THE ESTIMATE OF $\sum \delta(a, f)^\alpha$ FOR MEROMORPHIC FUNCTIONS $f(z)$

Dedicated to Professor Mitsuru Ozawa on the occasion of his 60th birthday

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1. Introduction.

It is known that a meromorphic function $f(z)$ of finite lower order satisfies $\sum \delta(a, f)^\alpha < \infty$ as long as $\alpha \geq 1/3$, where $\delta(a, f)$ denotes the deficiency of $f(z)$ at a and $\sum$ the summation over all extended complex numbers ([3]). In this assertion, we cannot replace α by any positive number less than $1/3$ ([2] p. 98). These facts naturally pose the problem to determine the upper bound $A(\alpha, \mu)$ of $\sum \delta(a, f)^\alpha$ over all meromorphic functions $f(z)$ of lower order $\leq \mu$ ($1/3 \leq \alpha < 1$). To do this, we begin with studying the behaviour of $A(\alpha, \mu)$ as a function of μ . The purpose of this paper is to show

THEOREM. *Let $1/3 < \alpha < 1$. Then*

$$(1) \quad \mu^{1-\alpha} \leq A(\alpha, \mu) \leq C_\alpha \mu^{1-\alpha} \quad (\mu \geq 1),$$

where C_α is a constant depending only on α .

COROLLARY 1. $\lim_{\mu \rightarrow \infty} \log A(\alpha, \mu) / \log \mu = 1 - \alpha \quad (1/3 < \alpha < 1)$.

COROLLARY 2. *Let $1/3 < \alpha < 1$ and $f(z)$ be a meromorphic function of lower order $\mu \geq 1$ such that $\sum \delta(a, f) = s > 0$. Then the sum of deficiencies satisfying $\geq C(\alpha, s)/\mu$ is larger than $s/2$, where $C(\alpha, s) = (s/2C_\alpha)^{1/(1-\alpha)}$.*

Corollary 1 is an immediate consequence of (1). Let $\lambda = C(\alpha, s)/\mu$. Then our theorem shows that

$$\sum_{\delta(a, f) < \lambda} \delta(a, f) < \sum_{\delta(a, f) < \lambda} \delta(a, f)^\alpha \lambda^{1-\alpha} \leq C_\alpha \mu^{1-\alpha} \lambda^{1-\alpha} \leq s/2,$$

which gives Corollary 2.

2. Proof of Theorem.

2.1. The first inequality in (1) is deduced from a well-known example ([2])

Received January 19, 1983

p. 46). For a positive integer n , we put $f_n(z) = \int_0^z \exp(-w^n) dw$. Then $f_n(z)$ is of order n and satisfies $\delta(b_j, f_n) = 1/n$ ($1 \leq j \leq n$), where

$$b_j = \exp(2\pi i/n) \int_0^\infty \exp(-x^n) dx.$$

Hence $\Delta(\alpha, n) \geq 1 + n^{1-\alpha}$ ($n \geq 1$). This gives

$$(2) \quad \Delta(\alpha, \mu) \geq \Delta(\alpha, [\mu]) \geq 1 + [\mu]^{1-\alpha} \geq \mu^{1-\alpha} \quad (\mu \geq 1),$$

where $[\mu]$ denotes the integral part of μ .

2.2. To prove the second inequality in (1), we use the notation in the Nevanlinna theory: $T(r, f)$, $N(r, a, f)$, $n(r, a, f)$, $m(r, a, f)$. (See [2].) We denote by C absolute constants. The value of C differs in general from one occasion to another. Here are two lemmas necessary for the proof; the first lemma is an immediate consequence of the Poisson-Jensen formula ([2] p. 1, p. 22) and the second lemma is a special case of the Milloux theorem ([2] p. 55).

LEMMA 3. Let $g(z)$ be a meromorphic function and let $(a_k)_{k=1}^K$ be its zeros and poles in a disk $|z| < R$. Then, for any z ($|z| < R$) with $g(z) \neq 0, \infty$,

$$(3) \quad \frac{\partial}{\partial t} \log |g(z)| = \Phi(z; R, g) + \phi(z; R, g) + \psi(z; R, g) \quad (z = re^{it}),$$

where

$$(4) \quad \begin{cases} \Phi(z; R, g) = \operatorname{Re} \left\{ (iz/2\pi) \int_0^{2\pi} \log |g(Re^{is})| \{2Re^{is}/(Re^{is}-z)^2\} ds \right\} \\ \phi(z; R, g) = \operatorname{Re} \sum_{k=1}^K \varepsilon_k \bar{a}_k / (R^2 - 2\bar{a}_k z) \\ \quad (\varepsilon_k = 1 \text{ if } g(a_k) = 0, \quad \varepsilon_k = -1 \text{ if } g(a_k) = \infty) \\ \psi(z; R, g) = \operatorname{Re} \sum_{k=1}^K \varepsilon_k / (z - a_k). \end{cases}$$

LEMMA 4. For a meromorphic function $g(z)$, there exists a set $E = E(g)$ in $(0, \infty)$ with $|E| < \infty$ ($|\cdot|$: the 1-dimensional Lebesgue measure) such that

$$(5) \quad T(r, g') \leq 2T(r, g) + o\{T(r, g)\} \quad (r \rightarrow \infty, r \notin E).$$

2.3. Now let $f(z)$ be a meromorphic function of lower order μ_0 and $(b_j)_{j=1}^\infty$ be finite complex numbers with positive deficiency; if such b_j 's do not exist, we have $\Sigma \delta(a, f)^\alpha = \delta(\infty, f)^\alpha \leq 1$. It is sufficient to show that

$$(6) \quad \sum_{j=1}^J \delta(b_j, f)^\alpha \leq C'_\alpha \mu^{1-\alpha} \quad (J \geq 1),$$

where C'_α is a constant depending only on α and $\mu = \max\{2\mu_0, 1\}$. Hence we fix arbitrarily a positive integer J . Without loss of generality, we may assume

that $|b_j - b_k| \geq 2(\pi + 1) (1 \leq j < k \leq J)$; otherwise, we choose $N > 0$ so that $N|b_j - b_k| \geq 2(\pi + 1) (1 \leq j < k \leq J)$ and discuss $\sum_{j=1}^J \delta(Nb_j, Nf)^\alpha (= \sum_{j=1}^J \delta(b_j, f)^\alpha)$. There exists a set F in $(0, \infty)$ with $|F| < \infty$ satisfying the following property: to every $r \in (0, \infty) - F$, there corresponds a finite number of mutually disjoint intervals $\mathcal{L}_r = (I_l)_{l=1}^L$ in $[0, 2\pi)$ such that

$$(7) \quad \sum_{j=1}^J m(r, b_j, f)^\alpha \leq \sum_{l=1}^L \left\{ (1/2\pi) \int_{I_l} \log 1/|rf'(re^{it})| dt \right\}^\alpha + o\{T(r, f)^\alpha\} \quad (r \rightarrow \infty, r \notin F),$$

$r|f'(re^{it})| \leq 1$ ($t \in \bigcup_{l=1}^L I_l$) and, at two end points of every I_l , $r|f'(re^{it})| = 1$, here I is called an interval if $\{e^{it}; t \in I\}$ is an arc on $|z| = 1$. (See [2] p. 96.) Hence the problem is reduced to estimating $\gamma_l = -\int_{I_l} \log |g(re^{it})| dt$ ($1 \leq l \leq L$), where $g(z) = zf'(z)$.

To do this, we choose a sequence $(R_\sigma)_{\sigma=1}^\infty$ of positive numbers and a sequence $(U_\sigma)_{\sigma=1}^\infty$ of intervals in $(0, \infty)$ in the following manner. Since $g(z)$ is of lower order μ_0 , we can choose a sequence $(\tilde{R}_\sigma)_{\sigma=1}^\infty$ of positive numbers tending to infinity so that

$$(8) \quad T(\tilde{R}_\sigma, g)/\tilde{R}_\sigma^\mu = \min\{T(r, g)/r^\mu; 1 \leq r \leq \tilde{R}_\sigma\}.$$

We put $R_\sigma = \tilde{R}_\sigma e^{-1/\mu}$, $U_\sigma = [r_\sigma^*, R_\sigma^*]$ ($\sigma \geq 1$), where $r_\sigma^* = \tilde{R}_\sigma e^{-3/\mu}$, $R_\sigma^* = \tilde{R}_\sigma e^{-2/\mu}$. Since $\lim_{\sigma \rightarrow \infty} |U_\sigma| = \infty$, we may assume that $|U_\sigma^*| \geq |U_\sigma|/2$, where $U_\sigma^* = U_\sigma - (U_\sigma \cap G)$ and $G = F \cup E(g) \cup \{r; g(re^{i\theta}) = \infty \text{ or } 0 \text{ for some } \theta\}$. We may also assume that $r_\sigma^* \in U_\sigma^{*+}$ ($\sigma \geq 1$).

Fixed $\sigma \geq 1$, we write simply $R, U = [r^*, R^*]$ and U^* . Let $r \in U^*$ and $I_l \in \mathcal{L}_r$. The estimate of γ_l requires us to distinguish the following two cases: (a) $|I_l| > 1/\mu$, (b) $|I_l| \leq 1/\mu$. In the case of (a), we have $\gamma_l = \int_{I_l} |\log |g(re^{it})|| dt$. Since the number of I_l 's with (a) is at most $C\mu$, the Hölder inequality and Lemma 4 give

$$(9) \quad \Gamma(r) = \sum_{(a)} \gamma_l^\alpha \leq C\mu^{1-\alpha} \left\{ \sum_{(a)} \int_{I_l} |\log |g(re^{it})|| dt \right\}^\alpha \\ \leq C\mu^{1-\alpha} \{T(r, g)^\alpha + O(1)\} \leq C\mu^{1-\alpha} T(r, f)^\alpha + o\{T(r, f)^\alpha\} \quad (r \rightarrow \infty, r \in G),$$

where $\sum_{(a)}$ denotes the summation over all I_l 's with (a).

In the case of (b), we use Lemma 3. We have, with $I_l = (\alpha_l, \beta_l)$ and $t_l =$ (the midpoint of I_l),

†) Otherwise, we choose $3 \leq p_\sigma \leq 4$ so that $r_\sigma^* = \tilde{R}_\sigma \exp(-p_\sigma/\mu) \in G$.

$$\begin{aligned}
(10) \quad \gamma_l &= - \int_{I_l} \log |g(re^{it})| dt = - \int_{I_l} (t-t_l) \frac{\partial}{\partial t} \log |g(re^{it})| dt \\
&= - \int_{I_l} (t-t_l) \Phi(re^{it}; R, g) dt - \int_{I_l} (t-t_l) \phi(re^{it}; R, g) dt \\
&\quad - \int_{I_l} (t-t_l) \psi(re^{it}; R, g) dt (= -\gamma_l^{(1)} - \gamma_l^{(2)} - \gamma_l^{(3)}).
\end{aligned}$$

2.4. Put $I^{(j)}(r) = \sum_{(b)} |\gamma_l^{(j)}|^\alpha$ ($j=1, 2, 3$), where $\sum_{(b)}$ denotes the summation over all I_l 's with (b). In this paragraph, we estimate $I^{(j)}(r)$ ($j=1, 2$). We have

$$\begin{aligned}
(11) \quad |\gamma_l^{(1)}| &= \left| (|I_l|^2/8) \{ \Phi(re^{i\beta l}; R, g) - \Phi(re^{i\alpha l}; R, g) \} \right. \\
&\quad \left. - (1/2) \int_{I_l} (t-t_l)^2 \frac{\partial}{\partial t} \Phi(re^{it}; R, g) dt \right| \\
&\leq C |I_l|^2 \int_{I_l} \left| \frac{\partial}{\partial t} \Phi(re^{it}; R, g) \right| dt.
\end{aligned}$$

Since

$$\left| \frac{\partial}{\partial t} \Phi(re^{it}; R, g) \right| \leq (1/2\pi) \int_0^{2\pi} |\log |g(Re^{is})|| \left\{ \frac{2Rr}{|Re^{is}-re^{it}|^2} + \frac{4Rr^2}{|Re^{is}-re^{it}|^3} \right\} ds,$$

the Hölder inequality, Lemma 4, (8) and the formula $\int_0^{2\pi} (1-x^2)/|e^{is}-x|^2 ds = 2\pi$ ($0 < x < 1$) yield

$$\begin{aligned}
(12) \quad I^{(1)}(r) &= \sum_{(b)} |\gamma_l^{(1)}|^\alpha \\
&\leq C \sum_{(b)} |I_l|^{2\alpha} \left(\int_{I_l} dt \int_0^{2\pi} |\log |g(Re^{is})|| \left\{ \frac{2Rr}{|Re^{is}-re^{it}|^2} + \frac{4Rr^2}{|Re^{is}-re^{it}|^3} \right\} ds \right)^\alpha \\
&\leq C \left(\sum_{(b)} |I_l| \right)^{1-\alpha} \\
&\quad \times \left(\sum_{(b)} |I_l|^{(3\alpha-1)/\alpha} \int_{I_l} dt \int_0^{2\pi} |\log |g(Re^{is})|| \left\{ \frac{2Rr}{|Re^{is}-re^{it}|^2} + \frac{4Rr^2}{|Re^{is}-re^{it}|^3} \right\} ds \right)^\alpha \\
&\leq C \mu^{1-3\alpha} \left(\int_0^{2\pi} \int_0^{2\pi} |\log |g(Re^{is})|| \left\{ \frac{2Rr}{|Re^{is}-re^{it}|^2} + \frac{4Rr^2}{|Re^{is}-re^{it}|^3} \right\} dt ds \right)^\alpha \\
&\leq C \mu^{1-3\alpha} \left\{ \frac{2Rr}{R^2-r^2} + \frac{4Rr^2}{(R-r)(R^2-r^2)} \right\}^\alpha \left\{ \int_0^{2\pi} |\log |g(Re^{is})|| ds \right\}^\alpha \\
&\leq C \mu^{1-\alpha} T(R, g)^\alpha + O(1) \leq C \mu^{1-\alpha} T(r, g)^\alpha + O(1) \\
&\leq C \mu^{1-\alpha} T(r, f)^\alpha + o\{T(r, f)^\alpha\} \quad (r \rightarrow \infty, r \in G).
\end{aligned}$$

In the same manner as above, we have

$$\begin{aligned}
I^{(2)}(r) &= \sum_{(b)} \left| \int_{I_l} (t-t_l) \phi(re^{it}; R, g) dt \right|^\alpha \leq C \sum_{(b)} |I_l|^{2\alpha} \left\{ \int_{I_l} \left| \frac{\partial}{\partial t} \phi(re^{it}; R, g) \right| dt \right\}^\alpha \\
&\leq C \mu^{1-3\alpha} \left\{ \int_0^{2\pi} \left| \frac{\partial}{\partial t} \phi(re^{it}; R, g) \right| dt \right\}^\alpha \leq C \mu^{1-3\alpha} \left\{ \int_0^{2\pi} \sum_{k=1}^K R^3 r / |R^2 - \bar{a}_k r e^{it}|^2 dt \right\}^\alpha \\
&\leq C \mu^{1-3\alpha} \left\{ \sum_{k=1}^K R^3 r / (R^4 - |a_k|^2 r^2) \right\}^\alpha \leq C \mu^{1-3\alpha} K^\alpha r^\alpha / (R-r)^\alpha \leq C \mu^{1-2\alpha} K^\alpha.
\end{aligned}$$

Lemma 4 and (8) show that

$$\begin{aligned}
(13) \quad K &\leq \frac{1}{\log(\tilde{R}/R)} \int_R^{\tilde{R}} \{n(x, 0, g) + n(x, \infty, g)\} / x \, dx \\
&\leq C \mu \{N(\tilde{R}, 0, g) + N(\tilde{R}, \infty, g)\} + O(1) \leq C \mu T(\tilde{R}, g) + O(1) \\
&\leq C \mu T(r, g) + O(1) \leq C \mu T(r, f) + o\{T(r, f)\} \quad (r \rightarrow \infty, r \in G).
\end{aligned}$$

Thus

$$(14) \quad I^{(2)}(r) \leq C \mu^{1-\alpha} T(r, f)^\alpha + o\{T(r, f)^\alpha\} \quad (r \rightarrow \infty, r \in G).$$

2.5. In this paragraph, we estimate $I^{(3)}(r) = \sum_{(b)} |\gamma l^{(3)}|^\alpha$. Let $a_k = \rho_k e^{i\theta_k}$ ($\rho_k \geq 0, \theta_k \in [0, 2\pi); 1 \leq k \leq K$). Then

$$\begin{aligned}
(15) \quad |\gamma l^{(3)}| &\leq \sum_{k=1}^K \left| \int_{I_l} (t-t_l) r / \{r - \rho_k e^{-i(t-\theta_k)}\} dt \right| = \Sigma^* + \Sigma_* \\
&\leq \Sigma^* \left| \int_{I_l} (t-t_l) r / \{r - \rho_k e^{-i(t-\theta_k)}\} dt \right| \\
&\quad + \Sigma_* \left| \int_{I_l} (t-t_l) h_k(t, r) dt \right| + \Sigma^* \left| \int_{I_l} (t-t_l) H_k(t, r) dt \right|,
\end{aligned}$$

where Σ_* denotes the summation over all k with $\rho_k \leq r^*/2$, $\Sigma^* = \sum_{k=1}^K - \Sigma_*$, $h_k(t, r) = r / \{(r - \rho_k) + i\rho_k(t - \theta_k)\}$ and $H_k(t, r) = r / \{r - \rho_k e^{-i(t-\theta_k)}\} - h_k(t, r)$. We have, in the same manner as in the estimate of $I^{(2)}(r)$,

$$\begin{aligned}
(16) \quad I_*^{(3)}(r) &= \sum_{(b)} \left| \Sigma_* \int_{I_l} (t-t_l) r / \{r - \rho_k e^{-i(t-\theta_k)}\} dt \right|^\alpha \\
&\leq C \mu^{1-2\alpha} K^\alpha \leq C \mu^{1-\alpha} T(r, f)^\alpha + o\{T(r, f)^\alpha\} \quad (r \rightarrow \infty, r \in G).
\end{aligned}$$

Since $|\partial H_k(t, \partial)/\partial t| \leq Cr / |r - \rho_k e^{-i(t-\theta_k)}|$, we have

$$\begin{aligned}
(17) \quad I^{(3)}(r) &= \sum_{(b)} \left| \Sigma^* \int_{I_l} (t-t_l) H_k(t, r) dt \right|^\alpha \\
&\leq C \sum_{(b)} |I_l|^{2\alpha} \left\{ \Sigma^* \int_{I_l} \left| \frac{\partial}{\partial t} H_k(t, r) \right| dt \right\}^\alpha \leq C \mu^{1-3\alpha} \left\{ \Sigma^* \int_0^{2\pi} \left| \frac{\partial}{\partial t} H_k(t, r) \right| dt \right\}^\alpha
\end{aligned}$$

$$\leq C\mu^{1-3\alpha} \left\{ \sum^* \int_0^{2\pi} r/|r-\rho_k e^{-i(t-\theta_k)}| dt \right\}^\alpha.$$

We do not estimate directly the last quantity in (17) but estimate the average over U^* . Note that (13) is valid with $r(r \rightarrow \infty, r \in G)$ replaced by $r^* = r_\sigma^*(\sigma \rightarrow \infty)$. Since $r^*/(C\mu) \leq |U^*| \leq Cr^*/\mu$, we have

$$\begin{aligned} (18) \quad & (1/|U^*|) \int_{U^*} \Gamma^{(3)}(r) dr \\ &= C(\mu/r^*) \int_{U^*} \left\{ \mu^{1-3\alpha} \left(\sum^* \int_0^{2\pi} r/|r-\rho_k e^{-i(t-\theta_k)}| dt \right)^\alpha \right\} dr \\ &\leq C(\mu^{2-3\alpha}/r^*) \left\{ \int_{U^*} dr \right\}^{1-\alpha} \left\{ \sum^* \int_{U^*} \left(\int_0^{2\pi} r/|r-\rho_k e^{-it}| dt \right) dr \right\}^\alpha \\ &\leq C\mu^{1-2\alpha} \left\{ \sum^* \int_{U^*} \left(\int_0^{2\pi} |r-\rho_k e^{-it}|^{-2} dt \right)^{1/2} dr \right\}^\alpha \\ &= C\mu^{1-2\alpha} \left\{ \sum^* \int_{U^*} |r^2 - \rho_k^2|^{-1/2} dr \right\}^\alpha \\ &\leq C(\mu^{1-2\alpha} K^\alpha / r^{*\alpha/2}) \left\{ \int_0^{2|U^*|} 1/\sqrt{x} dx \right\}^\alpha \leq C\mu^{1-5\alpha/2} K^\alpha \\ &\leq C\mu^{1-3\alpha/2} T(r_\sigma^*, f)^\alpha + o\{T(r_\sigma^*, f)^\alpha\} \quad (\sigma \rightarrow \infty). \end{aligned}$$

It remains to estimate $\bar{F}^{(3)}(r) = \sum_{(b)} |\sum^* P_{k,l}(r)|^\alpha$, where $P_{k,l}(r) = \int_{I_l} (t-t_l) h_k(t, r) dt$.

To do this we prepare

LEMMA 5. For an interval $I = (\alpha, \beta) \subset [0, 2\pi)$, $\rho \geq 0$ and $\theta \in [0, 2\pi)$, we put $h(t, r) = r/\{(r-\rho) + i\rho(t-\theta)\}$ and $P(r) = \int_I (t-t_0) h(t, r) dt$ ($r > 0, t \in [0, 2\pi)$), where $t_0 = (\alpha + \beta)/2$. Then, for any $0 < \delta < 1$,

$$(19) \quad |P(r)| \leq C(\rho/r)^\delta |I|^{1+\delta} \int_I |h(t, r)|^{1+\delta} dt \quad (r > 0, r \neq \rho).$$

Proof. We define u by $u = \alpha$ if $|\theta - \alpha| > |I|/2$, $u = \beta$ if $|\theta - \alpha| \leq |I|/2$. Then $|h(u, r)| \leq C|h(t, r)|$ ($t \in I$). We put $\eta = 1 - \delta$. Then

$$\begin{aligned} |P(r)| &= \left| \int_I (t-t_0) \{h(t, r) - h(u, r)\} dt \right| \\ &\leq |I| \int_I |h(t, r) - h(u, r)|^\eta \left| \int_u^t \partial h(s, r) / \partial s ds \right|^\delta dt \\ &\leq C|I| \int_I |h(t, r)|^\eta \left| \int_u^t \partial h(s, r) / \partial s ds \right|^\delta dt. \end{aligned}$$

We distinguish the following two cases: (c) θ is not contained in the interval Y with end points t and u , (d) $\theta \in Y$. In the case of (c), we have $|\partial h(s, r)/\partial s| = (\rho/r) |h(s, r)|^2 \leq C(\rho/r) |h(t, r)|^2 (s \in Y)$. Hence

$$|P(r)| \leq C |I| \int_I |h(t, r)|^{\eta+2\delta} |Y|^\delta dt \leq C |I|^{1+\delta} \int_I |h(t, r)|^{1+\delta} dt.$$

In the case of (d), we have $|\partial h(s, r)/\partial s| \leq (\rho/r) |h(t, r)|^2 (s \in Y - [\theta - (t - \theta), \theta + (t - \theta)])$. Hence $\left| \int_{Y - [\theta - (t - \theta), \theta + (t - \theta)]} \partial h(s, r)/\partial s ds \right| \leq (\rho/r) |I| |h(t, r)|^2$. Let $S = \{z; |z - \sigma| = |t - \theta|, (\operatorname{Re} z)/\sigma \geq 1\}$, where $\sigma = (r/\rho) - 1$. Then

$$\begin{aligned} \left| \int_{\theta - (t - \theta)}^{\theta + (t - \theta)} \partial h(s, r)/\partial s ds \right| &= (r/\rho) \left| \int_{-t+\theta}^{t-\theta} \{\sigma + is\}^{-2} ds \right| \\ &= (r/\rho) \left| \int_S z^{-2} dz \right| \leq (r/\rho) \pi |t - \theta| \max\{|z|^{-2}; z \in S\} \\ &\leq (r/\rho) \pi |t - \theta| / |\sigma + i(t - \theta)|^2 \leq C(\rho/r) |I| |h(t, r)|^2. \end{aligned}$$

Thus, in the same manner as above, we have

$$(20) \quad |P(r)| \leq C(\rho/r) |I|^{1+\delta} \int_I |h(t, r)|^{1+\delta} dt.$$

Consequently, in any case, (20) is valid.

Q. E. D.

Now we put $\delta = \max\{(1/\alpha) - 2, 1/2\} (< 1)$ and use Lemma 5 with $P(r) = P_{kl}(r)$. Then we have, by the Hölder inequality,

$$\begin{aligned} \bar{I}^{(3)}(r) &= \sum_{(b)} |\sum^* P_{kl}(r)|^\alpha \\ &\leq C \sum_{(b)} |I_l|^{\alpha+\delta\alpha} \left\{ \sum^* (\rho_k/r)^\delta \int_{I_l} |h_k(t, r)|^{1+\delta} dt \right\}^\alpha \\ &\leq C \left(\sum_{(b)} |I_l| \right)^{1-\alpha} \left\{ \sum_{(b)} |I_l|^{((2+\delta)\alpha-1)/\alpha} \sum^* \int_{I_l} |h_k(t, r)|^{1+\delta} dt \right\}^\alpha \\ &\leq C \mu^{1-(2+\delta)\alpha} \left\{ \sum^* \int_0^{2\pi} |h_k(t, r)|^{1+\delta} dt \right\}^\alpha. \end{aligned}$$

We have

$$\begin{aligned} (21) \quad & (1/|U^*|) \int_{U^*} \bar{I}^{(3)}(r) dr \\ & \leq C (\mu^{2-(2+\delta)\alpha}/r^*) \int_{U^*} \left\{ \sum^* \int_0^{2\pi} |h_k(t, r)|^{1+\delta} dt \right\}^\alpha dr \\ & \leq C (\mu^{1-(1+\delta)\alpha} r^{*\delta\alpha}) \left\{ \sum^* \int_{U^*} \left(\int_0^\infty |(r - \rho_k) + i\rho_k t|^{-1-\delta} dt \right) dr \right\}^\alpha \end{aligned}$$

$$\begin{aligned}
&\leq \left\{ C \left(\int_0^\infty |1+it|^{-1-\delta} dt \right)^\alpha \right\} \mu^{1-(1+\delta)\alpha} r^{*\delta\alpha} \left\{ \Sigma^*(1/\rho_k) \int_{v^*} |r-\rho_k|^{-\delta} dr \right\}^\alpha \\
&\leq \left\{ C \left(\int_0^\infty |1+it|^{-1-\delta} dt \right)^\alpha \right\} \mu^{1-(1+\delta)\alpha} K^{\alpha} r^{*\delta\alpha-\alpha} \left\{ \int_0^{2/U^*} x^{-\delta} dx \right\}^\alpha \\
&\leq \{C\delta^{-\alpha}(1-\delta)^{-\alpha}\} \mu^{1-2\alpha} K^\alpha \\
&\leq \{C\delta^{-\alpha}(1-\delta)^{-\alpha}\} \mu^{1-\alpha} T(r_\sigma^*, f)^\alpha + o\{T(r_\sigma^*, f)^\alpha\} \quad (\sigma \rightarrow \infty).
\end{aligned}$$

2.6. Now we prove (6). By (7), (9), (12), (14) and (16), we have

$$\begin{aligned}
\sum_{j=1}^J m(r, b_j, f)^\alpha &\leq C \{I(r) + I^{(1)}(r) + I^{(2)}(r) + I_*^{(3)}(r) + \underline{I}^{(3)}(r) + \bar{I}^{(3)}(r)\} \\
&\leq C \mu^{1-\alpha} T(r, f)^\alpha + o\{T(r, f)^\alpha\} + \underline{I}^{(3)}(r) + \bar{I}^{(3)}(r) \quad (r \rightarrow \infty, r \notin G).
\end{aligned}$$

By (18) and (21), we have, with a constant C'_α ,

$$\begin{aligned}
(1/|U_\sigma^*|) \int_{v_\sigma^*} \left\{ \sum_{j=1}^J m(r, b_j, f)^\alpha / T(r, f)^\alpha \right\} dr \\
\leq C \mu^{1-\alpha} + (1/|U_\sigma^*|) \int_{v_\sigma^*} \{\underline{I}^{(3)}(r) + \bar{I}^{(3)}(r)\} dr / T(r_\sigma^*, f)^\alpha + o(1) \\
\leq C'_\alpha \mu^{1-\alpha} + o(1) \quad (\sigma \rightarrow \infty).
\end{aligned}$$

Thus

$$\sum_{j=1}^J \delta(b_j, f)^\alpha \leq \liminf_{\sigma \rightarrow \infty} (1/|U_\sigma^*|) \int_{v_\sigma^*} \left\{ \sum_{j=1}^J m(r, b_j, f)^\alpha / T(r, f)^\alpha \right\} dr \leq C'_\alpha \mu^{1-\alpha}.$$

This completes the proof of our theorem.

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