ON THE GROWTH OF SUBHARMONIC FUNCTIONS IN $\{\operatorname{Re} z > 0\}$

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0. Let C be the complex plane. If $u(z)$ is subharmonic in a region $\Omega \subset C$, we put

$$m^*(r, u) = \inf_{\substack{|z|=r \\ z \in \Omega}} u(z), \quad M(r, u) = \sup_{\substack{|z|=r \\ z \in \Omega}} u(z).$$

If it is clear from the context what subharmonic function we consider, we sometimes write $M(r)$ instead of $M(r, u)$. Let $bd\Omega$ be the boundary of Ω . If $\zeta \in bd\Omega$ and $u(z)$ is subharmonic in Ω , we define

$$u(\zeta) = \overline{\lim}_{\substack{z \rightarrow \zeta \\ z \in \Omega}} u(z).$$

Further if the region Ω is unbounded, we also introduce the order and type of $u(z)$. The order of $u(z)$ is defined by

$$\rho = \overline{\lim}_{\substack{r \rightarrow \infty \\ M(r, u) > 0}} \frac{\log M(r, u)}{\log r}.$$

If $u(z)$ is of order ρ ($0 < \rho < +\infty$), it is further classified as having maximal, mean, or minimal type according as

$$\tau = \overline{\lim}_{r \rightarrow \infty} \frac{M(r, u)}{r^\rho}$$

is infinite, positive, or zero, respectively.

In [3], Essén proved the following result.

THEOREM A. *Let $u(z)$ be subharmonic in $\{\operatorname{Re} z > 0\}$. Let λ be given, $0 < \lambda < 2$. If $u(0) < +\infty$, and*

$$u(iy) \leq \cos\left(\frac{\pi\lambda}{2}\right) M^+(|y|) \quad (-\infty < y < +\infty, y \neq 0),$$

then either $u(z) \leq 0$ in $\{\operatorname{Re} z > 0\}$ or

$$\lim_{r \rightarrow \infty} \frac{M(r)}{r^\lambda}$$

exists and is positive or $+\infty$.

The case $\lambda=1$ is in Ahlfors-Heins [1].

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In this note we shall prove two results which are related to Theorem A in some sense.

1. In this section we prove the following

THEOREM 1. *Let $u(z)$ be subharmonic of order ρ ($0 < \rho < 1$) and mean type in $\{\operatorname{Re} z > 0\}$, and assume that $u(iy) < +\infty$ ($-\infty < y < +\infty$). If $h(r)$ is a positive slowly varying function such that $h(r) \rightarrow 0$ ($r \rightarrow \infty$), $h'(r) > -O(r^{-1})$ ($r \rightarrow \infty$), and*

$$(1.1) \quad \int_0^\infty \frac{h(t)}{t} dt = +\infty,$$

then

$$(1.2) \quad u(iy) > \cos\left(\frac{\pi\rho}{2}\right)(1 - h(|y|))M(|y|, u)$$

on a sequence of $|y| \rightarrow \infty$.

We remark that if $u(z)$ is subharmonic of order ρ ($0 < \rho < 2$) and minimal type in $\{\operatorname{Re} z > 0\}$, and if $u(z)$ satisfies $u(iy) < +\infty$ ($-\infty < y < +\infty$), then Theorem A implies that

$$u(iy) > \cos\left(\frac{\pi\rho}{2}\right)M(|y|, u)$$

on a sequence of $|y| \rightarrow \infty$.

For convenience, let us here state two theorems which will be used in this section.

THEOREM B. (cf. [6, p. 111]) *Let $u(z)$ be subharmonic in $\{\operatorname{Re} z > 0\}$ and assume that $u(iy) \leq 0$ ($-\infty < y < +\infty$). Then either $u(z) \leq 0$ in $\{\operatorname{Re} z > 0\}$ or else $\lim_{r \rightarrow \infty} M(r, u)/r > 0$.*

THEOREM C. ([2, p. 82]) *If $f(x)$ is such that both integrals*

$$\int_0^1 t^{-\alpha} |f(t)| dt, \quad \int_1^\infty t^\alpha |f(t)| dt$$

exist for some $\alpha > 0$, then

$$\int_0^\infty f(t) L(xt) dt \sim L(x) \int_0^\infty f(t) dt \quad (x \rightarrow \infty)$$

for every slowly varying function $L(x)$.

In order to prove Theorem 1, we need the following lemma.

LEMMA 1. *Let $H(z)$ be positive in $\{\operatorname{Re} z \geq 0\}$, and harmonic in $\{\operatorname{Re} z > 0\}$. Suppose, further, that $M(r, H)$ has order less than one. If $u(z)$ is a subharmonic*

function of order ρ ($0 < \rho < 1$) in $\{\operatorname{Re} z > 0\}$ which satisfies $u(iy) < +\infty$ ($-\infty < y < +\infty$) and $M(r, u) = o(H(\pm ir))$ ($r \rightarrow \infty$), then for given $\varepsilon > 0$, there are two numbers $a(\varepsilon)$, $y(\varepsilon)$ such that

$$(1.3) \quad u(iy(\varepsilon)) - \frac{H(iy(\varepsilon)) - 1}{H(z)} u(z) \geq a(\varepsilon) \left[1 - \frac{H(iy(\varepsilon)) - 1}{H(z)} \right]$$

$$(|z| = |y(\varepsilon)|; a(\varepsilon), |y(\varepsilon)| \rightarrow \infty \text{ as } \varepsilon \rightarrow 0).$$

Proof of Lemma 1. By assumption, for any $\varepsilon > 0$, $u(iy) - \varepsilon H(iy) < 0$ ($|y| \geq r_1(\varepsilon)$). Hence we note that

$$a(\varepsilon) \equiv \sup_{-\infty < y < +\infty} \{u(iy) - \varepsilon H(iy)\} < +\infty.$$

Here we show that $\lim_{\varepsilon \rightarrow 0} a(\varepsilon) = +\infty$. If this were not the case, there would be a positive constant A and a sequence $\{\varepsilon_n\} \downarrow 0$ such that $u(iy) - \varepsilon_n H(iy) < A$ ($|y| < +\infty$). Thus as $n \rightarrow \infty$, we would have

$$(1.4) \quad u(iy) \leq A \quad (|y| < +\infty).$$

Since $u(z)$ is of order ρ (< 1), $\lim_{r \rightarrow \infty} M(r, u)/r = 0$. Hence by (1.4) and Theorem B, $u(z) \leq A$ in $\{\operatorname{Re} z > 0\}$. This is impossible. Now, $u(z) - \varepsilon H(z)$ is of order less than one, and hence using Theorem B once more, it follows that

$$(1.5) \quad u(z) - \varepsilon H(z) - a(\varepsilon) \leq 0 \quad (z \in \{\operatorname{Re} z \geq 0\}).$$

On the other hand, since $a(\varepsilon) \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, there exists a $y(\varepsilon)$ such that $|y(\varepsilon)| \rightarrow \infty$ as $\varepsilon \rightarrow 0$ and

$$(1.6) \quad u(iy(\varepsilon)) - \varepsilon H(iy(\varepsilon)) \geq a(\varepsilon) - \varepsilon.$$

Combining (1.5) and (1.6), we have

$$u(iy(\varepsilon)) - \frac{H(iy(\varepsilon)) - 1}{H(z)} u(z) \geq a(\varepsilon) \left[1 - \frac{H(iy(\varepsilon)) - 1}{H(z)} \right]$$

$$(|z| = |y(\varepsilon)|, \varepsilon (> 0): \text{small enough}).$$

This proves (1.3).

Proof of Theorem 1. Put

$$(1.7) \quad \begin{aligned} H(z) &= \frac{1}{\pi} \int_{-\infty}^{+\infty} A(|t|) \operatorname{Re} \frac{1}{z - it} dt \\ &= \frac{1}{\pi} \int_0^\infty A(t) \left[\frac{r \cos \theta}{r^2 + t^2 + 2tr \sin \theta} + \frac{r \cos \theta}{r^2 + t^2 - 2tr \sin \theta} \right] dt \\ &\equiv \frac{1}{\pi} \int_0^\infty A(rt) A(\theta, t) dt \quad (z = re^{i\theta}, |\theta| < \pi/2), \end{aligned}$$

where

$$(1.8) \quad \begin{cases} \Lambda(r) = r^\rho L(r), \\ L(r) = \exp\left\{\delta \int_1^r \frac{h(t)}{t} dt\right\} \quad (\delta > 0: \text{small enough}). \\ A(\theta, t) = \frac{\cos \theta}{1+t^2+2t \sin \theta} + \frac{\cos \theta}{1+t^2-2t \sin \theta} > 0 \quad (|\theta| < \pi/2). \end{cases}$$

Then $H(z)$ is a positive harmonic function of order ρ in $\{\operatorname{Re} z > 0\}$ with boundary values

$$H(iy) = \Lambda(|y|) \quad (-\infty < y < +\infty).$$

By (1.1), $M(r, u) = o(\Lambda(r)) = o(H(\pm ir))$ ($r \rightarrow \infty$). We claim here that

$$(1.9) \quad H(\pm ir) \leq H(re^{i\theta}) \leq H(r) \quad (|\theta| \leq \pi/2).$$

Using (1.8), we have

$$(1.10) \quad r \frac{d\Lambda(r)}{dr} = \Lambda(r)[\rho + \delta h(r)],$$

$$(1.11) \quad \begin{aligned} r \frac{d}{dr} \left(r \frac{d\Lambda(r)}{dr} \right) &= \Lambda(r)[(\rho + \delta h(r))^2 + \delta r h'(r)] \\ &\geq \Lambda(r)[\rho^2 + \delta r h'(r)] \geq 0, \end{aligned}$$

if $r h'(r) \geq -\rho^2/\delta$, which is true if δ is small enough. Using the Beppo Levi theorem, we deduce from (1.7) and (1.10) that

$$(1.12) \quad r \frac{\partial}{\partial r} H(re^{i\theta}) = \frac{1}{\pi} \int_0^\infty A(\theta, t) (rt) \frac{d}{d(rt)} \Lambda(rt) dt.$$

By (1.11), $r \frac{d}{dr} \Lambda(r)$ is nondecreasing, and so is $r \frac{\partial}{\partial r} H(re^{i\theta})$ by (1.12). Since $H(z)$ is harmonic in $\{\operatorname{Re} z > 0\}$, follows that

$$r \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} H(re^{i\theta}) \right) \geq 0 \quad (|\theta| < \pi/2).$$

Hence

$$(1.13) \quad \frac{\partial^2}{\partial \theta^2} H(re^{i\theta}) \leq 0 \quad (|\theta| < \pi/2).$$

Noting that $H(re^{i\theta}) = H(re^{-i\theta})$ ($|\theta| \leq \pi/2$), we deduce (1.9) from (1.13). In view of (1.3), (1.6) and (1.9), we easily see that

$$(1.14) \quad \frac{u(iy)}{M(|y|, u)} \geq \frac{H(iy) - 1}{H(|y|)}$$

for a sequence of $|y| \rightarrow \infty$. It remains to estimate this least term. Now,

$$(1.15) \quad \begin{aligned} H(r) &= \frac{1}{\pi} \int_0^\infty (rt)^\rho L(r) A(0, t) dt + \frac{1}{\pi} \int_0^\infty (rt)^\rho [L(rt) - L(r)] A(0, t) dt \\ &= \Lambda(r) \sec\left(\frac{\pi\rho}{2}\right) + I(r), \quad \text{say.} \end{aligned}$$

To estimate $I(r)$, we use the formula

$$(t^2+1)^{-1/2} = \sum_{n=0}^{\infty} P_n(0)t^n \quad (|t| < 1),$$

where $P_n(x)$ ($n=0, 1, 2, \dots$) is the Legendre's polynomial. Then

$$\begin{aligned} I(r) &= \frac{2r^\rho}{\pi} \int_0^1 [L(rt) - L(r)] t^\rho \sum_{m,n=0}^{\infty} P_m(0)P_n(0)t^{m+n} dt \\ &\quad + \frac{2r^\rho}{\pi} \int_1^\infty [L(rt) - L(r)] t^{\rho-2} \sum_{m,n=0}^{\infty} P_m(0)P_n(0)t^{-m-n} dt \\ &= -\frac{2r^\rho}{\pi} \sum_{m,n=0}^{\infty} P_m(0)P_n(0) \frac{1}{m+n+1+\rho} \int_0^1 t^{m+n+\rho} rt L'(rt) dt \\ &\quad + \frac{2r^\rho}{\pi} \sum_{m,n=0}^{\infty} P_m(0)P_n(0) \frac{1}{m+n+1-\rho} \int_1^\infty t^{-m-n-2+\rho} rt L'(rt) dt \\ &= -\frac{2r^\rho}{\pi} \int_0^1 rt L'(rt) \sum_{m,n=0}^{\infty} \frac{P_m(0)P_n(0)}{m+n+1+\rho} t^{m+n+\rho} dt \\ &\quad + \frac{2r^\rho}{\pi} \int_1^\infty rt L'(rt) \sum_{m,n=0}^{\infty} \frac{P_m(0)P_n(0)}{m+n+1-\rho} t^{-m-n-2+\rho} dt. \end{aligned}$$

For the legitimacy of the inversions in the order of integration and summation, cf. [8]. Here we use Theorem C to obtain

$$\begin{aligned} (1.16) \quad I(r) &\sim -\frac{2r^\rho}{\pi} \delta h(r) L(r) \int_0^1 \sum_{m,n=0}^{\infty} \frac{P_m(0)P_n(0)}{m+n+1+\rho} t^{m+n+\rho} dt \\ &\quad + \frac{2r^\rho}{\pi} \delta h(r) L(r) \int_1^\infty \sum_{m,n=0}^{\infty} \frac{P_m(0)P_n(0)}{m+n+1-\rho} t^{-m-n-2+\rho} dt \\ &= \frac{2\delta}{\pi} h(r) A(r) \sum_{m,n=0}^{\infty} P_m(0)P_n(0) \left\{ \frac{1}{(m+n+1-\rho)^2} - \frac{1}{(m+n+1+\rho)^2} \right\} \\ &\equiv \delta h(r) A(r) C(\rho) \quad (r \rightarrow \infty). \end{aligned}$$

Hence by (1.14)-(1.16), for a sequence of $|y| \rightarrow \infty$,

$$\begin{aligned} \frac{u(iy)}{M(|y|, u)} &\geq \frac{A(|y|) - 1}{A(|y|) \left[\sec \frac{\pi\rho}{2} + \delta \{1 + o(1)\} |C(\rho)| h(|y|) \right]} \\ &> \left\{ 1 - \frac{1}{A(|y|)} \right\} \cos \frac{\pi\rho}{2} \left[1 - \delta \{1 + o(1)\} |C(\rho)| \cos \left(\frac{\pi\rho}{2} \right) h(|y|) \right] \\ &> \cos \left(\frac{\pi\rho}{2} \right) [1 - h(|y|)], \end{aligned}$$

if δ is chosen sufficiently small. This completes the proof of Theorem 1.

2. Let Ω denote an unbounded region in the plane $\mathbf{C}$ and suppose that

$$bd\Omega \cap \{|z|=r\} \neq \emptyset \quad (0 < r < +\infty).$$

(Such an Ω has been considered in [5], [7].) Let $u(z)$ be subharmonic in Ω and satisfy the conditions $u(0) < +\infty$ and

$$u(\zeta) < M^+(|\zeta|) - \pi^2 \sigma \quad (\sigma: \text{a positive constant, } \zeta \in bd\Omega, 0 < |\zeta| < +\infty).$$

In this section we prove the following

THEOREM 2. *If Ω and $u(z)$ satisfy the above hypotheses, then either $u(z) \leq 0$ in Ω or*

$$\lim_{r \rightarrow \infty} \frac{M(r) - \sigma(\log r)^2}{\log r}$$

exists and is finite or $+\infty$.

From this, we immediately deduce

COROLLARY. *Let $u(z)$ be subharmonic in $\{\operatorname{Re} z > 0\}$. If $u(z)$ satisfies the conditions $u(0) < +\infty$ and*

$$u(iy) < M^+(|y|) - \pi^2 \sigma \quad (\sigma: \text{a positive constant, } -\infty < y < +\infty, y \neq 0),$$

then either $u(z) \leq 0$ in $\{\operatorname{Re} z > 0\}$ or

$$\lim_{r \rightarrow \infty} \frac{M(r) - 4\sigma(\log r)^2}{\log r}$$

exists and is finite or $+\infty$.

This formulation is analogous to the one of Theorem A. In order to prove Theorem 2, we use the following fact due to Fenton.

THEOREM D. ([4, p. 473]) *Suppose that $u(z)$ is subharmonic in $\mathbf{C}$ and that σ is any positive number. Then either*

$$m^*(r, u) > M(r, u) - \pi^2 \sigma$$

for certain arbitrarily large values of r or, if this is not the case, then

$$\lim_{r \rightarrow \infty} \frac{M(r, u) - \sigma(\log r)^2}{\log r}$$

exists and is either finite or $+\infty$.

First, we prove a lemma, which will play a fundamental role in order to reduce our problem to a problem to which Theorem D is applicable.

LEMMA 2. *Let Ω and $u(z)$ satisfy the hypotheses of Theorem 2. Then $M^+(r)$*

is a nondecreasing convex function of $\log r$ ($0 < r < \infty$).

Proof of Lemma 2. We proceed by showing

- (a) $M^+(r)$ is nondecreasing,
- (b) $M^+(r)$ is continuous on the right,
- (c) $M^+(r)$ is convex in $\log r$.

Let $0 < t < +\infty$ and let $\Omega_t = \Omega \cap \{|z| < t\}$. Put $C = \sup_{z \in \Omega_t} u(z)$. If $\zeta \in bd\Omega \cap bd\Omega_t$ and $\zeta \neq 0$, then

$$u|_{\Omega_t}(\zeta) \leq u(\zeta) < M^+(|\zeta|) - \pi^2 \sigma \leq C^+ - \pi^2 \sigma.$$

Moreover, since $u(z)$ is upper semicontinuous in Ω , we have

$$u|_{\Omega_t}(\zeta) \leq u(\zeta) \leq M^+(t) \quad (\zeta \in \Omega \cap bd\Omega_t).$$

Using these facts and the Phragmén-Lindelöf maximum principle, we deduce that

$$u(z) \leq \max\{C^+ - \pi^2 \sigma, M^+(t)\} \quad (z \in \Omega_t).$$

Hence, if $0 < s < t < +\infty$, then

$$M(s) \leq C \leq \max\{C^+ - \pi^2 \sigma, M^+(t)\},$$

and so $M^+(s) \leq M^+(t)$. This proves (a).

To prove (b), let $0 < s < +\infty$ and note that

$$M(s) \leq \lim_{r \rightarrow s+0} M^+(r) = C_1.$$

If $C_1 = 0$, then (b) is trivial. Hence we assume that $C_1 > 0$. In this case there exists a sequence $\{z_n\}_1^\infty$, where $|z_n| \downarrow s$ ($n \rightarrow \infty$) and $z_n \in \Omega$, such that $u(z_n) = M(|z_n|)$, $\lim_{n \rightarrow \infty} z_n = z$ ($|z| = s$), as we see from the inequality:

$$u(\zeta) < M^+(|\zeta|) - \pi^2 \sigma < M(|\zeta|) \quad (\zeta \in bd\Omega, s < |\zeta| < +\infty).$$

Moreover, $z \in \Omega$. If this is not the case and $z \in bd\Omega$, we know that if $0 < \varepsilon < \pi^2 \sigma$ is given and n is large enough, then

$$(2.1) \quad M(s) \leq M(|z_n|) = u(z_n) < u(z) + \varepsilon < M^+(s) - \pi^2 \sigma + \varepsilon.$$

By the choice of ε , we easily see that $M^+(s) - \pi^2 \sigma + \varepsilon < 0$. Hence (2.1) implies $M(|z_n|) < 0$, which contradicts the assumption $C_1 > 0$.

Using the above facts and the upper semicontinuity of u at $z \in \Omega$, we obtain $u(z) \geq \lim_{n \rightarrow \infty} u(z_n) = C_1$. Hence $M(s) \geq C_1$. We know that $M(s) \leq C_1$, therefore $M(s) = C_1$. This proves (b).

To prove (c), let $0 < s < +\infty$ and suppose that $M(s) > 0$. We assert that

$$(2.2) \quad \lim_{r \rightarrow s-0} M(r) > (M(s) - \pi^2 \sigma)^+.$$

Choose $z_0 \in \Omega$ such that $|z_0| = s$ and $u(z_0) = M(s)$. Let $\varepsilon > 0$ be a small positive number and let $z_0 + \varepsilon e^{i\theta_1}$, $z_0 + \varepsilon e^{i\theta_2}$ denote the points of intersection of the circle $\{|z - z_0| = \varepsilon\}$ with the circle $\{|z| = s\}$. We may assume that $0 < \theta_1 < \theta_2 < 2\pi$ and $\{z_0 + \varepsilon e^{i\theta}; \theta_1 \leq \theta \leq \theta_2\} \subset \{|z| \leq s\}$. Clearly $\theta_2 - \theta_1 \rightarrow \pi$ as $\varepsilon \rightarrow 0$. Hence if (2.2) is false,

$$\begin{aligned} M(s) = u(z_0) &\leq \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \varepsilon e^{i\theta}) d\theta \\ &= \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} u(z_0 + \varepsilon e^{i\theta}) d\theta + \frac{1}{2\pi} \int_{\theta_2}^{\theta_1 + 2\pi} u(z_0 + \varepsilon e^{i\theta}) d\theta \\ &\leq (M(s) - \pi^2 \sigma) + \frac{\theta_2 - \theta_1}{2\pi} + M(s + \varepsilon) \frac{2\pi + \theta_1 - \theta_2}{2\pi} \\ &\rightarrow \frac{1}{2} \{(M(s) - \pi^2 \sigma)^+ + M(s)\} \quad (\varepsilon \rightarrow 0). \end{aligned}$$

We have reached a contradiction. Hence (2.2) holds. Using (2.2), (a) and (b), we see that there exist r_1, r_2 such that $0 < r_1 < s < r_2 < +\infty$ and $M(r_1) > (M(r_2) - \pi^2 \sigma)^+$.

Let

$$v(z) = u(z) - (M(r_2) - \pi^2 \sigma) \quad (r_1 < |z| < r_2).$$

Then

$$v(\zeta) = u(\zeta) - (M(r_2) - \pi^2 \sigma) < M^+(|\zeta|) - M(r_2) \leq 0 \quad (\zeta \in bd\Omega, r_1 < |\zeta| < r_2).$$

This inequality implies that $v^+(z)$ has a subharmonic extension to annulus $\{r_1 < |z| < r_2\}$. Hence $M(r, v^+)$ is a convex function of $\log r$ in (r_1, r_2) . However, since

$$M(r, v) = M(r, u) - (M(r_2) - \pi^2 \sigma) \geq M(r_1) - (M(r_2) - \pi^2 \sigma) > 0 \quad (r_1 < r < r_2),$$

it follows that $M(r, u)$ is convex in $\log r$ in (r_1, r_2) .

This completes the proof of Lemma 2.

Proof of Theorem 2. Set

$$(2.3) \quad W(z) = \begin{cases} \max\{u(z), M^+(|z|, u) - \pi^2 \sigma\} & (z \in \Omega) \\ M^+(|z|, u) - \pi^2 \sigma & (z \in C - \Omega - \{0\}). \end{cases}$$

We assert that $W(z)$ is subharmonic in $C - \{0\}$. To verify this assertion note that, by Lemma 2, $M^+(|z|, u)$ is subharmonic in $C - \{0\}$. Hence if $z_0 \in bd\Omega \cup \{0\}$, then $W(z)$ is subharmonic at z_0 . If $z_0 \in bd\Omega - \{0\}$, then from Lemma 2 we deduce for small $s > 0$ that $u(z) < M^+(|z|, u) - \pi^2 \sigma$ when $z \in \Omega \cap \{|z - z_0| < s\}$. It follows from this inequality that $W(z) = M^+(|z|, u) - \pi^2 \sigma$ in $\{|z - z_0| < s\}$. Hence in this case $W(z)$ is subharmonic at z_0 . Therefore $W(z)$ is subharmonic in $C - \{0\}$. Next, we observe that $W(z)$ has a subharmonic extension to C . Define

$$(2.4) \quad \tilde{W}(z) = \begin{cases} W(z) & (z \in \mathbf{C} - \{0\}) \\ \overline{\lim}_{z \rightarrow 0} W(z) & (z = 0). \end{cases}$$

Let $0 < R < +\infty$ and let $h_R(z)$ ($|z| < R$) denote the Poisson integral whose boundary values are $h_R(Re^{i\theta}) = W(Re^{i\theta})$ ($0 \leq \theta \leq 2\pi$). Then $W(z) - h_R(z)$ is subharmonic in $\{0 < |z| < R\}$, and satisfies $\sup_{0 < |z| < R} (W(z) - h_R(z)) < +\infty$, $(W - h_R)(Re^{i\theta}) = 0$ ($0 \leq \theta \leq 2\pi$). Hence using the Phragmén-Lindelöf maximum principle, we deduce that $W(z) \leq h_R(z)$ ($0 < |z| < R$). Thus

$$(2.5) \quad \tilde{W}(0) = \overline{\lim}_{z \rightarrow 0} W(z) \leq \overline{\lim}_{z \rightarrow 0} h_R(z) = h_R(0) = \frac{1}{2\pi} \int_0^{2\pi} W(Re^{i\theta}) d\theta.$$

By (2.4) and (2.5) we see that $\tilde{W}(z)$ is subharmonic in $\mathbf{C}$. Now assume that $M(r_0, u) > 0$ for some $r_0 > 0$. Then by Lemma 2 and (2.3)

$$\begin{cases} M(r, \tilde{W}) = M(r, u) \\ m^*(r, \tilde{W}) = M(r, u) - \pi^2 \sigma \end{cases} \quad (r \geq r_0).$$

Here we use Theorem D to obtain

$$\alpha = \lim_{r \rightarrow \infty} \frac{M(r, u) - \sigma(\log r)^2}{\log r} \quad (-\infty < \alpha \leq +\infty).$$

This completes the proof of Theorem 2.

Remark. Let Ω be as in Theorem 2, and let $u(z)$ be a subharmonic function in Ω satisfying $u(0) < +\infty$ and

$$u(\zeta) < M(|\zeta|) - \pi^2 \sigma \quad (\sigma : \text{a positive constant, } \zeta \in bd\Omega, 0 < |\zeta| < +\infty).$$

Then we conclude that $M(r)$ is a nondecreasing convex function of $\log r$ ($0 < r < +\infty$). (The proof is similar to the one of Lemma 2.) Using this fact, we can prove that

$$\lim_{r \rightarrow \infty} \frac{M(r, u) - \sigma(\log r)^2}{\log r} = \alpha, \quad -\infty < \alpha \leq +\infty.$$

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