M. OZAWA
KODAI MATH. J.
6 (1983), 250—260

ON THE GROWTH OF MEROMORPHIC FUNCTIONS

BY MITSURU OZAWA

§1. **Introduction.** By making use of Fourier series method Miles and Shea [4], [5] recently obtained a better estimate for the $\kappa(\lambda)$ and related results. It seems to the present author that the method contains more. In this paper we shall discuss some of them.

For completeness we shall list up several known results, which will be used later. For a meromorphic function $f(z)$ we define $m_2(r, f)$ by

$$m_2(r, f)^2=\frac{1}{2\pi}\int_0^{2\pi}(\log|f(re^{i\theta})|)^2 d\theta .$$

In what follows we only consider entire or meromorphic functions of the following form:

$$f(z)=\Pi E\left(\frac{z}{z_\nu}, q\right)$$

or

$$f(z)=\Pi E\left(\frac{z}{z_\nu}, q\right)\Big/\Pi E\left(\frac{z}{w_\nu}, q\right),$$

where

$$E(x, q)=(1-x)\exp(x+x^2/2+\cdots+x^q/q), \qquad q=[\lambda]$$

and λ is the order of $f(z)$, $\lambda<\infty$.

Let $c_m(r)$ be the m-th Fourier coefficient:

$$c_m(r)=\frac{1}{2\pi}\int_0^{2\pi}(\log|f(re^{i\theta})|)e^{-im\theta}d\theta .$$

Then

$$m_2(r, f)^2=\sum_{m=-\infty}^{\infty}|c_m(r)|^2 .$$

Edrei and Fuchs [1] had shown that, with $r_\nu=|z_\nu|$ and $s_\nu=|w_\nu|$,

$$c_m(r)=\frac{1}{2m}\sum_{r_\nu\leqq r}\left\{\left(\frac{r}{z_\nu}\right)^m-\left(\frac{\bar{z}_\nu}{r}\right)^m\right\}-\frac{1}{2m}\sum_{s_\nu\leqq r}\left\{\left(\frac{r}{w_\nu}\right)^m-\left(\frac{\bar{w}_\nu}{r}\right)^m\right\}$$

for $m\geqq 1$ and, for $m\geqq q+1$,

$$c_m(r)=-\frac{1}{2m}\left\{\sum_{r_\nu>r}\left(\frac{r}{z_\nu}\right)^m-\sum_{s_\nu>r}\left(\frac{r}{w_\nu}\right)^m+\sum_{r_\nu\leqq r}\left(\frac{\bar{z}_\nu}{r}\right)^m-\sum_{s_\nu\leqq r}\left(\frac{\bar{w}_\nu}{r}\right)^m\right\}.$$

Received January 7, 1983

Obviously $c_m(r)=\bar{c}_{-m}(r)$ for $m\leqq -1$ and $c_0(r)=N(r, 0, f)-N(r, \infty, f)$. Further

$$|c_m(r)|\leqq 2T(r, f)-N(r, 0, f)-N(r, \infty, f)\leqq m_2(r, f).$$

For $m\geqq q+1$

$$-\frac{1}{2m}\left\{\sum_{r_\nu>r}\left(\frac{r}{r_\nu}\right)^m+\sum_{r_\nu\leqq r}\left(\frac{r_\nu}{r}\right)^m\right\}$$

$$=N(r, 0)-\frac{m}{2}\int_0^r \frac{N(t, 0)}{t}\left(\frac{t}{r}\right)^m dt-\frac{m}{2}\int_r^\infty \frac{N(t, 0)}{t}\left(\frac{r}{t}\right)^m dt$$

and for $m\leqq q$

$$\frac{1}{2m}\sum_{r_\nu\leqq r}\left\{\left(\frac{r}{r_\nu}\right)^m-\left(\frac{r_\nu}{r}\right)^m\right\}$$

$$=N(r, 0)+\frac{m}{2}\int_0^r \frac{N(t, 0)}{t}\left\{\left(\frac{r}{t}\right)^m-\left(\frac{t}{r}\right)^m\right\}dt.$$

§ 2. Discussion of results.

Our first result is the following

THEOREM 1. *Let $f(z)$ be the canonical product formed by $\{z_\nu\}$, which satisfies*

$$\sum |z_\nu|^{-q}=\infty, \qquad \sum |z_\nu|^{-q-1}<\infty$$

for a positive integer q and

$$|\arg z_\nu|\leqq\omega, \qquad 0\leqq\omega\leqq(\pi-\varepsilon)/2q, \quad 0<\varepsilon\leqq\pi.$$

Then

$$\varlimsup_{r\to\infty}\frac{N(r, 0, f)}{m_2(r, f)}\leqq\frac{1}{A(q, \omega)}$$

and

$$\varlimsup_{r\to\infty}\frac{m(r, f)}{m_2(r, f)}\leqq\frac{1}{2}+\frac{1}{2A(q, \omega)},$$

where

$$A(q, \omega)=\left\{q+1+\frac{\cos(q+1)\omega\sin q\omega}{\sin\omega}\right\}^{1/2},$$

Since $m_2(r, f)\geqq N(r, 0, f)$ for entire functions with $f(0)=1$,

$$\varlimsup_{r\to\infty}\frac{N(r, 0, f)}{m_2(r, f)}\leqq B$$

loses its effectivity when $B\geqq 1$. We shall prove that the estimate given in Theorem 1 loses its efficiency when $q=1$ and $\omega=\pi/2$. Further for entire f

$$\varlimsup_{r\to\infty}\frac{N(r, 0, f)}{m_2(r, f)}\leqq\varlimsup_{r\to\infty}\frac{N(r, 0, f)}{2m(r, f)-N(r, 0, f)-O(1)}$$

$$=\frac{1-\delta(0, f)}{1+\delta(0, f)}.$$

This estimate is effective when $\delta(0, f)>0$. In this direction Kobayashi [2] had shown the following

THEOREM A. *Under the same assumptions as in Theorem* 1 *with* $\omega=\pi/2(q+1)$

$$\delta(0, f)>0.$$

If $\omega>\pi/2(q+1)$, *then there is an entire function such that* $\delta(0, f)=0$.

If $0<\varepsilon<\pi/(q+1)$, then $(\pi-\varepsilon)/2q>\pi/2(q+1)$. Therefore $\delta(0, f)=0$ does not always imply the inefficiency of our Theorem 1. The opening of ω in Theorem 1 is equal to the one of the following result due to Kobayashi [3].

THEOREM B. *Under the same assumptions as in Theorem* 1

$$q\leqq\mu\leqq\lambda\leqq q+1,$$

where λ *and* μ *are the order and the lower order of* f, *respectively. This is best possible.*

Theorem 1 can be extended to a wider opening if $q\geqq 2$. For example, if $q=2$ and $\omega=(\pi-\varepsilon)/2$ $(\varepsilon>0)$, then

$$m_2(r, f)^2\geqq|c_0(r)|^2+2|c_1(r)|^2$$
$$\geqq N(r, 0, f)^2(1+2\sin^2\varepsilon).$$

This gives an estimate of desired type.

THEOREM 2. *Let* $f(z)$ *be the canonical product formed by the set of zeros* $\{a_\nu, -a_\nu\}$. *Assume that*

$$\sum|a_\nu|^{-q}=\infty, \qquad \sum|a_\nu|^{-q-1}<\infty,$$
$$|\arg a_\nu|\leqq\omega, \qquad 0\leqq\omega\leqq(\pi-\varepsilon)/2q, \qquad \varepsilon>0.$$

Then

$$\varlimsup_{r\to\infty}\frac{N(r, 0, f)}{m_2(r, f)}\leqq\frac{1}{A},$$

where

$$A^2=1+\left[\frac{q}{2}\right]+\frac{\cos([q/2]+1)\omega\sin[q/2]\omega}{\sin 2\omega}.$$

This theorem has its meaning only if $q\geqq 2$.

THEOREM 3. *Let* $f(z)$ *be a meromorphic function being representable as*

$f_1(z)/f_2(z)$, where f_1 and f_2 are canonical products formed by $\{a_n\}$ and $\{b_n\}$, respectively. Assume that

$$\sum|a_n|^{-q}=\infty, \qquad \sum|b_n|^{-q}=\infty, \qquad \sum|a_n|^{-q-1}+\sum|b_n|^{-q-1}<\infty$$

and

$$|\arg a_n|\leqq\omega, \qquad |\pi-\arg b_n|\leqq\omega$$

with $0\leqq\omega\leqq(\pi-\varepsilon)/2q$. Then

$$\varlimsup_{r\to\infty}\frac{N(r,0,f)+N(r,\infty,f)}{m_2(r,f)}\leqq\frac{1}{A},$$

where

$$A^2=s+1+\frac{\sin 4(s+1)\omega}{2\sin 2\omega}, \qquad s=\max\{p\,|\,2p+1\leqq q\}.$$

In [5] Miles and Shea indicated that $m_2(r,f)\leqq 4\sqrt{q+1}\,m(r,f)$, if f is entire, of finite genus q with only positive zeros. More precisely,

$$m_2(r,f)^2\leqq 8(q+s_q)m(r,f)^2-8(q+s_q)m(r,f)N(r,0,f)+(2q+2s+1)N(r,0,f)^2$$
$$\leqq 8(q+s_q)m(r,f)^2,$$

where

$$s_q=(q+1)^2\sum_{j=0}^{\infty}\frac{1}{(q+1+j)^2}=(q+1)^2\left(\frac{\pi^2}{6}-\sum_{j=1}^{q}\frac{1}{j^2}\right).$$

It is very easy to prove $q+1<s_q<q+2$.

Let f be the canonical product of genus q. When does the estimate $m_2(r,f)\leqq Km(r,f)$, $0<K<\infty$, hold? Of course this does not hold in general. Let us denote

$$f(z)=\prod E\left(\frac{z}{a_\nu},q\right), \qquad F(z)=\prod E\left(\frac{z}{|a_\nu|},q\right).$$

If the Valiron deficiency $\Delta(0,F)$ of F satisfies $\Delta(0,F)<1$, then $m_2(r,f)\leqq Km(r,f)$ and $(1-\Delta(0,F)-\varepsilon)m(r,F)<N(r,0,F)=N(r,0,f)\leqq m(r,f)$ give the result. Since $m_2(r,F)\leqq km(r,F)$ holds without any condition, $\Delta(0,F)<1$ is not a necessary condition. So it is hoped to give a more appropriate condition for the above problem.

In the above results we do not make use of the concept of Pólya peaks of any kind. Under the assumptions of Theorem 1 we can prove

$$\varlimsup_{r\to\infty}\frac{N(r,0,f)}{m_2(r,f)}\leqq\frac{1}{A^{1/2}}, \qquad A=1+2\sum_{1}^{q}\left(\frac{\rho^2}{\rho^2-m^2}\right)^2\cos^2 m\omega$$

by making use of Pólya peaks of the second kind, order ρ, for $N(r,0,f)$. We can also prove similar results corresponding to Theorem 2 and Theorem 3 quite similarly.

Let M_ρ be the class of meromorphic functions $f(z)$ of order ρ defined by $f_1(z)/f_1(-z)$ with the canonical product

$$f_1(z)=\Pi E\left(\frac{z}{a_n}, q\right), \qquad q=[\rho].$$

Let $F(z)$ be $F_1(z)/F_1(-z)$ with

$$F_1(z)=\Pi E\left(\frac{z}{|a_n|}, q\right).$$

THEOREM 4. *Let $f(z)$ belong to M_ρ. Then $m_2(r, f)\leqq m_2(r, F)$ and*

$$\varlimsup_{r\to\infty}\frac{N(r, 0, f)}{m_2(r, f)}\geqq\frac{\sqrt{2}}{\sqrt{\pi\rho}}\frac{|\cos\pi\rho/2|}{(\pi\rho-\sin\pi\rho)^{1/2}}.$$

This is best possible.

THEOREM 5. *Under the same assumptions as in Theorem 3 with $\omega=0$*

$$\varlimsup_{r\to\infty}\frac{N(r, 0, f)+N(r, \infty, f)}{m_2(r, f)}\leqq\frac{2\sqrt{2}}{\sqrt{\pi\rho}}\frac{|\cos\pi\rho/2|}{(\pi\rho-\sin\pi\rho)^{1/2}}.$$

This is best possible.

§3. **Proof of Theorem 1.** For $1\leqq m\leqq q$

$$\mathfrak{R}c_m(r)=\frac{1}{2m}\sum_{r_\nu\leqq r}\left\{\left(\frac{r}{r_\nu}\right)^m-\left(\frac{r_\nu}{r}\right)^m\right\}\cos m\varphi_\nu,$$

$$z_\nu=r_\nu e^{i\varphi_\nu}, \qquad |\varphi_\nu|\leqq\omega.$$

Hence

$$\mathfrak{R}c_m(r)\geqq\frac{1}{2m}\sum_{r_\nu\leqq r}\left\{\left(\frac{r}{r_\nu}\right)-\left(\frac{r_\nu}{r}\right)^m\right\}\cdot\cos m\omega$$

$$=\cos m\omega\left[N(r, 0, f)+\frac{m}{2}\int_0^r\left\{\left(\frac{r}{t}\right)^m-\left(\frac{t}{r}\right)^m\right\}\frac{N(t, 0, f)}{t}dt\right.$$

$$\geqq N(r, 0, f)\cos m\omega.$$

Thus

$$m_2(r, f)^2\geqq N(r, 0, f)^2\left(1+2\sum_{m=1}^{q}\cos^2 m\omega\right)$$

$$=N(r, 0, f)^2\left(q+1+\frac{\cos(q+1)\omega\sin q\omega}{\sin\omega}\right).$$

This gives the first desired result. By $m_2(r, f)\geqq 2m(r, f)-N(r, 0, f)$ we have the second desired result.

If $\omega=0$, the $A(q, 0)^2=2q+1$ and

$$\varlimsup_{r\to\infty}\frac{N(r, 0, f)}{m_2(r, f)}\leqq\frac{1}{\sqrt{2q+1}}.$$

§4. **Proof of Theorem 2.** In this case for $2s+1\leqq q$ $c_{2s+1}(r)=0$. Further

$$\mathfrak{R}c_{2s}(r)=\frac{1}{2s}\mathfrak{R}\sum_{|a_\nu|\leqq r}\left\{\left(\frac{r}{a_\nu}\right)^{2s}-\left(\frac{a_\nu}{r}\right)^{2s}\right\}$$

$$\geqq\frac{\cos 2s\omega}{2s}\sum_{|a_\nu|\leqq r}\left\{\left(\frac{r}{|a_\nu|}\right)^{2s}-\left(\frac{|a_\nu|}{r}\right)^{2s}\right\},$$

where $\sum_{|a_\nu|\leqq r}$ means the summation over all a_ν but not over any $-a_\nu$. Hence

$$\mathfrak{R}c_{2s}(r)\geqq\cos 2s\omega\left[N(r,0,f)+s\int_0^r\left\{\left(\frac{r}{t}\right)^{2s}-\left(\frac{t}{r}\right)^{2s}\right\}\frac{N(t,0,f)}{t}dt\right.$$

$$\geqq N(r,0,f)\cos 2s\omega.$$

Therefore

$$m_2(r,f)^2\geqq\left\{1+2\sum_1^{[q/2]}\cos^2 2s\omega\right\}N(r,0,f)^2$$

$$=\left\{1+\left[\frac{q}{2}\right]+\frac{\cos([q/2]+1)\omega\sin[q/2]\omega}{\sin 2\omega}\right\}N(r,0,f)^2.$$

This gives the desired result.

If $\omega=0$ holds, then we can prove that

$$m_2(r,f)^2\leqq 4(2s_0-2+2s_q)(m(r,f)^2-m(r,f)N(r,0,f))$$

$$+(2s_0-1+2s_q)N(r,0,f)^2,$$

where $s_0=\min\{s\,|\,2s\geqq q+1\}$ and

$$s_q=s_0^2\sum_{j=0}^{\infty}(s_0+j)^{-2}.$$

§5. **An example.** Let a_j be

$$2^{j^j/\rho}.$$

Let $f(z)$ be the canonical product formed by $\{a_j,-a_j\}_{j=1,2,\dots}$ with their multiplicities a_j^ρ. Evidently for $\varepsilon>0$

$$\sum\frac{a_j^\rho}{a_j^\rho}=\infty,\qquad \sum\frac{a_j^\rho}{a_j^{\rho+\varepsilon}}\leqq\sum a_j^{-\varepsilon}<\infty.$$

Hence the exponent of convergence of the given series is equal to ρ. Assume that $1<\rho<2$. Then $f(z)$ is of the first genus. In the present case

$$f(z)=\prod_{j=1}^{\infty}\left(1-\frac{z^2}{a_j^2}\right)^{a_j^\rho}.$$

Let $n(r)$ be the number of zeros of $f(z)$ in $|z|\leqq r$ and $N(r)$ the counting function of zeros of $f(z)$ there. Then for $a_p\leqq r<a_{p+1}$

$$n(r)=2\sum_{j=1}^{p} a_j^{\rho}$$

and

$$N(r)=\int_0^r \frac{n(t)}{t}dt$$

$$=2\sum_{k=1}^{p-1} \log \frac{a_{k+1}}{a_k} \sum_{j=1}^{k-1} a_j^{\rho}+2 \log \frac{r}{a_p} \sum_{j=1}^{p} a_j^{\rho}.$$

Now we put $r_p=(a_p a_{p+1}^{2-\rho})^{1/2}$. Then

$$N(r_p)\sim a_p^{\rho} \log (a_p^{-1} a_{p+1}^{2-\rho}),$$

which is very easy to prove. Let $M(r, f)$ and $m^*(r, f)$ be the maximum modulus and the minimum modulus of $f(z)$ on $|z|=r$. Then

$$\log M(r, f)=\log |f(ir)|,$$

$$\log m^*(r, f)=\log |f(r)|.$$

Then we can prove that

$$\log M(r_p, f)\sim a_p^{\rho} \log (a_p^{-1} a_{p+1}^{2-\rho})$$

and

$$\log m^*(r_p, f)\sim a_p^{\rho} \log (a_p^{-1} a_{p+1}^{2-\rho}).$$

Therefore along the sequence $\{r_p\}$

$$m_2(r, f)\sim \log M(r, f)\sim \log m^*(r, f)\sim N(r, 0, f).$$

Hence

$$\overline{\lim_{r\to\infty}} \frac{N(r, 0, f)}{m_2(r, f)} \geqq \overline{\lim_{p\to\infty}} \frac{N(r_p, 0, f)}{m_2(r_p, f)}=1.$$

Thus we have the desired non-effectivity of estimations in Theorem 1 for $q=1$, $\omega=\pi/2$ and in Theorem 2 for $q=1$, $\omega=0$.

§ 6. Proof of Theorem 3.

In this case for $2p+1\leqq q$

$$\Re c_{2p+1}(r)\geqq \frac{1}{2(2p+1)}\left[\sum_{r_\nu \leqq r}\left\{\left(\frac{r}{r_\nu}\right)^{2p+1}-\left(\frac{r_\nu}{r}\right)^{2p+1}\right\}\cos (2p+1)\theta_\nu\right.$$

$$\left.-\sum_{t_\nu \leqq r}\left\{\left(\frac{r}{t_\nu}\right)^{2p+1}-\left(\frac{t_\nu}{r}\right)^{2p+1}\right\}\cos (2p+1)\varphi_\nu\right],$$

where $|\theta_\nu|\leqq \omega$, $|\pi-\varphi_\nu|\leqq \omega$. Hence

$$\Re c_{2p+1}(r)\geqq (N(r, 0)+N(r, \infty)) \cos (2p+1)\omega.$$

Further with $s=\max \{p\,|\,2p+1\leqq q\}$

$$m_2(r, f)^2 \geqq 2\sum_0^s |c_{2p+1}(r)|^2$$

$$\geqq 2(N(r, 0)+N(r, \infty))^2 \sum_0^s \cos^2(2p+1)\omega$$

$$=(N(r, 0)+N(r, \infty))^2\Big(s+1+\frac{\sin 4(s+1)\omega}{2\sin 2\omega}\Big).$$

This is the desired result.

§ 7. **Proof of Theorem 4.** In this case $c_{2p}(r)=0$ and for $2p+1\leqq q$

$$|c_{2p+1}(r)|\leqq\frac{1}{2p+1}\Big[\sum_{r_\nu\leqq r}\Big\{\Big(\frac{r}{r_\nu}\Big)^{2p+1}-\Big(\frac{r_\nu}{r}\Big)^{2p+1}\Big\}\Big]\equiv\gamma_{2p+1}(r)$$

and for $2p+1\geqq q+1$

$$|c_{2p+1}(r)|\leqq\frac{1}{2p+1}\Big[\sum_{r_\nu>r}\Big(\frac{r}{r_\nu}\Big)^{2p+1}+\sum_{r_\nu\leqq r}\Big(\frac{r_\nu}{r}\Big)^{2p+1}\Big\}\equiv\gamma_{2p+1}(r).$$

Hence

$$m_2(r, f)^2=2\sum_{p=1}^{\infty}|c_{2p-1}(r)|^2$$

$$\leqq 2\sum_{p=1}^{\infty}\gamma_{2p-1}(r)^2=m_2(r, F)^2.$$

Therefore it is sufficient to prove the result for F instead of f. Then we can make use of the integral representation of $\gamma_{2p-1}(r)$. Let $\{t_n\}$ be a sequence of Pólya peaks of the first kind, order ρ, for $N(r)\equiv N(r, 0)$. Then

$$N(t)t_n^{\rho-\varepsilon}\leqq N(t_n)t^{\rho-\varepsilon}, \qquad 0<t\leqq t_n$$

$$N(t)t_n^{\rho+\varepsilon}\leqq N(t_n)t^{\rho+\varepsilon}, \qquad t_n\leqq t<\infty.$$

Then we have

$$\gamma_{2p-1}(t_n)\leqq 2N(t_n)\frac{\rho^2+\varepsilon(2p-1-\varepsilon)}{(2p-1-\varepsilon)^2-\rho^2}, \qquad 2p-1\geqq q+1$$

and

$$\gamma_{2p-1}(t_n)\leqq 2N(t_n)\frac{(\rho-\varepsilon)^2}{(\rho-\varepsilon)^2-(2p-1)^2}, \qquad 2p-1\leqq q.$$

Hence

$$m_2(t_n, F)^2\leqq 8N(t_n)^2\Big[\sum_{p=1}^{s}\Big\{\frac{(\rho-\varepsilon)^2}{(\rho-\varepsilon)^2-(2p-1)^2}\Big\}^2$$

$$+\sum_{p=s+1}^{\infty}\Big\{\frac{\rho^2+\varepsilon(2p-1-\varepsilon)}{(2p-1-\varepsilon)^2-\rho^2}\Big\}^2\Big].$$

If ε tends to zero, then the term in the bracket tends to

$$\sum_{p=1}^{\infty} \frac{(\rho/2)^4}{\{(\rho/2)^2-(p-1/2)^2\}^2},$$

which is equal to

$$\frac{1}{8}\frac{\pi\rho-\sin\pi\rho}{\cos^2\frac{\pi}{2}\rho}\cdot\frac{\pi}{2}\rho.$$

This gives the desired result:

$$\varliminf_{r\to\infty}\frac{N(r, 0, f)}{m_2(r, f)} \geqq \frac{\sqrt{2}}{\sqrt{\pi\rho}} \frac{\left|\cos\frac{\pi}{2}\rho\right|}{(\pi\rho-\sin\pi\rho)^{1/2}}.$$

§ 8. **Proof of Theorem 5.** Firstly we have

$$m_2(r, f)^2 \geqq 2\sum |c_{2p+1}(r)|^2.$$

Let us put $N(r)=N(r, 0, f)+N(r, \infty, f)$. We can make use of the integral representation of $c_{2p+1}(r)$. Let $\{t_n\}$ be a sequence of Pólya peaks of the second kind, order ρ, for $N(r)$. Let $\{s_n\}$ and $\{S_n\}$ be the associated sequences such that $s_n\to\infty$, $t_n/s_n\to\infty$, $S_n/t_n\to\infty$ and

$$N(t) \geqq (1+o(1))(t/t_n)^\rho N(t_n)$$

for $s_n \leqq t \leqq S_n$. Similarly as in the proof of Theorem 4

$$m_2(t_n, f)^2 \geqq 2N(t_n)^2 \sum_{p=1}^{\infty}\left(\frac{\rho^2}{\rho^2-(2p-1)^2}\right)^2(1+o(1)).$$

Hence we have the desired result.

§ **9.** In this section we shall give an extension of Theorem 4. Let $f(z)$ be the canonical product formed by zeros $\{r_\nu e^{i\theta_\nu}\}$ and $g(z)$ be the canonical product formed by zeros $\{r_\nu e^{i(\theta_\nu+\alpha)}\}$, where α is a constant satisfying $0<\alpha\leqq\pi$. Let $F(z)$ be $f(z)/g(z)$. Then

$$|c_m(r)| = \frac{1}{2m}\left|\sum_{r_\nu\leqq r}\left\{\left(\frac{r}{r_\nu}\right)^m-\left(\frac{r_\nu}{r}\right)^m\right\}(1-e^{-im\alpha})e^{-im\theta_\nu}\right|$$
$$\leqq \frac{\sqrt{2(1-\cos m\alpha)}}{2m}\sum_{r_\nu\leqq r}\left\{\left(\frac{r}{r_\nu}\right)^m-\left(\frac{r_\nu}{r}\right)^m\right\}.$$

Again by making use of Pólya peaks of the first kind, order ρ, for $N(r)=N(r, 0, f)$, we have

$$\varliminf_{r\to\infty}\frac{m_2(r, F)^2}{N(r)^2} \leqq 16\sum_{m=1}^{\infty}(1-\cos m\alpha)\frac{\rho^4}{(m^2-\rho^2)^2}.$$

This is best possible. Especially, if $\alpha=\pi$, then we have Theorem 4.

§ 10. Let $F(z)$ be $f(z)g(z)$, where f and g are defined in § 9. Then

$$|c_m(r)|=\frac{1}{2m}\left|\sum_{r_\nu\leqq r}\left\{\left(\frac{r}{r_\nu}\right)^m-\left(\frac{r_\nu}{r}\right)^m\right\}(1+e^{-imα})e^{-im\theta_\nu}\right|$$

$$\leqq\frac{1}{2m}(2+2\cos mα)^{1/2}\sum_{r_\nu\leqq r}\left\{\left(\frac{r}{r_\nu}\right)^m-\left(\frac{r_\nu}{r}\right)^m\right\}.$$

Hence by the same method as in § 9

$$\varliminf_{r\to\infty}\frac{m_2(r,\ F)^2}{N(r,\ 0,\ F)^2}\leqq 1+\sum_{m=1}^{\infty}(1+\cos mα)\frac{\rho^4}{(m^2-\rho^2)^2}.$$

Especially for $α=\pi$

$$\varliminf_{r\to\infty}\frac{m_2(r,\ F)^2}{N(r,\ 0,\ F)^2}\leqq\frac{1}{8}\frac{\pi\rho(\pi\rho+\sin\pi\rho)}{\sin^2\frac{\pi}{2}\rho}.$$

Of course this is best possible. The last part is due to the following identities:

$$1+2\sum_{m=1}^{\infty}\frac{\rho^4}{(\rho^2-m^2)^2}=\frac{1}{2}\frac{\pi\rho(\pi\rho+\cos\pi\rho\sin\pi\rho)}{\sin^2\pi\rho},$$

$$2\sum_{p=1}^{\infty}\frac{\rho^4}{(\rho^2-(2p-1)^2)^2}=\frac{1}{4}\frac{\pi\rho(\pi\rho-\sin\pi\rho)}{2\cos^2\frac{\pi}{2}\rho}.$$

§ 11. Let $f(z)$ be the canonical product formed by zeros $\{r_\nu\}$ and $g(z)$ the canonical product formed by zeros $\{s_\nu e^{iα}\}$, where $α$ is a constant satisfying $0\leqq α\leqq\pi$. Let $F(z)$ be $f(z)g(z)$. Let $f_1(z)$ be the canonical product formed by zeros $\{r_\nu,\ s_\nu\}$ and $f_{1α}(z)$ be the canonical product formed by zeros $\{r_\nu e^{iα},\ s_\nu e^{iα}\}$. Let $F_1(z)$ be $f_1(z)f_{1α}(z)$. Then

$$|c_m(r\,;\ F)|=|A_m+B_m e^{-imα}|.$$

It is very easy to prove that

$$2|A_m+B_m e^{-imα}|\geqq|A_m+B_m+(A_m+B_m)e^{-imα}|.$$

This gives

$$4m_2(r,\ F)^2\geqq m_2(r,\ F_1)^2.$$

By making use of Pólya peaks of the second kind, order ρ, for $N(r,\ 0,\ F_1)=2N(r,\ 0,\ F)$, we have

$$\varliminf_{r\to\infty}\frac{N(r,\ 0,\ F)^2}{m_2(r,\ F)^2}\leqq\left(1+\sum_{m=1}^{\infty}(1+\cos mα)\frac{\rho^4}{(m^2-\rho^2)^2}\right)^{-1/2}.$$

Especially for $α=\pi$

$$\varliminf_{r\to\infty}\frac{N(r,\ 0,\ F)}{m_2(r,\ F)}\leqq\frac{2\sqrt{2}\left|\sin\frac{\pi}{2}\rho\right|}{\sqrt{\pi\rho(\pi\rho+\sin\pi\rho)}}.$$

Again this is best possible.

REFERENCES

[1] EDREI, A. AND W.H.J. FUCHS, On the growth of meromorphic functions with several deficient values, Trans. Amer. Math. Soc. **93** (1959) 292-328.

[2] KOBAYASHI. T., On the deficiency of an entire function of finite genus, Kōdai Math. Sem. Rep. **27** (1976), 320-328.

[3] KOBAYASHI, T., On the lower order of an entire function, Kōdai Math. Sem. Rep. **27** (1976), 484-495.

[4] MILES, J. AND D.F. SHEA, An extremal problem in value-distribution theory, Quart. J. Math. Oxford **24** (1973), 377-383.

[5] MILES, J. AND D.F. SHEA, On the growth of meromorphic functions having at least one deficient value, Duke Math. J. **43** (1976), 171-186.

[6] SHEA, D.F., On the Valiron deficiencies of meromorphic functions of finite order, Trans. Amer. Math. Soc. **124** (1966), 201-227.

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
OH-OKAYAMA, MEGURO-KU, TOKYO, JAPAN