. We need an explicit description of the elements of $\mathcal{M}_K(\mathfrak{f}, \mathcal{O}(\mathfrak{f}); n)$.

PROPOSITION. *The notation being as above, let p be a prime number and $\mathfrak{g} = p^{-\alpha}\mathfrak{f}$ where $p^\alpha \parallel \mathfrak{f}$.*

$$(1) \quad t(\phi, n) = \sum_{M \in \mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f}); n)} \phi(M) \quad \text{for } n \geq 1 \quad (3.8)$$

and $t(\phi, n)$ is multiplicative with respect to $n \geq 1$.

(2) Let $(p) = \mathfrak{p}\bar{\mathfrak{p}}$ (resp. $(p) = \mathfrak{p}^2$; $(p) = \mathfrak{p}$) be the prime decomposition of p in $\mathcal{O}_K(\mathfrak{g})$ if $\chi(p) = 1$ (resp. $\chi(p) = 0$; $\chi(p) = -1$). Then the following assertions hold.

(i) There exists $P \in \mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{g}))$ such that $P\mathcal{O}_K(\mathfrak{g}) = \mathfrak{p}$. Moreover P can be taken so that $P = \bar{P}$ if $\chi(p) \neq 1$.

(ii) Let

$$\{M \in \mathcal{M}_K(\mathfrak{f}) \mid M\mathcal{O}_K(\mathfrak{g}) = \mathcal{O}_K(\mathfrak{g})\} = \{M_j \mid j = 1, \dots, \kappa\}, \quad (3.9)$$

which is in fact a finite set and $\kappa = \kappa(p^\alpha) = p^\alpha \prod_{p|\mathfrak{p}^\alpha} (1 - \chi(p)p^{-1})$. Then, for $\delta \geq 2\alpha$,

the elements of $\mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f}); p^\delta)$ are of the following form:

$$\begin{aligned} & \{p^\alpha M_j P^{\delta-2\alpha-t} \bar{P}^t \mid 1 \leq j \leq \kappa, 0 \leq t \leq \delta-2\alpha\} & \text{if } \chi(p)=1; \\ & \{p^\alpha M_j P^{\delta-2\alpha} \mid 1 \leq j \leq \kappa\} & \text{if } \chi(p)=0; \\ & \left. \begin{aligned} & \{p^\alpha M_j P^{(\delta/2)-\alpha} \mid 1 \leq j \leq \kappa\} & \text{for } \delta \text{ even} \\ & \emptyset & \text{for } \delta \text{ odd} \end{aligned} \right\} & \text{if } \chi(p)=-1. \end{aligned}$$

(3) Suppose $0 \leq \delta \leq 2\alpha$. Then:

$$\mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f}); p^\delta) = \{p^{\delta/2} M \mid M \in \mathcal{M}_K(\mathfrak{f}) \text{ such that } M\mathcal{O}_K(\mathfrak{f}p^{-\delta/2}) = \mathcal{O}_K(\mathfrak{f}p^{-\delta/2})\}$$

for δ even;

$$\mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f}); p^\delta) = \emptyset \quad \text{for } \delta \text{ odd.}$$

Moreover, for $0 \leq \delta \leq 2\alpha$, δ even, we have a bijection:

$$\begin{aligned} \mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f}); p^\delta) & \longleftrightarrow (\mathcal{O}_K(\mathfrak{f}p^{-\delta/2})^\times / \mathcal{O}_K(\mathfrak{f})^\times) \times \text{Ker}(\nu(\mathfrak{f}, \mathfrak{f}p^{-\delta/2})) \\ p^{\delta/2} \zeta M & \longleftrightarrow (\zeta \mathcal{O}_K(\mathfrak{f})^\times, \text{the class containing } M). \end{aligned}$$

(4) The notation being as in (2), suppose $\delta \geq 2\alpha$. If ϕ is non-trivial on $\text{Ker}(\nu(\mathfrak{f}, \mathfrak{g}))$, then $t(\phi, p^\delta) = 0$; if ϕ is trivial on $\text{Ker}(\nu(\mathfrak{f}, \mathfrak{g}))$, then

$$t(\phi, p^\delta) = \begin{cases} \kappa(p^\alpha) \sum_{t=0}^{\delta-2\alpha} \phi(P)^{\delta-2\alpha-t} \phi(\bar{P})^t & \text{for } \chi(p)=1, \\ \kappa(p^\alpha) \phi(P)^\delta & \text{for } \chi(p)=0, \\ \kappa(p^\alpha) \phi(P)^{\delta/2} & \text{for } \delta \text{ even, } \chi(p)=-1, \\ 0 & \text{for } \delta \text{ odd, } \chi(p)=-1. \end{cases}$$

(5) Suppose $0 \leq \delta \leq 2\alpha$. If ϕ is trivial on $\text{Ker}(\nu(\mathfrak{f}, \mathfrak{f}p^{-\delta/2}))$ and δ is even, then

$$t(\phi, p^\delta) = \begin{cases} p^{\delta/2} & \text{for } \delta < 2\alpha \\ \kappa(p^\alpha) & \text{for } \delta = 2\alpha \end{cases};$$

otherwise, $t(\phi, p^\delta) = 0$.

Proof. (1) In (2.34), each T_j represents the norm of each module in $\mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f})) \cap C_j$ exactly $w(\mathfrak{f})$ times; here $C_j \in \mathcal{C}_K(\mathfrak{f})$ is the class of modules which corresponds to T_j . Hence we have (3.8). The multiplicativity of $t(\phi, n)$ follows, as we have noted, from Lemma 3 and (2.36).

(2) By the surjectivity of $\nu(\mathfrak{f}, \mathfrak{g})$, there exist $N \in \mathcal{M}_K(\mathfrak{f})$ and $\beta \in K^\times$ such that $\beta N \mathcal{O}_K(\mathfrak{g}) = \mathfrak{p}$. Since $\mathfrak{p} \subset \mathcal{O}_K(\mathfrak{g})$, we have $\beta N \subset \mathcal{O}_K(\mathfrak{g})$. Thus $P = \beta N$ is the desired one. If $\chi(p) \neq 1$, then $\bar{\mathfrak{p}} = \mathfrak{p}$. Replacing P by $P + \bar{P}$ if necessary, we may assume $P = \bar{P}$ in this case. This proves (i).

We prove (ii) in the case $\chi(p) = 1$; other cases are similar. For the finiteness

of the set (3.9), see, for example, Borevich-Shafarevich [2] (Problems 8, 9 of Sec. 7, Chap. 2). The coefficient ring of $p^\alpha M_j P^{\delta-2\alpha-t} \bar{P}^t$ is $\mathcal{O}_K(\mathfrak{f})$ since M_j and P lie in $\mathcal{M}_K(\mathfrak{f})$. Observe that the mapping $\mathcal{M}_K(\mathfrak{f}) \rightarrow \mathcal{M}_K(\mathfrak{g})$ ($N \mapsto N\mathcal{O}_K(\mathfrak{g})$) preserves the norm. Now the module

$$(p^\alpha M_j P^{\delta-2\alpha-t} \bar{P}^t) \mathcal{O}_K(\mathfrak{g}) = \mathfrak{p}^{\alpha+\delta-2\alpha-t} \bar{\mathfrak{p}}^{\alpha+t}$$

has norm p^δ , so the norm of $p^\alpha M_j P^{\delta-2\alpha-t} \bar{P}^t$ is equal to p^δ . Conversely, take a module $N' \in \mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f}); p^\delta)$. Then:

(*) there exists some $N \in \mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{g}))$ such that $N' = p^\alpha N$.

Assuming (*), we proceed as follows. Noting that N has norm $p^{\delta-2\alpha}$, we have

$$N\mathcal{O}_K(\mathfrak{g}) = \mathfrak{p}^{\delta-2\alpha-t} \bar{\mathfrak{p}}^t \quad \text{for some } t, \quad 0 \leq t \leq \delta - 2\alpha.$$

Hence $NP^{-(\delta-2\alpha-t)} \bar{P}^{-t}$ coincides with some M_j . Thus we have

$$N' = p^\alpha N = p^\alpha M_j P^{\delta-2\alpha-t} \bar{P}^t.$$

To prove (*), we consider the generators of N' ; we have

$$N' = s \{a, (-b + \mathfrak{f} \sqrt{-\Delta(T)})/2\},$$

where a, b, s are integers such that $p^\delta = as^2$, $b^2 - 4ac = -\Delta(T)\mathfrak{f}^2$ with an integer c , and $\{x, y\} = \mathbf{Z}x + \mathbf{Z}y$. Cf. Borevich-Shafarevich [2] (Sec. 7, Chap. 2). By $p^\delta = as^2$, we have $p^\alpha | as$. Furthermore $(bs)^2 - 4(as^2)c = -\Delta(T)(gp^\alpha s)^2$ implies $p^\alpha | bs$. Putting $as = p^\alpha a'$, $bs = p^\alpha b'$, we have $N' = p^\alpha N$ with

$$N = \{a', (-b' + sg \sqrt{-\Delta(T)})/2\}.$$

One verifies easily that the right hand side is contained in $\mathcal{O}_K(\mathfrak{g})$. By $N = p^{-\alpha} N'$, the coefficient ring of N is $\mathcal{O}_K(\mathfrak{f})$. Thus (*) is proved.

(3) For a positive integer ν , $\mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f}); \nu) \neq \emptyset$ is equivalent to the existence of integers $s > 0$, $a > 0$, b, c satisfying the conditions:

$$\nu = as^2, \quad b^2 - 4ac = -\Delta(T)\mathfrak{f}^2, \quad (a, b, c) = 1, \quad -a \leq b < a \quad (3.10)$$

(Theorem 6 of Borevich-Shafarevich [2], Sec. 7, Chap. 2). Suppose such s, a, b, c exist for $\nu = p^\delta$, $0 \leq \delta \leq 2\alpha$, δ odd. Then $\alpha \geq 1$. By $p^\delta = as^2$, we have $a = p^\varepsilon$ for some odd integer ε ($< 2\alpha$). Hence by $b^2 - 4ac = -\Delta(T)p^{2\alpha}g^2$, we have $p^\varepsilon | b^2$. We write $b = p^\gamma b'$ with $2\gamma > \varepsilon$ and $p \nmid b'$. By $(a, b, c) = 1$, we have $p \nmid c$. Thus $4c = p(2^{2\gamma-\varepsilon-1}(b')^2 + \Delta(T)p^{2\alpha-\varepsilon-1}g^2)$, which is impossible for $p \geq 3$. If $p = 2$, then we have $2c = 2^{2\gamma-\varepsilon-1}(b')^2 + \Delta(T)2^{2\alpha-\varepsilon-1}g^2$. The right hand side is congruent to 0, 1, or 3 modulo 4, a contradiction. Thus $\mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f}); p^\delta) = \emptyset$ if δ is odd, $0 \leq \delta \leq 2\alpha$. Next suppose that δ is even, $0 \leq \delta \leq 2\alpha$. For $M \in \mathcal{M}_K(\mathfrak{f})$ such that $M\mathcal{O}_K(\mathfrak{f}p^{-\delta/2}) = \mathcal{O}_K(\mathfrak{f}p^{-\delta/2})$, the norm of $p^{\delta/2}M$ is p^δ by the same reason stated before (*) in the proof of (2). Conversely, take any $M' \in \mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f}); p^\delta)$. Then there exists some $M \in \mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f}p^{-\delta/2}))$ such that $M' = p^{\delta/2}M$; the proof is similar to that of

(*) in (2). Observe that M has norm 1 so that $M\mathcal{O}_K(\mathfrak{f}p^{-\delta/2}) \in \mathcal{M}_K(\mathfrak{f}p^{-\delta/2}, \mathcal{O}_K(\mathfrak{f}p^{-\delta/2}); 1)$. Since $M\mathcal{O}_K(\mathfrak{f}p^{-\delta/2}) \subset \mathcal{O}_K(\mathfrak{f}p^{-\delta/2})$, we have $M\mathcal{O}_K(\mathfrak{f}p^{-\delta/2}) = \mathcal{O}_K(\mathfrak{f}p^{-\delta/2})$. This proves the first assertion of (3). To prove the second assertion, let $M'_j \in \mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f}); p^\delta)$ ($j=1, 2$) be similar modules, i.e. $M'_1 = \beta M'_2$ for some $\beta \in K^\times$; here δ is even, $0 \leq \delta \leq 2\alpha$. There exist $M''_j \in \mathcal{M}_K(\mathfrak{f})$ such that $M'_j = p^{\delta/2} M''_j$ and $M''_j \mathcal{O}_K(\mathfrak{f}p^{-\delta/2}) = \mathcal{O}_K(\mathfrak{f}p^{-\delta/2})$, as above. Hence $M'_1 = \beta M'_2$. Then multiplication by $\mathcal{O}_K(\mathfrak{f}p^{-\delta/2})$ yields $\mathcal{O}_K(\mathfrak{f}p^{-\delta/2}) = \beta \mathcal{O}_K(\mathfrak{f}p^{-\delta/2})$, hence $\beta \in \mathcal{O}_K(\mathfrak{f}p^{-\delta/2})^\times$. Moreover $M'_1 = M'_2$ is equivalent to $\beta \in \mathcal{O}_K(\mathfrak{f}) \cap \mathcal{O}_K(\mathfrak{f}p^{-\delta/2})^\times = \mathcal{O}_K(\mathfrak{f})^\times$. Since each class in $\text{Ker}(\nu(\mathfrak{f}, \mathfrak{f}p^{-\delta/2}))$ contains a module $M \in \mathcal{M}_K(\mathfrak{f})$ such that $M\mathcal{O}_K(\mathfrak{f}p^{-\delta/2}) = \mathcal{O}_K(\mathfrak{f}p^{-\delta/2})$, the proof of (3) is completed.

Here we note the following. For $\delta=2\alpha$, the last assertion of (3) with (2) (ii) implies that: $\{M_j | 1 \leq j \leq \kappa\}$ in (3.9) may be taken as $\{\zeta M'_j | 1 \leq j \leq \kappa/(w(\mathfrak{g})/w(\mathfrak{f}))\}$, $\zeta \mathcal{O}_K(\mathfrak{f})^\times \in \mathcal{O}_K(\mathfrak{g})^\times / \mathcal{O}_K(\mathfrak{f})^\times$ such that M'_j ($1 \leq j \leq \kappa/(w(\mathfrak{g})/w(\mathfrak{f}))$) gives representatives of the classes in $\text{Ker}(\nu(\mathfrak{f}, \mathfrak{g}))$.

By the above remark, the assertions (4), (5) follow from (1), (2), (3). Note that: if $0 \leq \delta \leq 2\alpha$, δ even, then by the bijection in (3) we have

$$*\mathcal{M}_K(\mathfrak{f}, \mathcal{O}_K(\mathfrak{f}); p^\delta) = (w(\mathfrak{f}p^{-\delta/2})/w(\mathfrak{f})) \cdot (h(\mathfrak{f})/h(\mathfrak{f}p^{-\delta/2})) = \begin{cases} p^{\delta/2} & (\delta < 2\alpha) \\ \kappa(p^\alpha) & (\delta = 2\alpha) \end{cases}$$

(cf. (3.3)).

Remark 3. In Hirzebruch-Zagier [3] (Proposition 2, Chap. 1), the assertions (4) (5) for the trivial character are obtained by a different method.

Let the notation be as above. We put

$$L(s, \phi) = \sum_{n=1}^{\infty} t(\phi, n) n^{-s}.$$

By Proposition (1), $L(s, \phi)$ and $D(s, G_k, g_\phi^{(v)})$ have Euler-product expressions for $\text{Re}(s)$ sufficiently large:

$$L(s, \phi) = \prod_p L_p(s, \phi)$$

where

$$L_p(s, \phi) = \sum_{\delta=0}^{\infty} t(\phi, p^\delta) p^{-\delta s};$$

$$D(s, G_k, g_\phi^{(v)}) = \prod_p D_p(s, G_k, g_\phi^{(p^u)})$$

where

$$D_p(s, G_k, g_\phi^{(p^u)}) = \sum_{\delta=0}^{\infty} \sigma_{k-1}(p^\delta) t(\phi, p^{\delta+2u}) p^{-\delta s} \quad (3.11)$$

with the integer u such that $p^u \parallel v$.

Now we prove Lemma 5. Let $v > 0$ be an integer dividing $\mathfrak{f}$, and for each prime number p , let α and u be integers such that $p^\alpha \parallel \mathfrak{f}$, $p^u \parallel v$.

(1) Suppose first that $\phi = \phi_0$, the trivial character. Then, by (3.11) and Proposition (4) (5) we have (after some computation):

$$D(s, G_k, g_{\phi_0}^{(v)}) = \frac{\zeta_K(s)\zeta_K(s-k+1)}{L(2s-k+1, \chi)} \prod_{p|\mathfrak{f}} R_p^u(s) \quad (3.12)$$

where

$$\begin{aligned} R_p^u(s) = & (1-p^{k-1})^{-1}(1-\chi(p)p^{k-1-2s})^{-1} \\ & \cdot \left[(1-p^{-s})(1-p^{k-1-s})(1-\chi(p)p^{-s})(1-\chi(p)p^{k-1-s}) \right. \\ & \cdot \sum_{\delta=0}^{\alpha-u-1} (p^{\delta(1-2s)} - p^{k-1+\delta(2k-1-2s)}) + \kappa(p^\alpha)p^{-2(\alpha-u)s} \\ & \cdot \left. \{ (1-p^{k-1-s})(1-\chi(p)p^{k-1-s}) - p^{(2\alpha-2u+1)(k-1)}(1-p^{-s})(1-\chi(p)p^{-s}) \} \right]. \end{aligned} \quad (3.13)$$

Putting

$$V_p(s, \mathfrak{f}) = M(p^\alpha)D_p(s, G_k, g_{\phi_0}) + \sum_{u=1}^{\alpha} M(p^{\alpha-u})(D_p(s, G_k, g_{\phi_0}^{p^u}) - D_p(s, G_k, g_{\phi_0}^{p^{u-1}})),$$

we see

$$V(\phi_0) = \prod_p V_p(k-1, \mathfrak{f}). \quad (3.14)$$

By (3.12) we have

$$\prod_p V_p(s, \mathfrak{f}) = \frac{\zeta_K(s)\zeta_K(s-k+1)}{L(2s-k+1, \chi)} \prod_{p|\mathfrak{f}} \left\{ M(p^\alpha)R_p^0(s) + \sum_{u=1}^{\alpha} M(p^{\alpha-u})(R_p^u(s) - R_p^{u-1}(s)) \right\}. \quad (3.15)$$

The right hand side is a meromorphic function on the whole s -plane and holomorphic at $s=k-1$. By (3.13),

$$R_p^u(k-1) = \kappa(p^\alpha) = p^\alpha \prod_{p|\mathfrak{f}} (1-\chi(p)p^{-1}) \quad \text{for } 0 \leq u \leq \alpha.$$

Hence, by (3.14) and (3.15),

$$V(\phi_0) = \frac{\zeta_K(k-1)\zeta_K(0)}{L(k-1, \chi)} \prod_{p|\mathfrak{f}} M(p^\alpha)\kappa(p^\alpha),$$

which proves Lemma 5 (1).

(2) Next suppose that ϕ is non-trivial. In this case, it is sufficient to prove:

$$D(k-1, G_k, g_{\phi}^{(v)}) = 0 \quad \text{for any } v|\mathfrak{f}. \quad (3.16)$$

For each j , the Epstein zeta function $\sum_{(m,n) \in \mathbb{Z}^2} \left(T_j \begin{bmatrix} m \\ n \end{bmatrix} \right)^{-s}$ is meromorphically continued to the whole s -plane. In particular, it is holomorphic at $s=0$, and the value at $s=0$ does not depend on j . Hence $L(s, \phi)$ is holomorphic at $s=0$ and $L(0, \phi)=0$. Therefore, (3.16) follows from the following

LEMMA 6. $D(s, G_k, g_{\phi}^{(v)})/L(s-k+1, \phi)$ is holomorphic at $s=k-1$ for any $v|\mathfrak{f}$. (Here each function is considered as a meromorphic function on the whole

s-plane by the analytic continuation.)

Proof. We have

$$D(s, G_k, g_\phi^{(v)})/L(s-k+1, \phi) = \prod_p [D_p(s, G_k, g_\phi^{(p^u)})/L_p(s-k+1, \phi)] \quad (3.17)$$

and

$$\prod_{p \nmid \mathfrak{f}} [D_p(s, G_k, g_\phi^{(p^u)})/L_p(s-k+1, \phi)] = \prod_{p \nmid \mathfrak{f}} [(1-\chi(p)p^{k-1-2s})L_p(s, \phi)] \quad (3.18)$$

by Proposition (4) for $p \nmid \mathfrak{f}$. The right hand side of (3.18) converges absolutely and uniformly for $\text{Re}(s) > k/2$, hence holomorphic at $s=k-1$. Next we prove:

$$\prod_{p \mid \mathfrak{f}} [D_p(s, G_k, g_\phi^{(p^u)})/L_p(s-k+1, \phi)] \text{ is holomorphic at } s=k-1. \quad (3.19)$$

This, together with what we have mentioned, implies Lemma 6. Observe that

$$\begin{aligned} D_p(s, G_k, g_\phi^{(p^u)}) = (1-p^{1-k})^{-1} & \left[p^{2(s-k+1)u} \left\{ L_p(s-k+1, \phi) - \sum_{\delta=0}^{2u-1} t(\phi, p^\delta) p^{-\delta(s-k+1)} \right\} \right. \\ & \left. - p^{1-k+2us} \left\{ L_p(s, \phi) - \sum_{\delta=0}^{2u-1} t(\phi, p^\delta) p^{-\delta s} \right\} \right]. \end{aligned}$$

Hence, to prove (3.19), it is sufficient to see:

$$L_p(s, \phi)^{-1} \text{ is holomorphic at } s=0 \text{ for each } p \mid \mathfrak{f}. \quad (3.20)$$

Suppose first that ϕ is non-trivial on $\text{Ker}(\nu(\mathfrak{f}, \mathfrak{f}p^{-\alpha}))$. Then, by Proposition (4) (5), $L_p(s, \phi)$ is a polynomial of p^{-s} and $L_p(0, \phi) > 0$. Hence (3.20) holds. Next suppose that ϕ is trivial on $\text{Ker}(\nu(\mathfrak{f}, \mathfrak{f}p^{-\alpha}))$. We prove (3.20) in the case $\chi(p)=1$. (Other cases are similar.) By Proposition (4) (5), we have

$$L_p(s, \phi) = \sum_{\delta=0}^{\alpha-1} p^{\delta(1-2s)} + \kappa(p^\alpha) p^{-2\alpha s} (1-\phi(P)p^{-s})^{-1} (1-\phi(\bar{P})p^{-s})^{-1}.$$

Thus, if $\phi(P) \neq 1$, then $L_p(s, \phi)$ is holomorphic at $s=0$ and $L_p(0, \phi) > 0$, hence (3.20). If $\phi(P)=1$, then

$$L_p(s, \phi) = (1-p^{-s})^{-2} H(s),$$

where

$$H(s) = (1-p^{-s})^2 \sum_{\delta=0}^{\alpha-1} p^{\delta(1-2s)} + \kappa(p^\alpha) p^{-2\alpha s}.$$

Hence $H(0) \neq 0$ implies (3.20). This completes the proof of Lemma 6, (3.16), and Lemma 5, hence of Theorem 1.

§ 4. Proofs of Theorem 2 and Theorem 3.

The proof of Theorem 2 is entirely similar to that of Theorem 2 in Part I (cf. Remark 2 in Part I, § 4); observe that each $\mathcal{G}_f^{(v)}$ for $v \mid \mathfrak{f}$ is a modular form

of weight 1 and character $\left(\frac{-|2T|}{*}\right)$ with respect to the congruence subgroup $\Gamma_0(|2T|)$ of Γ_1 (Li [9], Lemma 1).

Before the proof of Theorem 3, we note the following facts. If $\Phi f \neq 0$, then $a(T, [f]) \neq 0$ for all $T \geq 0$ by the formula of Siegel [16], Maaß [11, 12]. So we may suppose $\Phi f = 0$. In this case, the product expression (2.28) which converges absolutely for $k \geq 12$ shows that $\sum_{j=1}^{h(\mathfrak{f})} \phi(T_j) a(T_j, [f]) \neq 0$, hence $a(T_j, [f]) \neq 0$ for at least one j . This point was communicated by Professor Maaß, cf. § 1.

Now we prove Theorem 3. We may assume that $\Phi f = 0$ as we have just mentioned. If $\text{rank } T < 2$, then T is unimodularly equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ by the assumption, hence $a(T, [f]) = a(1) \neq 0$. So we assume $T > 0$. Then by the assumption T is primitive. By (2.23), we have:

$$a(T, [f]) = \eta \sum_{\nu=1}^{\infty} \sum_{\substack{d \geq 0 \\ d^2 | \nu}} N(T, \nu) a(\nu d^{-2}) \nu^{1-k} r(\nu, d), \quad (4.1)$$

where

$$\eta = 2^{-1} w(\mathfrak{f}) \xi^{\frac{k}{2}} |T|^{k - (3/2)} \prod_p A(p), \quad N(T, \nu) = \#\left\{ (r, s) \in \mathcal{P} \mid T \begin{bmatrix} r \\ s \end{bmatrix} = \nu \right\},$$

and

$$r(\nu, d) = \prod_{p|\nu} A(p)^{-1} B(p) \prod_{p|d} B(p)^{-1} S_p(\nu, [d]_p).$$

By the assumption,

$$N(T, 1) = w(\mathfrak{f})/2. \quad (4.2)$$

By (2.25), for each $\nu \geq 1$ we have:

$$N(T, \nu) \leq 2^{-1} w(\mathfrak{f}) \#\{x \bmod 2\nu \mid x^2 \equiv -|2T| \pmod{4\nu}\} \leq w(\mathfrak{f})\nu. \quad (4.3)$$

Next we prove:

$$|r(\nu, d)| \leq \varphi(d) \prod_{p|\nu} (1 - p^{2-k})^{-1}. \quad (4.4)$$

To prove this, it is sufficient to see the case where $\nu = p^\delta$ and $d = p^u$ ($\delta \geq 2u$) with a prime number p . It is clear that (4.4) holds for $\nu = d = 1$, so we may assume $\delta > 0$. Suppose first that $u = 0$. Then

$$r(p^\delta, 1) = A(p)^{-1} B(p). \quad (4.5)$$

Here $B(p)$, which is the right hand side of (2.21) with $\gamma = 0$, satisfies $|B(p)| \leq 1$. And $A(p)$, which is the right hand side of (2.20) with $\gamma = 0$, may also be written

$$A(p) = (1 - p^{2-2k}) \sum_{a=0}^j p^{a(3-2k)} + (1 - \chi(p) p^{1-k})^{-1} (1 - p^{2-2k}) \chi(p)^2 p^{(j+1)(3-2k)}.$$

Here $j \geq -1$, and $j = -1$ implies $\chi(p) \neq 0$. Thus

$$A(p) \geq 1 - p^{1-k}. \quad (4.6)$$

Hence by (4.5) we have $|r(p^\delta, 1)| \leq (1 - p^{1-k})^{-1}$, thus (4.4) is valid if $\nu = p^\delta$, $d=1$. Next suppose $u > 0$. Then by (2.5),

$$|A(p^u, p^r)| \leq p^r \varphi(p^r) \varphi(p^u) \quad \text{for } r \geq 0.$$

Hence by (2.16) we have

$$\begin{aligned} |S_p(p^\delta, p^u)| &\leq \sum_{r=0}^{\infty} p^{-kr} |A(p^u, p^r)| \leq \varphi(p^u) \left(1 + \sum_{r=1}^{\infty} p^{(2-k)r} (1 - p^{-1})\right) \\ &= \varphi(p^u) (1 - p^{1-k}) (1 - p^{2-k})^{-1}. \end{aligned}$$

Combining this with (4.6), we obtain $|r(p^\delta, p^u)| \leq \varphi(p^u) (1 - p^{2-k})^{-1}$. Hence (4.4) is proved.

We put

$$S = \sum_{\nu=2}^{\infty} \left\{ \nu \sum_{\substack{d^2 | \nu \\ d > 0}} |a(\nu d^{-2})| \nu^{1-k} \varphi(d) \prod_{p|\nu} (1 - p^{2-k})^{-1} \right\}. \quad (4.7)$$

We shall prove: $S < 1/2$. If this inequality holds, then by (4.1), (4.2), (4.3), (4.4) we get $a(T, [f]) \neq 0$.

By (4.7),

$$1 + S = \prod_p S_p,$$

where

$$\begin{aligned} 1 < S_p &= \sum_{\delta=0}^{\infty} \left\{ p^{\delta(2-k)} \prod_{p|p^\delta} (1 - p^{2-k})^{-1} \left(\sum_{\substack{d^2 | p^\delta \\ d > 0}} |a(p^\delta d^{-2})| \varphi(d) \right) \right\} \\ &\leq (1 - p^{2-k})^{-1} \sum_{\delta=0}^{\infty} \left\{ p^{\delta(2-k)} \sum_{\substack{d^2 | p^\delta \\ d > 0}} \sigma_0(p^\delta d^{-2}) (p^\delta d^{-2})^{(k-1)/2} \varphi(d) \right\} \\ &\leq (1 - p^{2-k})^{-1} \sum_{\delta=0}^{\infty} p^{\delta(3-k)/2} \sum_{u=0}^{[\delta/2]} (\delta - 2u + 1) \varphi(p^u) p^{u(1-k)}. \end{aligned}$$

By $|\varphi(p^u) p^{u(1-k)}| \leq 1$ ($0 \leq u \leq [\delta/2]$) we have

$$S_p \leq (1 - p^{2-k})^{-1} \sum_{\delta=0}^{\infty} p^{\delta(3-k)/2} (\delta + 1) ([\delta/2] + 1).$$

It is easy to see: $(\delta + 1)([\delta/2] + 1) \leq p^{3\delta/2}$ for $p \geq 2$, $\delta \geq 0$. Thus

$$S_p \leq (1 - p^{2-k})^{-1} (1 - p^{3-(k/2)})^{-1}.$$

Hence

$$S = \prod_p S_p - 1 \leq \zeta(k-2) \zeta((k/2)-3) - 1 \leq \zeta(10) \zeta(3) - 1 < 1/2.$$

This completes the proof of Theorem 3.

Remark 4. By the above proof and the formula of Siegel [16], Maaß [11, 12], we see the following. Under the same assumption as in Theorem 3, we have $(-1)^{(r-1)k/2} a(T, [f]) > 0$, where $r = \text{rank } T$.

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Note added in proof

Details of [20] are contained in: Mizumoto, S., On integrality of certain algebraic numbers associated with modular forms. *Math. Ann.*, **265** (1983), 119-135.