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A SUBHARMONIC FUNCTION RELATED TO THEOREMS OF BARRY

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Introduction. Let $u(z)$ be a nonconstant subharmonic function in the finite plane and write

$$m^*(r, u)=\inf_{|z|=r} u(z), \qquad M(r, u)=\max_{|z|=r} u(z).$$

The order and lower order of $u(z)$, $\rho(u)$ and $\mu(u)$ respectively, are by definition

$$\rho(u)=\limsup_{r\to\infty}\frac{\log M(r, u)}{\log r}, \qquad \mu(u)=\liminf_{r\to\infty}\frac{\log M(r, u)}{\log r}.$$

If E is a Lebesgue measurable set on the positive real axis, we use the notation $E_r=E\cap[1, r]$, and define the upper logarithmic and lower logarithmic densities, respectively, of E by

$$\overline{\log \text{dens}}\, E=\limsup_{r\to\infty}\frac{1}{\log r}\int_{E_r} t^{-1}dt,$$

$$\underline{\log \text{dens}}\, E=\liminf_{r\to\infty}\frac{1}{\log r}\int_{E_r} t^{-1}dt.$$

Kjellberg [5] proved that, if $0<\mu(u)<1$,

$$\limsup_{r\to\infty}\frac{m^*(r, u)}{M(r, u)}\geqq\cos \pi\mu(u).$$

Barry showed that, if $0\leqq\rho(u)<\alpha<1$,

$$\underline{\log \text{dens}}\{r\,;\, m^*(r, u)>\cos \pi\alpha M(r, u)\}\geqq 1-\rho(u)/\alpha, \tag{1}$$

and that, if $0\leqq\mu(u)<\alpha<1$,

$$\overline{\log \text{dens}}\{r\,;\, m^*(r, u)>\cos \pi\alpha M(r, u)\}\geqq 1-\mu(u)/\alpha. \tag{2}$$

(For (1) see [1]; for (2) see [2]). The above estimates (1) and (2) are both sharp in the sense that the sign $\geqq$ cannot be replaced by $>$. In fact, the following result was proved by Hayman [3].

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THEOREM A. *Given any numbers ρ, α such that $0<\rho<\alpha<1$, there exists a subharmonic function $u(z)$ in the finite plane satisfying the following conditions*:

(i) $\rho(u)=\mu(u)=\rho$,

(ii) $\underline{\text{log dens}}\, E=\overline{\text{log dens}}\, E=1-\rho/\alpha$, *where E is defined by*

$$E=\{r\,;\ m^*(r,u)>\cos\pi\alpha M(r,u)\}\ . \tag{3}$$

In § 1, we show that the relation (1) ((2)) is the only essential restriction imposed on $\underline{\text{log dens}}\, E(\overline{\text{log dens}}\, E)$, where E is the set defined by (3).

THEOREM 1. *Given any numbers ρ, α, γ such that $0<\rho<\alpha<1$ and $0\leqq\gamma\leqq1$ there exists a subharmonic function $u(z)$ in the finite plane satisfying the following conditions*:

(i) $\rho(u)=\mu(u)=\rho$,

(ii) $\underline{\text{log dens}}\, E=\overline{\text{log dens}}\, E=1-\gamma\rho/\alpha$, *where E is defined by* (3).

Since Hayman has given examples for $\gamma=1$, we may consider the cases $0\leqq\gamma<1$. In § 1, we suppose $\gamma>0$. The remaining case $-\gamma=0-$ is handled by minor technical variations of our arguments, and we will omit the proof.

In [6], we showed the following result complementing Theorem A.

THEOREM B. *Given any numbers μ, ρ, α such that $0\leqq\mu<\rho<\alpha<1$, there exists a subharmonic function $u(z)$ in the finite plane satisfying the following conditions*:

(i) $\rho(u)=\rho$,

(ii) $\mu(u)=\mu$,

(iii) $\underline{\text{log dens}}\, E=1-\rho/\alpha$,

(iv) $\overline{\text{log dens}}\, E=1-\mu/\alpha$,

where E is the set defined by (3).

Now, it is natural to ask whether only the relations (1) and (2) are essential restrictions imposed on $\underline{\text{log dens}}\, E$ and $\overline{\text{log dens}}\, E$. I do not know the answer. In § 2, we give examples in this direction.

THEOREM 2. *Let μ, ρ, α, β be any numbers such that $0<\mu<\rho<\alpha<1$ and $\beta>\alpha$. If $\beta\leqq1$, let λ be a number satisfying*

$$\frac{\rho}{\mu} \leqq \lambda < \frac{\beta(\rho-\mu)+(\alpha-\rho)\mu}{\mu(\alpha-\mu)}.$$

If $\beta>1$, *let* λ *be a number satisfying*

$$\frac{\beta(\rho-\mu)+(1-\rho)\mu}{\mu(1-\mu)} < \lambda < \frac{\beta(\rho-\mu)+(\alpha-\rho)\mu}{\mu(\alpha-\mu)}.$$

Then there exists a subharmonic function $u(z)$ *in the finite plane satisfying the following conditions:*

(i) $\rho(u)=\rho$,

(ii) $\mu(u)=\mu$,

(iii) $\overline{\text{log dens}}\, E=1-\mu/\beta$,

(iv) $\underline{\text{log dens}}\, E=1-\lambda\mu/\beta$,

where E *is defined by* (3).

§1. Proof of Theorem 1.

1. Let $\{\alpha_m\}_0^\infty$ be a decreasing sequence tending to α such that $\alpha_0<1$, and set

(1.1) $$\beta_m=\alpha_m\rho(1-\gamma)/(\alpha_m-\gamma\rho) \quad (m=0, 1, 2, \cdots).$$

Define a sequence $\{r_m\}_0^\infty$ by

(1.2) $$r_0=1, \qquad K_m\equiv r_{m+1}/r_m=4+m \quad (m\geqq 0).$$

Further let r'_m be the number satisfying

(1.3) $$\left(\frac{r'_m}{r_m}\right)^{\alpha_m}=\left(\frac{r_{m+1}}{r_m}\right)^{\gamma\rho} \quad (m=0, 1, 2, \cdots).$$

Then, since $\gamma\rho<\rho<\alpha<\alpha_m$, we deduce from (1.2) and (1.3) that

$$r_m<r'_m<r_{m+1} \quad (m=0, 1, 2, \cdots).$$

Now, we put $\nu(t)$ as follows:

(1.4) $$\begin{cases} \nu(t)=0 \quad (0\leqq t<1), \qquad \nu(r_m)=r_m^\rho \quad (m=0, 1, 2, \cdots), \\ \nu(t)/t^{\alpha_m}=r_m^{\rho-\alpha_m} \quad (r_m<t\leqq r'_m), \qquad \nu(t)/t^{\beta_m}=r_{m+1}^{\rho-\beta_m} \quad (r'_m<t<r_{m+1}) \end{cases}$$

It is easy to see from (1.1)—(1.4) that $\nu(t)$ $(t\geqq 1)$ is a continuous increasing function.

LEMMA 1. $\nu(t)$ *has order and lower order equal to* ρ.

Proof. Assume that $r_m\leqq t<r_{m+1}$. From (1.4) and (1.2) we have

$$\nu(t) \leqq r_{m+1}^{\rho} \leqq t^{\rho}\left(\frac{r_{m+1}}{r_m}\right)^{\rho} = t^{\rho}(4+m)^{\rho}$$

and

$$\nu(t) \geqq r_m^{\rho} \geqq t^{\rho}\left(\frac{r_m}{r_{m+1}}\right)^{\rho} \geqq t^{\rho}(4+m)^{-\rho}.$$

By (1.2), we have for $m \geqq 1$

$$r_m = (3+m)(2+m)\cdots 4 \geqq m! = \Gamma(m+1) \sim \sqrt{2\pi}\,(m+1)^{m+1/2}e^{-m-1} \quad (m\to\infty),$$

so that

$$r_m \geqq (m/e)^m \quad (m\text{: large enough}).$$

Hence, for all sufficiently large t

$$t^{\rho}(4+\log t)^{-\rho} < \nu(t) < t^{\rho}(4+\log t)^{\rho}.$$

This proves Lemma 1.

2. Put

$$K_m = (\log K'_m)^{2/\rho}. \tag{2.1}$$

In view of (1.2), (1.3) and (2.1), we have $r'_m/K_m > K_m r_m$ and $r_{m+1}/K_m > K_m r'_m$ $(m \geqq m_0)$. Now, we define F_1 and F_2 as follows:

$$F_1 = \bigcup_{m=m_0}^{\infty} [K_m r_m,\ r'_m/K_m], \qquad F_2 = \bigcup_{m=m_0}^{\infty} [K_m r'_m,\ r_{m+1}/K_m]. \tag{2.2}$$

Then we have the following

LEMMA 2. $\log\text{dens}\, F_1 = \gamma\rho/\alpha$, $\quad \log\text{dens}\, F_2 = 1-\gamma\rho/\alpha$.

Proof. Let R be a large positive number and let m_1 be the integer such that $r'_{m_1}/K_{m_1} \leqq R < r'_{m_1+1}/K_{m_1+1}$. Suppose first that $r'_{m_1}/K_{m_1} \leqq R < K_{m_1+1}r_{m_1+1}$ $(m_1 \geqq m_0)$. Then we have from (2.2), (1.2) and (1.3) that

$$\int_{(F_1)_R}\frac{dt}{t} = \sum_{m=m_0}^{m_1}\int_{K_m r_m}^{r'_m/K_m}\frac{dt}{t}$$

$$= \sum_{m=m_0}^{m_1}\left\{\log\left(\frac{r'_m}{r_m}\right) - 2\log K_m\right\} = \sum_{m=m_0}^{m_1}\left\{\frac{\gamma\rho}{\alpha_m}\log K'_m - 2\log K_m\right\}.$$

In view of (2.2)

$$\log K_m = o(\log K'_m) \quad (m\to\infty). \tag{2.3}$$

Also $\alpha_m \downarrow \alpha (m\to\infty)$. Hence given $\varepsilon>0$, we can choose $N=N(\varepsilon)$, so that for $m_1 \geqq N$

$$\frac{\gamma\rho}{\alpha}\log\frac{r_{m_1+1}}{r_{m_0}}>\int_{(F_1)_R}\frac{dt}{t}$$

$$>\frac{\gamma\rho}{\alpha}(1-\varepsilon)\sum_{m=N}^{m_1}\log K'_m=\frac{\gamma\rho}{\alpha}(1-\varepsilon)\log\frac{r_{m_1+1}}{r_N}.$$

Since $R\in[r'_{m_1}/K_{m_1},\ K_{m_1+1}r_{m_1+1}]$, we have

(2.4)
$$\frac{(\gamma\rho/\alpha)(\log r_{m_1+1}-\log r_{m_0})}{\log r'_{m_1}-\log K_{m_1}}>\frac{1}{\log R}\int_{(F_1)_R}\frac{dt}{t}$$

$$>\frac{(\gamma\rho/\alpha)(1-\varepsilon)(\log r_{m_1+1}-\log r_N)}{\log K_{m_1+1}+\log r_{m_1+1}}.$$

By (1.2), (1.3)

(2.5)
$$\log K'_m=o(\log r_m)\quad(m\to\infty).$$

Using (2.3) and (2.5), we deduce from (2.4) that

(2.6)
$$\frac{\gamma\rho}{\alpha}(1+\varepsilon)>\frac{1}{\log R}\int_{(F_1)_R}\frac{dt}{t}>\frac{\gamma\rho}{\alpha}(1-\varepsilon)^2$$

for all sufficiently large $R\in\bigcup_{m\geqq m_0}[r'_m/K_m,\ K_{m+1}r_{m+1}]$.

Next suppose that $K_{m_1+1}r_{m_1+1}\leqq R<r'_{m_1+1}/K_{m_1+1}\,(m_1\geqq m_0)$. In this case

$$\int_{(F_1)_R}\frac{dt}{t}=\sum_{m=m_0}^{m_1}\left\{\frac{\gamma\rho}{\alpha_m}\log K'_m-2\log K_m\right\}+\log\frac{R}{K_{m_1+1}r_{m_1+1}}.$$

Since $R<r'_{m_1+1}/K_{m_1+1}$, we have

$$\frac{1}{\log R}\int_{(F_1)_R}\frac{dt}{t}<\frac{(\gamma\rho/\alpha)\log(r_{m_1+1}/r_{m_0})+\log R-\log r_{m_1+1}}{\log R}$$

$$<1-\frac{(1-\gamma\rho/\alpha)\log r_{m_1+1}}{\log R}<1-\frac{(1-\gamma\rho/\alpha)\log r_{m_1+1}}{\log r'_{m_1+1}-\log K_{m_1+1}}.$$

On the other hand, since $R>K_{m_1+1}r_{m_1+1}$

$$\frac{1}{\log R}\int_{(F_1)_R}\frac{dt}{t}>\frac{(\gamma\rho/\alpha)(1-\varepsilon)\log(r_{m_1+1}/r_N)+\log(R/K_{m_1+1}r_{m_1+1})}{\log R}$$

$$=\frac{\log R-\log K_{m_1+1}-\{1-(1-\varepsilon)(\gamma\rho/\alpha)\}\log r_{m_1+1}-(1-\varepsilon)(\gamma\rho/\alpha)\log r_N}{\log R}$$

$$>1-o(1)-\{1-(1-\varepsilon)\gamma\rho/\alpha\}\frac{\log r_{m_1+1}}{\log K_{m_1+1}r_{m_1+1}}\quad(m_1\to\infty).$$

Thus for all sufficiently large $R\in\bigcup_{m\geqq m_0}[K_{m+1}r_{m+1},\ r_{m+1}/K_{m+1}]$

(2.7)
$$\frac{\gamma\rho}{\alpha}(1-\varepsilon)^2<\frac{1}{\log R}\int_{(F_1)_R}\frac{dt}{t}<\frac{\gamma\rho}{\alpha}+\varepsilon(1-\gamma\rho/\alpha).$$

Combining (2.6) and (2.7), we have

$$\frac{\gamma\rho}{\alpha}(1-\varepsilon)^2<\frac{1}{\log R}\int_{(F_1)_R}\frac{dt}{t}<\frac{\gamma\rho}{\alpha}+\varepsilon$$

for all sufficiently large R. Hence

$$\frac{\gamma\rho}{\alpha}(1-\varepsilon)^2\leqq\underline{\log\text{dens}}\,F_1\leqq\overline{\log\text{dens}}\,F_1\leqq\frac{\gamma\rho}{\alpha}+\varepsilon\,.$$

Since ε is an arbitrary positive number, we deduce that

$$\log\text{dens}\,F_1=\gamma\rho/\alpha\,.$$

The proof of $\log\text{dens}\,F_2=1-\gamma\rho/\alpha$ is quite similar to the above one.

3. Define $\nu(t)\,(t\geqq0)$ by (1.4). It follows from Lemma 1 that

$$\int_1^\infty\frac{d\nu(t)}{t}=-1+\int_1^\infty\frac{\nu(t)}{t^2}dt<\infty\,.$$

Hence

$$u(z)\equiv\int_0^\infty\log\left|1+\frac{z}{t}\right|d\nu(t)=\text{Re}\left\{\int_1^\infty\frac{z\nu(t)}{t(t+z)}dt\right\} \tag{3.1}$$

is subharmonic in the finite plane (See [4, Theorems 4.1 and 4.2].). Clearly

$$M(r,\,u)=r\int_1^\infty\frac{\nu(t)}{t(t+r)}dt\,. \tag{3.2}$$

By Lemma 1 and (3.2), $\rho(u)=\mu(u)=\rho$.

LEMMA 3. *Suppose that* $z=re^{i\theta}$, *Then if* $u(z)$ *is defined by* (3.1), *we have the following estimates.*

$$\left|u(z)-\frac{\pi\nu(r)}{\sin\pi\alpha_m}\cos\alpha_m\theta\right|<O\Big(\Big(\frac{\log K'_m}{K_m^\rho}+\frac{1}{K_m^{1-\alpha_0}}\Big)\nu(r)\Big) \tag{3.3}$$
$$(K_mr_m\leqq r\leqq r'_m/K_m)\,,$$

$$\left|u(z)-\frac{\pi\nu(r)}{\sin\pi\beta_m}\cos\beta_m\theta\right|<O\Big(\Big(\frac{\log K'_m}{K_m^\rho}+\frac{1}{K_m^{1-\beta}}\Big)\nu(r)\Big) \tag{3.4}$$
$$(K_mr'_m\leqq r\leqq r_{m+1}/K_m)\,,$$

where $\beta=\lim_{m\to\infty}\beta_m=\alpha\rho(1-\gamma)/(\alpha-\gamma\rho)$.

Proof. We prove only (3.3). By (3.1)

$$u(z)=\mathrm{Re}\left\{z\int_1^\infty \frac{\nu(t)}{t(t+z)}dt\right\}=\mathrm{Re}\left\{z\int_1^{r_m}\frac{\nu(t)}{t(t+z)}dt\right\}$$

$$(3.5)\qquad +\mathrm{Re}\left\{z\int_{r_m}^{r_m'}\frac{\nu(t)}{t(t+z)}dt\right\}+\mathrm{Re}\left\{z\int_{r_m'}^{\infty}\frac{\nu(t)}{t(t+z)}dt\right\}\equiv I_1+I_2+I_3,\quad \text{say}.$$

For $t\leqq r_m$, we have $|z/(t+z)|\leqq r/(r-r_m)\leqq K_m/(K_m-1)\leqq 2$, so that

$$|I_1(z)|\leqq 2\int_1^{r_m}\frac{\nu(t)}{t}dt=2\sum_{\mu=1}^{m}\int_{r_{\mu-1}}^{r_\mu}\frac{\nu(t)}{t}dt\leqq 2\sum_{\mu=1}^{m}\nu(r_\mu)\log K'_{\mu-1}$$

$$(3.6)\qquad \leqq 2\log K'_{m-1}\sum_{\mu=1}^{m}r_\mu^\rho\leqq 2(\log K'_{m-1})r_m^\rho\sum_{\mu=0}^{\infty}4^{-\mu\rho}=2r_m^\rho(\log K'_{m-1})\frac{1}{1-4^{-\rho}}$$

$$<\frac{8}{\rho}r_m^\rho(\log K'_{m-1})=\frac{8}{\rho}(r_m/r)^{\alpha_m}\nu(r)(\log K'_{m-1})$$

$$\leqq\frac{8}{\rho}K_m^{-\alpha_m}\nu(r)(\log K'_{m-1})<\frac{8}{\rho}K_m^{-\rho}\nu(r)(\log K'_{m-1}).$$

Assume next that $r_m\leqq t\leqq r_m'$. Then, we have $\nu(t)=r_m^{\rho-\alpha_m}t^{\alpha_m}$. Hence from Lemma 1 in [3] we have for $|\theta|<\pi$

$$(3.7)\qquad \left|I_2(z)-\frac{\pi\nu(r)}{\sin\pi\alpha_m}\cos\alpha_m\theta\right|<\left\{\frac{2}{\alpha_m K_m^{\alpha_m}}+\frac{2}{(1-\alpha_m)K_m^{1-\alpha_m}}\right\}\nu(r).$$

Finally for $t\geqq r_m'$, $|(t+z)/t|\geqq(r_m'-r)/r_m'\geqq 1-1/K_m\geqq 1/2$, so that

$$|I_3(z)|<2r\int_{r_m'}^{\infty}\nu(t)/t^2dt.$$

Since $\nu(t)/t^{\alpha_0}$ decreases for all t, we deduce that

$$\int_{r_m'}^{\infty}\frac{\nu(t)}{t^2}dt\leqq\int_{r_m'}^{\infty}\frac{\nu(r)t^{\alpha_0}}{r^{\alpha_0}}\frac{1}{t^2}dt=\frac{\nu(r)}{r^{\alpha_0}(1-\alpha_0)(r_m')^{1-\alpha_0}}.$$

Hence

$$(3.8)\qquad |I_3(z)|\leqq\frac{2\nu(r)}{1-\alpha_0}\left(\frac{r}{r_m'}\right)^{1-\alpha_0}\leqq\frac{2K_m^{\alpha_0-1}}{1-\alpha_0}\nu(r).$$

Combining (3.5)—(3.8), we obtain (3.3) for $|\theta|<\pi$.

Now, we consider the case $|\theta|=\pi$. Since $\log\left|1+\frac{re^{i\theta}}{t}\right|$ is a decreasing function of θ in $(0,\pi)$, and since $\log\left|1+\frac{re^{i\theta}}{t}\right|\leqq\log\left(1+\frac{r}{t}\right)$ $(0<\theta\leqq\pi)$ with $\int_0^\infty\log\left(1+\frac{r}{t}\right)d\nu(t)<\infty$, the monotone convergence theorem shows that $\lim_{\theta\to\pi-}u(re^{i\theta})=\lim_{\theta\to-\pi+}u(re^{i\theta})=u(-r)$. Hence (3.3) is valid also for $|\theta|=\pi$.

Assume now that $K_m r_m \leqq r \leqq r'_m/K_m (m \geqq m_0)$. We deduce from Lemma 3 that

$$M(r, u)=\left\{\frac{\pi}{\sin \pi\alpha_m}+O\left(\frac{\log K'_m}{K_m^\rho}+\frac{1}{K_m^{1-\alpha_0}}\right)\right\}\nu(r),$$

and that

$$m^*(r, u)=\left\{\frac{\pi \cos \pi\alpha_m}{\sin \pi\alpha_m}-O\left(\frac{\log K'_m}{K_m^\rho}+\frac{1}{K_m^{1-\alpha_0}}\right)\right\}\nu(r).$$

Here we choose $\alpha_m=\alpha+\frac{1-\alpha}{2}(1+\log^+ \log^+ m)^{-1}$. Then

$$\frac{m^*(r, u)}{M(r, u)}\leqq \cos \pi\alpha_m+O\left(\frac{1}{\log m}+\frac{1}{(\log m)^{2(1-\alpha_0)/\rho}}\right)+O\left(\frac{\log r}{\nu(r)}\right)$$

$$\leqq \cos \pi\alpha_m+O\left(\frac{1}{\log m}+\frac{1}{(\log m)^{2(1-\alpha_0/\rho)}}\right)<\cos \pi\alpha_m+o(\alpha_m-\alpha)<\cos \pi\alpha .$$

Hence for all sufficiently large $r\in F_1$

(3.9) $$m^*(r, u)<\cos \pi\alpha\, M(r, u).$$

Next assume that $K_m r'_m \leqq r \leqq r_{m+1}/K_m (m\geqq m_0)$. By (1.1) and the definition of β, $\alpha_m-\alpha=O(\beta-\beta_m)\,(m\to\infty)$. Hence

$$\frac{m^*(r, u)}{M(r, u)}\geqq \cos \pi\beta_m-O\left(\frac{1}{\log m}+\frac{1}{(\log m)^{2(1-\beta)/\rho}}\right)-O\left(\frac{\log r}{\nu(r)}\right)$$

$$\geqq \cos \pi\beta_m-o(\alpha_m-\alpha)\geqq \cos \pi\beta_m-o(\beta-\beta_m)>\cos \pi\beta .$$

Thus for all sufficiently large $r\in F_2$

(3.10) $$m^*(r, u)>\cos \pi\beta M(r, u)>\cos \pi\rho M(r, u)>\cos \pi\alpha M(r, u).$$

4. Define E by (3). Then by (3.9) and (3.10)

$$F_2\cap[R_0, \infty)\subset E\subset[1, \infty)\backslash F_1\cap[R_0, \infty)$$

for a large positive constant R_0. Hence

$$\underline{\log \mathrm{dens}}\, F_2\leqq \underline{\log \mathrm{dens}}\, E\leqq \overline{\log \mathrm{dens}}\, E\leqq 1-\underline{\log \mathrm{dens}}\, F_1 .$$

From this and Lemma 2 we deduce that $\log \mathrm{dens}\, E=1-\gamma\rho/\alpha$. This completes the proof of Theorem 1.

§ 2. Proof of Theorem 2.

5. Put

(5.1) $$\gamma=\frac{(\beta-\mu\lambda)(\rho-\mu)}{(\lambda-1)\mu}+\rho$$

and

$$\delta=\frac{\lambda\mu-\rho}{\lambda-1}. \tag{5.2}$$

The choice of β and λ implies $\alpha<\gamma<1$ and $0\leqq\delta<\mu$. Define two sequences $\{r_m\}_0^\infty$, $\{r'_m\}_1^\infty$ as follows:

$$r_0=1,\quad r_1=3,\quad r_{m+1}=r_m^{\lambda(\beta-\mu)/(\beta-\mu\lambda)}\quad (m=1,2,\cdots), \tag{5.3}$$

$$r'_m=r_m^{(\beta-\mu)/(\beta-\mu\lambda)}\quad (m=1,2,\cdots). \tag{5.4}$$

It is easy to see that $r_m<r'_m<r_{m+1}$ $(m=1, 2, \cdots)$.

Now, we define $\nu(t)$ so that

$$\begin{cases} \nu(t)=0\quad (0\leqq t<1),\qquad \nu(r_m)=r_m^\mu\quad (m=0,1,2,\cdots), \\ \nu(t)=r_m^{\mu-\gamma}t^\gamma\quad (r_m\leqq t\leqq r'_m;\ m\geqq 1),\quad \nu(t)=r_{m+1}^{\mu-\delta}t^\delta\quad (r'_m\leqq t<r_{m+1};\ m\geqq 1). \end{cases} \tag{5.5}$$

We deduce from (5.1)—(5.5) that $\nu(t)\,(t\geqq 1)$ is a continuous increasing function.

LEMMA 4. *$\nu(t)$ has order ρ and lower order μ.*

Proof. We note that $\nu(r_m)=r_m^\mu$ and $\nu(r'_m)=(r'_m)^\rho\,(m=1, 2, \cdots)$. Hence it suffices to show that $t^\mu\leqq\nu(t)\leqq t^\rho\,(r_m\leqq t<r_{m+1};\ m=1, 2, \cdots)$. Assume first that $r_m\leqq t\leqq r'_m$. From (5.5) we have $\nu(t)/t^\mu=(t/r_m)^{\gamma-\mu}\geqq 1$. On the other hand, we deduce from (5.5), (5.4) and (5.1) that $\nu(t)/t^\rho=r_m^{\mu-\gamma}t^{\gamma-\rho}\leqq r_m^{\mu-\gamma}(r'_m)^{\gamma-\rho}=r_m^{\mu-\gamma+(\beta-\mu)(\gamma-\rho)/\beta-\mu\lambda}=r_m^0=1$. Assume next that $r'_m\leqq t<r_{m+1}$. By (5.5) $\nu(t)/t^\mu=(r_{m+1}/t)^{\mu-\delta}\geqq 1$, and from (5.2)—(5.5) we have

$$\nu(t)/t^\rho=r_{m+1}^{\mu-\delta}t^{\delta-\rho}\leqq r_{m+1}^{\mu-\delta}(r'_m)^{\delta-\rho}=r_m^{(\mu-\delta)\lambda(\beta-\mu)/(\beta-\mu\lambda)+(\beta-\mu)(\delta-\rho)/(\beta-\mu\lambda)}=r_m^0=1.$$

6. We set $K'_m=r_{m+1}/r_m\,(m=1, 2, \cdots)$ and define

$$K_m=(\log K'_m)^{2/\mu}. \tag{6.1}$$

In view of (5.3), (5.4) and (6.1), we have $r'_m/K_m>K_m r_m$ and $r_{m+1}/K_m>K_m r'_m$ $(m\geqq m_0)$. Define two sets F_1 and F_2 by (2.2). Then the same reasoning as in the proof of Lemma 2 gives the following.

LEMMA 5. $\underline{\text{log dens}}\, F_1=\mu/\beta$, $\overline{\text{log dens}}\, F_1=\lambda\mu/\beta$, $\underline{\text{log dens}}\, F_2=1-\lambda\mu/\beta$, $\overline{\text{log dens}}\, F_2=1-\mu/\beta$.

7. Define $\nu(t)$ by (5.5). Then from Lemma 4 we deduce that

$$u(z)\equiv\int_0^\infty \log\left|1+\frac{z}{t}\right|d\nu(t)=\mathrm{Re}\left\{\int_1^\infty \frac{z\nu(t)}{t(t+z)}dt\right\} \tag{7.1}$$

is subharmonic in the finite plane. Clearly

$$M(r,\ u)=r\int_1^\infty \frac{\nu(t)}{t(t+r)}\,dt\,. \tag{7.2}$$

Here we prove the following

LEMMA 6. $\rho(u)=\rho\,,\quad \mu(u)=\mu\,.$

Proof. As noted in the proof of Lemma 4, $t^{\mu}\leqq\nu(t)\leqq t^{\rho}(t\geqq 1)$. From this and (7.2) it is easy to see that $\mu\leqq\mu(u)\leqq\rho(u)\leqq\rho$. We proceed to show that $\rho(u)\geqq\rho$. For this purpose, note that $N(r)=\int_1^r \nu(t)t^{-1}dt$ has the same order as $\nu(r)$, and so by Lemma 4 it has order ρ. Further by the subharmonic from of Jensen's Theorem

$$M(r,\ u)\geqq\frac{1}{2\pi}\int_{-\pi}^{\pi}u(re^{i\theta})d\theta=N(r)\,.$$

Thus we have $\rho(u)\geqq\rho$. It remains to prove that $\mu(u)\leqq\mu$. By (7.2)

$$M(r_m,\ u)=r_m\left(\int_1^{r_m}+\int_{r_m}^\infty\right)\frac{\nu(t)}{t(t+r_m)}\,dt\equiv J_1+J_2\,,\quad \text{say}\,. \tag{7.3}$$

Clearly

$$J_1\leqq\int_1^{r_m}\frac{\nu(t)}{t}\,dt\,.$$

Computing as in (3.6), we have

$$J_1\leqq\frac{4}{\mu}r_m^{\mu}\log K'_{m-1}=\frac{4(\lambda-1)\beta}{\mu\lambda(\beta-\mu)}r_m^{\mu}\log r_m\,. \tag{7.4}$$

Since $\nu(t)/t^{\gamma}$ decreases for all t, we have

$$\begin{aligned}J_2&\leqq r_m\int_{r_m}^\infty\frac{\nu(t)}{t^2}\,dt\leqq r_m\nu(r_m)\frac{1}{r_m^{\gamma}}\int_{r_m}^\infty t^{\gamma-2}dt\\&=\frac{\nu(r_m)}{1-\gamma}=\frac{r_m^{\mu}}{1-\gamma}\,.\end{aligned} \tag{7.5}$$

Combining (7.3)—(7.5), we obtain

$$M(r_m,\ u)\leqq O(r_m^{\mu}\log r_m)\quad(m\to\infty)\,.$$

Hence

$$\mu(u)\leqq\liminf_{m\to\infty}\frac{\log M(r_m,\ u)}{\log r_m}\leqq\mu\,.$$

8. We first suppose that $\lambda>\rho/\mu$. By (5.2) $\delta>0$. In this case the same arguments as in the proof of Lemma 3 give

(8.1)
$$\left|u(z)-\frac{\pi\nu(r)}{\sin\pi\gamma}\cos\gamma\theta\right|<O\Big(\Big(\frac{\log K'_m}{K^{\mu}_m}+\frac{1}{K^{1-\gamma}_m}\Big)\nu(r)\Big)$$
$$(z=re^{i\theta},\ K_m r_m\leqq r\leqq r'_m/K_m,\ m\geqq m_0),$$

and

(8.2)
$$\left|u(z)-\frac{\pi\nu(r)}{\sin\pi\delta}\cos\delta\theta\right|<O\Big(\Big(\frac{\log K'_m}{K^{\mu}_m}+\frac{1}{K^{1-\delta}_m}\Big)\nu(r)\Big)$$
$$(z=re^{i\theta},\ K_m r'_m\leqq r\leqq r_{m+1}/K_m,\ m\geqq m_0).$$

Since $\alpha<\gamma<1$, we deduce from (8.1) that

(8.3)
$$m^*(r,\ u)<\cos\pi\alpha M(r,\ u)\quad(r\in F_1,\ r\geqq R_0).$$

Similarly, by (8.2) and the fact that $0<\delta<\alpha$

(8.4)
$$m^*(r,\ u)>\cos\pi\alpha M(r,\ u)\quad(r\in F_2,\ r\geqq R_0).$$

From (8.3) and (8.4) it follows that

$$F_2\cap[R_0,\ \infty)\subset E\subset[1,\ \infty)\backslash(F_1\cap[R_0,\ \infty))$$

for a large positive constant R_0. Hence

(8.5)
$$\underline{\log\text{dens}}\,F_2\leqq\underline{\log\text{dens}}\,E\leqq1-\overline{\log\text{dens}}\,F_1,$$

and

(8.6)
$$\overline{\log\text{dens}}\,F_2\leqq\overline{\log\text{dens}}\,E\leqq1-\underline{\log\text{dens}}\,F_1.$$

Combining (8.5), (8.6) with Lemma 5, we have

$$\underline{\log\text{dens}}\,E=1-\lambda\mu/\beta,\qquad\overline{\log\text{dens}}\,E=1-\mu/\beta.$$

Next, we suppose that $\lambda=\rho/\mu$. It is easy to see that (8.1) remains true in this case. We estimate $u(re^{i\theta})$ for $K_m r'_m\leqq r\leqq r_{m+1}/K_m$ $(m\geqq m_0)$. We write

(8.7)
$$u(z)=\mathrm{Re}\Big\{\Big(\int_1^{r_m}+\int_{r_m}^{r'_m}+\int_{r'_m}^{r_{m+1}}+\int_{r_{m+1}}^{\infty}\Big)\frac{z\nu(t)}{t(t+z)}dt\Big\}$$
$$\equiv I_1+I_2+I_3+I_4,\ \text{say}.$$

For $t\leqq r_m$,

$$|I_1(z)|\leqq2\int_1^{r_m}\frac{\nu(t)}{t}dt.$$

Hence by (7.4)

(8.8)
$$|I_1(z)|\leqq\frac{8(\lambda-1)\beta}{\mu\lambda(\beta-\mu)}r^{\mu}_m\log r_m=\frac{8(\lambda-1)\beta(\beta-\mu\lambda)}{\mu(\beta-\mu)^2\lambda^2}r_{m+1}^{(\mu/\lambda)(\beta-\mu\lambda)/(\beta-\mu)}\log r_{m+1}.$$

Next,

$$(8.9)\qquad |I_2(z)|\leqq 2\int_{r_m}^{r'_m}\frac{\nu(t)}{t}dt=2\int_{r_m}^{r'_m} r_m^{\mu-\gamma}t^{\gamma-1}dt<\frac{2}{\gamma}r_m^{\mu-\gamma}(r'_m)^\gamma=\frac{2}{\gamma}r_{m+1}^\mu .$$

Since $\nu(t)/t^\gamma$ is a decreasing function, we have

$$(8.10)\qquad \begin{aligned}|I_4(z)|&\leqq 2r\int_{r_{m+1}}^\infty \frac{\nu(t)}{t^2}dt\leqq 2r\int_{r_{m+1}}^\infty r_{m+1}^{\mu-\gamma}t^{\gamma-2}dt\\ &=\frac{2r\cdot r_{m+1}^{\mu-1}}{1-\gamma}\leqq\frac{2r_{m+1}^\mu}{(1-\gamma)K_m}.\end{aligned}$$

Finally, for $|\arg z|<\pi$

$$\begin{aligned}I_3(z)&=r_{m+1}^\mu \operatorname{Re}\int_{r'_m}^{r_{m+1}}\frac{z}{t(t+z)}dt=r_{m+1}^\mu\log\left|\frac{r_{m+1}}{r_{m+1}+z}\frac{r'_m+z}{r'_m}\right|\\ &=r_{m+1}^\mu\left\{\log\frac{r}{r'_m}+\log\left|\frac{1+(r'_m/z)}{1+(z/r_{m+1})}\right|\right\}.\end{aligned}$$

Hence

$$(8.11)\qquad \left|I_3(z)-r_{m+1}^\mu\log\frac{r}{r'_m}\right|\leqq r_{m+1}^\mu\log\frac{1+K_m^{-1}}{1-K_m^{-1}}\leqq\frac{3}{K_m}r_{m+1}^\mu .$$

Combining (8.7)—(8.11) we deduce that for $|\theta|<\pi$

$$(8.12)\qquad \left|u(re^{i\theta})-\left(\log\frac{r}{r'_m}\right)r_{m+1}^\mu\right|\leqq O(r_{m+1}^\mu)\quad (K_m r'_m\leqq r\leqq r_{m+1}/K_m) .$$

However as noted in 4, $\lim_{\theta\to\pi-}u(re^{i\theta})=\lim_{\theta\to-\pi+}u(re^{i\theta})=u(-r)$, so that (8.12) holds also for $|\theta|=\pi$.

Since $\alpha<\gamma<1$, we obtain from (8.1) that

$$(8.13)\qquad m^*(r,\ u)<\cos\pi\alpha M(r,\ u)\quad (r\in F_1,\ r\geqq R_0) .$$

On the other hand, (8.12) gives

$$(8.14)\qquad m^*(r,\ u)>\cos\pi\alpha M(r,\ u)\quad (r\in F_2,\ r\geqq R_0) .$$

Combining (8.13), (8.14) with Lemma 5, we have

$$\underline{\log\text{dens}}\, E=1-\lambda\mu/\beta ,\qquad \overline{\log\text{dens}}\, E=1-\mu/\beta .$$

This completes the proof of Theorem 2.

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