K. SEKIGAWA
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ALMOST COMPLEX SUBMANIFOLDS OF A 6-DIMENSIONAL SPHERE

BY KOUEI SEKIGAWA

1. Introduction

Among all submanifolds of an almost Hermitian manifold, there are two typical classes: one is the class of almost complex submanifolds, and the other is the class of totally real submanifolds. A Riemannian submanifold (M, ψ) (or briefly M) of an almost Hermitian manifold $(\tilde{M}, J, \langle , \rangle)$ (or briefly $\tilde{M}$) is called an almost complex submanifold provided that $J_{\psi(p)}((d\psi)_p(X)) \in (d\psi)_p(T_p(M))$ for any $X \in T_p(M)$, $p \in M$. The most typical example of nearly Kaehlerian manifolds is a 6-dimensional sphere S^6. In fact, Fukami and Ishihara [3] proved that there exists a nearly Kaehlerian structure on a 6-dimensional sphere S^6 by making use of the properties of the Cayley division algebra. We shall call it the canonical nearly Kaehlerian structure on S^6. In this paper, we shall study almost complex submanifolds of a 6-dimensional unit sphere S^6 with the canonical nearly Kaehlerian structure. First of all, Gray [1] proved that with respect to the canonical nearly Kaehlerian structure, S^6 has no 4-dimensional almost complex submanifolds. We shall prove the following Theorems and some related results. In the following Theorems, we assume that $M=(M, \psi)$ is an almost complex submanifold of S^6. Then it follows that $\dim M=2$. We denote by K the Gaussian curvature of M.

THEOREM A. *If (M, ψ) is not totally geodesic, then the degree of ψ is* 3.

THEOREM B. *If K is constant on M, then $K=1$ or $1/6$ or* 0.

THEOREM C. *Assume that M is compact. If $K>1/6$ on M, then $K=1$ on M, and if $1/6 \leqq K<1$ on M, then $K=1/6$ on M.*

In the last section of this paper, we shall give some examples of almost complex submanifolds of S^6 corresponding to the cases, $K=1$, $1/6$ and 0 in Theorem B. We note that the result of Theorem B is a special case of the result obtained by Kenmotsu under more general situation ([6]).

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2. Riemannian submanifolds

Let $(\tilde{M}, \langle\, ,\rangle)$ (or briefly $\tilde{M}$) be a Riemannian manifold and (M, ψ) (or brieflv M) be a Riemannian submanifold of $\tilde{M}$ with isometric immersion ψ. Let ∇ (resp. $\tilde{\nabla}$) be the Riemannian connection on M (resp. $\tilde{M}$) and R (resp. $\tilde{R}$) be the curvature tensor of M (resp. $\tilde{M}$). We denote by σ the second fundamental form of M in $\tilde{M}$. Since ψ is locally an imbedding, we may identify $p \in M$ with $\psi(p) \in \tilde{M}$ locally, and $T_p(M)$ with the subspace $(d\psi)_p(T_p(M))$ of $T_{\psi(p)}(\tilde{M})$. Then, the Gauss formula, Weingarten formula are given respectively by

$$\sigma(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y\,, \tag{2.1}$$

$$\tilde{\nabla}_X \xi=-A_\xi X+\nabla^\perp_X \xi\,, \qquad X, Y \in \mathfrak{X}(M)\,, \tag{2.2}$$

where ξ is a local field of normal vector to M and $-A_\xi X$ (resp. $\nabla^\perp_X \xi$) denotes the tangential part (resp. normal part) of $\tilde{\nabla}_X \xi$.

The tangential part $A_\xi X$ is related to the second fundamental form σ as follows:

$$\langle \sigma(X, Y), \xi\rangle=\langle A_\xi X, Y\rangle\,, \qquad X, Y \in \mathfrak{X}(M)\,. \tag{2.3}$$

We denote by $R^\perp$ the curvature tensor of the normal connection, i.e., $R^\perp(X, Y)=[\nabla^\perp_X, \nabla^\perp_Y]-\nabla^\perp_{[X,Y]}$. Then, the Gauss, Codazzi and Ricci equations are given respectively by

$$\langle R(X, Y)Z, Z'\rangle=\langle \tilde{R}(X, Y)Z, Z'\rangle+\langle \sigma(X, Z'), \sigma(Y, Z)\rangle \tag{2.4}$$
$$-\langle \sigma(X, Z), \sigma(Y, Z')\rangle\,,$$

$$(\tilde{R}(X, Y)Z)^\perp=(\nabla'_X \sigma)(Y, Z)-(\nabla'_Y \sigma)(X, Z)\,, \tag{2.5}$$

$$\langle \tilde{R}(X, Y)\xi, \eta\rangle=\langle R^\perp(X, Y)\xi, \eta\rangle-\langle [A_\xi, A_\eta]X, Y\rangle\,, \tag{2.6}$$

for $X, Y, Z, Z' \in \mathfrak{X}(M)$, where $(\nabla'_X \sigma)(Y, Z)=\nabla^\perp_X \sigma(Y, Z)-\sigma(\nabla_X Y, Z)-\sigma(Y, \nabla_X Z)$ and ξ, η are local fields of normal vectors to M.

In the sequel, the following convention for the notations will be used unless otherwise specified:

$$X, Y, Z, \cdots, \in \mathfrak{X}(M)\,, \qquad U, V, W, \cdots, \in \mathfrak{X}(\tilde{M})$$

and $\mathfrak{X}(M)$ (resp. $\mathfrak{X}(\tilde{M})$) denotes the set of all tangential vector fields to M (resp. $\tilde{M}$).

For the definition of the degree of the isometric immersion ψ, we refere to [8].

3. 6-dimensional nearly Kaehlerian manifolds

In this section, for the sake of later uses, we shall recall some elementary formulas in a 6-dimensional nearly Kaehlerian manifold and furthermore the canonical nearly Kaehlerian structure on a 6-dimensional unit sphere S^6. Let $\tilde{M}$ be an almost Hermitian manifold with the almost Hermitian structure $(J, \langle\,,\rangle)$. We denote by N the Nijienhuis tensor of J and by $\tilde{\nabla}$ the Riemannian connection of $\tilde{M}$. It is known that the tensor field N satisfies

$$N(JU, V)=N(U, JV)=-JN(U, V), \qquad U, V\in\mathfrak{X}(\tilde{M}). \tag{3.1}$$

Especially, if $\tilde{M}$ is a nearly Kaehlerian manifold (i.e., $(\tilde{\nabla}_U J)U=0$, for any $U\in\mathfrak{X}(\tilde{M})$), then the tensor field N is written in the following form (cf. [13]):

$$N(U, V)=-4J(\tilde{\nabla}_U J)V, \qquad U, V\in\mathfrak{X}(\tilde{M}). \tag{3.2}$$

From (3.2), we get

$$\langle N(U, V), W\rangle=-\langle N(U, W), V\rangle, \qquad U, V, W\in\mathfrak{X}(\tilde{M}). \tag{3.3}$$

An almost complex submanifold M of an almost Hermitian manifold $\tilde{M}$ is called to be a σ-submanifold if the second fundamental form σ is complex linear, i.e.,

$$\sigma(JX, Y)=\sigma(X, JY)=J\sigma(X, Y), \qquad \text{for} \quad X, Y\in\mathfrak{X}(M), \tag{3.4}$$

(cf. [12]). From (3.4), any σ-submanifold is necessarily minimal. Vanhecke [12] proved that if $\tilde{M}$ is a nearly Kaehlerian manifold, any almost complex submanifold is a σ-submanifold and is also a nearly Kaehlerian manifold. W now assume that $\tilde{M}$ is a 6-dimensional non-Kaehlerian, nearly Kaehlerian manifold. Then the followings hold in $\tilde{M}$ (cf. [7], [9]):

$$\begin{aligned}\tilde{\nabla}_U((\tilde{\nabla}_V J)W)&-(\tilde{\nabla}_{\tilde{\nabla}_U V} J)W-(\tilde{\nabla}_V J)(\tilde{\nabla}_U W)\\ &=-\frac{S}{30}(\langle U, V\rangle JW-\langle U, W\rangle JV+\langle JV, W\rangle U),\end{aligned} \tag{3.5}$$

$$\begin{aligned}(\tilde{\nabla}_U J)(\tilde{\nabla}_V J)W=-\frac{S}{30}(&\langle U, V\rangle W-\langle U, W\rangle V\\ &+\langle JU, V\rangle JW-\langle JU, W\rangle JV),\end{aligned} \tag{3.6}$$

$U, V, W\in\mathfrak{X}(\tilde{M})$, where S denotes the scalar curvature of $\tilde{M}$.

From (3.2), (3.5) and (3.6), we get

$$(\tilde{\nabla}_U N)(V, W)=\frac{2S}{15}(\langle JU, V\rangle JW-\langle JU, W\rangle JV+\langle JV, W\rangle JU), \tag{3.7}$$

$$N(U, N(V, W))=16(\tilde{\nabla}_U J)(\tilde{\nabla}_V J)W \tag{3.8}$$

$$=-\frac{8S}{15}(\langle U, V\rangle W-\langle U, W\rangle V+\langle JU, V\rangle JW-\langle JU, W\rangle JV),$$

(3.9) $$\langle N(U, V), N(U', V')\rangle=-16\langle V, (\tilde{\nabla}_U J)(\tilde{\nabla}_{U'} J)V'\rangle$$
$$=\frac{8S}{15}(\langle U, U'\rangle\langle V, V'\rangle-\langle U, V'\rangle\langle V, U'\rangle$$
$$+\langle JU, U'\rangle\langle JV', V\rangle-\langle JU, V'\rangle\langle JU', V\rangle),$$

$U, U', V, V', W\in\mathfrak{X}(\tilde{M})$.

We shall now recall the canonical nearly Kaehlerian structure on a 6-dimensional sphere S^6. Let $\mathcal{C}$ be the Cayley division algebra generated by $\{e_0=1, e_i(1\leqq i\leqq 7)\}$ over real number field $\boldsymbol{R}$ and $\mathcal{C}_+$ be the subspace of $\mathcal{C}$ consisting of all purely imaginary Cayley numbers. We may identify $\mathcal{C}_+$ with a 7-dimensional Euclidean space $\boldsymbol{R}^7$ with the canonical inner product $(\,,\,)$ (i.e., $(e_i, e_j)=\delta_{ij}$, $1\leqq i, j\leqq 7$). The automorphism group of $\mathcal{C}$ is the compact simple Lie group G_2 and the inner product $(\,,\,)$ is invariant under the action of the group G_2. A vector cross product for the vectors in $\mathcal{C}_+=\boldsymbol{R}^7$ is defined by

(3.10) $$x\times y=(x, y)e_0+xy\,,\qquad x, y\in\mathcal{C}_+\,.$$

Then the multiplication table is given by the following:

$e_j\times e_k=$

j \ k	1	2	3	4	5	6	7
1	0	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
2	$-e_3$	0	e_1	e_6	$-e_7$	$-e_4$	e_5
3	e_2	$-e_1$	0	$-e_7$	$-e_6$	e_5	e_4
4	$-e_5$	$-e_6$	e_7	0	e_1	e_2	$-e_3$
5	e_4	e_7	e_6	$-e_1$	0	$-e_3$	$-e_2$
6	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	0	e_1
7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	0

Considering S^6 as $\{x\in\mathcal{C}_+\,;\,(x, x)=1\}$, the canonical almost complex structure J on S^6 is defined by

(3.11) $$J_x U=x\times U\,,$$

where $x\in S^6$ and $U\in T_x(S^6)$ (the tangent space of S^6 at x).

The above almost complex structure J together with the induced Riemannian metric $\langle\,,\,\rangle$ on S^6 from the inner product $(\,,\,)$ on $\mathcal{C}_+=\boldsymbol{R}^7$ gives rise to a nearly Kaehlerian structure on S^6. The group G_2 acts on S^6 transitively as the group of automorphisms of the nearly Kaehlerian structure $(J, \langle\,,\,\rangle)$ (cf. [3]). It is well known that S^6 does not admit any Kaehlerian structures.

4. Proofs of Theorems A, B and C

Let M be an almost complex submanifold of a 6-dimensional unit sphere $\tilde{M}=S^6$ with the canonical nearly Kaehlerian structure $(J, \langle\ ,\ \rangle)$. Then it follows that $\dim M=2$ and hence M is a Kaehlerian manifold of complex dimension 1 with respect to the induced structure from S^6. We denote by K the Gaussian curvature of M. Then, from (2.4) and (3.4), we get

$$K=1-\frac{\|\sigma\|^2}{2}, \tag{4.1}$$

where $\|\sigma\|$ denotes the length of the second fundamental form σ.

Codazzi equation (2.5) implies in particular

$$(\nabla'_X\sigma)(Y, Z)=(\nabla'_Y\sigma)(X, Z). \tag{4.2}$$

From (2.1), (2.2) and (3.2), we get

$$\tilde{\nabla}_Z(J\sigma(X, Y))=\frac{1}{4}JN(Z, \sigma(X, Y))+J(-A_{\sigma(X,Y)}Z+\nabla_Z^{\perp}\sigma(X, Y)), \tag{4.3}$$
$$\tilde{\nabla}_Z(\sigma(X, JY))=-A_{\sigma(X,JY)}Z+\nabla_Z^{\perp}\sigma(X, JY).$$

From (4.3), taking account of (3.1), (3.3) and (3.4), we get

$$\frac{1}{4}JN(Z, \sigma(X, Y))=(\nabla'_Z\sigma)(X, JY)-J(\nabla'_Z\sigma)(X, Y). \tag{4.4}$$

Since $\dim M=2$, from (3.1) and (3.4), we get easily

$$N(Z, \sigma(X, Y))=N(Y, \sigma(X, Z)). \tag{4.5}$$

Let $M'=\{p\in M;\ \sigma\neq 0 \text{ at } p\}$. Then M' is an open set of M.

We now assume that $M'\neq\emptyset$ (i.e., M is not totally geodesic in S^6). Let $\{X_1, X_2=JX_1\}$ be a local field of orthonormal frame on a neighborhood of a point $p\in M'$ in M. If we put

$$\nabla_{X_i}X_j=\sum_{k=1}^{2}B_{ijk}X_k, \qquad 1\leqq i, j\leqq 2, \tag{4.6}$$

then we get

$$B_{ijk}=-B_{ikj}, \qquad 1\leqq i, j, k\leqq 2. \tag{4.7}$$

Taking account of (3.1), (3.3), (3.4) and (3.9), we may put

$$(\nabla'_{X_1}\sigma)(X_1, X_1)=a\sigma(X_1, X_1)+b\sigma(X_1, X_2)+\frac{c}{4}N(X_1, \sigma(X_1, X_1))+\frac{d}{4}N(X_2, \sigma(X_1, X_1)), \tag{4.8}$$

$$(\nabla'_{X_2}\sigma)(X_1, X_1)=a'\sigma(X_1, X_1)+b'\sigma(X_1, X_2)$$
$$+\frac{c'}{4}N(X_1, \sigma(X_1, X_1))+\frac{d'}{4}N(X_2, \sigma(X_1, X_1)).$$

Then, from (4.8), taking account of (2.5), (3.1), (3.4) and (4.4), we get

$$a'=-b,\quad b'=a,\quad c'=d,\quad d'=-c-1. \tag{4.9}$$

Thus, from (4.8), taking account of (3.3), (3.4) and (4.9), we get

$$a=\frac{1}{\|\sigma\|}X_1\|\sigma\|,\qquad b=-\frac{1}{\|\sigma\|}X_2\|\sigma\|. \tag{4.10}$$

From (4.6), (4.7) and (4.10), we get

$$\begin{aligned}[X_1, X_2]\|\sigma\| &= X_1(X_2\|\sigma\|)-X_2(X_1\|\sigma\|)\\ &= -X_1(b\|\sigma\|)-X_2(a\|\sigma\|)\\ &= -(X_1b+X_2a)\|\sigma\|,\end{aligned}$$

and hence

$$X_2a+X_1b+aB_{121}+bB_{212}=0. \tag{4.11}$$

Taking account of (3.4), (4.6) and (4.7), we get easily

$$\sum_{i=1}^{2}(\nabla'_X\sigma)(X_i, X_i)=0. \tag{4.12}$$

From (4.8) with (4.9), taking account of (2.5), (3.1), (3.3)~(3.6) and (4.12), we get

$$\begin{aligned}\|\nabla'\sigma\|^2 &= \sum_{1\leq i,j,k\leq 2}\langle(\nabla'_{X_i}\sigma)(X_j, X_k), (\nabla'_{X_i}\sigma)(X_j, X_k)\rangle\\ &= 4(\langle(\nabla'_{X_1}\sigma)(X_1, X_1), (\nabla'_{X_1}\sigma)(X_1, X_1)\rangle\\ &\quad +\langle(\nabla'_{X_2}\sigma)(X_1, X_1), (\nabla'_{X_2}\sigma)(X_1, X_1)\rangle)\\ &= (2(a^2+b^2)+2(c^2+c+d^2)+1)\|\sigma\|^2.\end{aligned} \tag{4.13}$$

From (4.10) and (4.13), we get

$$a^2+b^2=\|\mathrm{grad}(\log\|\sigma\|)\|^2, \tag{4.14}$$

$$c^2+c+d^2=\frac{1}{2\|\sigma\|^2}(\|\nabla'\sigma\|^2-2\|\mathrm{grad}\|\sigma\|\|^2-\|\sigma\|^2). \tag{4.15}$$

We put

$$F=\|\mathrm{grad}(\log\|\sigma\|)\|^2$$

and

$$G=\frac{1}{2\|\sigma\|^2}(\|\nabla'\sigma\|^2-2\|\mathrm{grad}\|\sigma\|\|^2-\|\sigma\|^2).$$

Then, from (4.15), we have easily

LEMMA 4.1. $G \geqq -\frac{1}{4}$ on M'.

From (2.6), taking account of (2.1), (2,2), (3.1)~(3.4), (3.7), (3.8), (4.1), (4.5)~(4.9), we get

$$-\frac{1}{8}\|\sigma\|^4 = \langle R^{\perp}(X_1, X_2)\sigma(X_1, X_1), \sigma(X_1, X_2)\rangle$$

$$= \frac{\|\sigma\|^2}{4}(X_1 a - X_2 b - bB_{121} + aB_{212} - 2G - 1$$

$$+ 2(X_1 B_{212} - X_2 B_{112} - B_{121}B_{112} + B_{212}B_{212}))$$

$$= \frac{\|\sigma\|^2}{4}(X_1 a - X_2 b - bB_{121} + aB_{212} - 1 - 2G - 2K),$$

and hence

(4.16) $$X_1 a - X_2 b - bB_{121} + aB_{212} = 2G + 3K.$$

Similarly, we get

(4.17) $$X_1 d - X_2 c = 3(2c+1)B_{121} - 6dB_{212} - 2ad - (2c+1)b,$$

(4.18) $$X_1 c + X_2 d = -6dB_{121} - 3(2c+1)B_{212} + 2bd - (2c+1)a.$$

LEMMA 4.2. $\Delta(\log\|\sigma\|) = 2G + 3K$ *on* M'.

Proof. From (4.6), (4.7), (4.10) and (4.16), we get

$$\Delta\|\sigma\| = X_1(X_1\|\sigma\|) + X_2(X_2\|\sigma\|) + B_{121}X_2\|\sigma\| + B_{212}X_1\|\sigma\|$$

$$= \|\sigma\|(X_1 a - X_2 b - bB_{121} + aB_{212} + a^2 + b^2)$$

$$= \|\sigma\|(F + 2G + 3K),$$

and hence

$$\Delta(\log\|\sigma\|) = (1/\|\sigma\|)\Delta\|\sigma\| - \|\text{grad}(\log\|\sigma\|)\|^2$$

$$= 2G + 3K.$$

Q.E.D.

Let $\{E_1, E_2 = JE_1\}$ be an orthonormal basis of $T_p(M)$, $p \in M'$ and $\gamma_i = \gamma_i(t_i)$ $(1 \leqq i \leqq 2)$ be the geodesics in M' such that

$$\gamma_i(0) = p \quad \text{and} \quad \frac{d\gamma_i}{dt_i}(0) = E_i, \qquad 1 \leqq i \leqq 2.$$

Then, we may easily see that there exists an orthonormal frame field $\{X_1, X_2 = JX_1\}$ near p in M' such that

(4.19) $$X_i=E_i \quad (1\leqq i\leqq 2) \qquad \text{at } p,$$

and

$$X_1=\frac{d\gamma_1}{dt_1} \quad \text{along } \gamma_1, \qquad X_2=\frac{d\gamma_2}{dt_2} \quad \text{along } \gamma_2.$$

From (4.19), we get

(4.20) $$B_{121}=0 \quad \text{along } \gamma_1 \quad \text{and} \quad B_{212}=0 \quad \text{along } \gamma_2.$$

From (4.17) and (4.18), taking account of (4.19) and (4.20), we get

(4.21) $$\begin{aligned} E_1(X_1d)-E_1(X_2c)&=-(2c+1)E_1d-2dE_1a \\ &\quad -6dE_1B_{212}-2bE_1c-2aE_1d, \\ E_2(X_1c)+E_2(X_2d)&=-(2c+1)E_2a+adE_2b \\ &\quad -6dE_2B_{121}-2aE_2c+2bE_2d. \end{aligned}$$

From (4.21), taking account of (4.11), (4.16) and (4.20), we get

(4.22) $$d=-4dG-2aE_1d+2bE_2d-2bE_1c-2aE_2c.$$

Similarly, we get

(4.23) $$c=-2(2c+1)G+2bE_1d+2aE_2d-2aE_1c+2bE_2c.$$

On one hand, from (4.17), (4.18) and (4.20), we get

(4.24) $$\begin{aligned} (E_1c)^2&=-(E_1c)(E_2d)-(2c+1)aE_1c+2bdE_1c, \\ (E_2c)^2&=(E_2c)(E_1d)+2adE_2c+(2c+1)bE_2c, \\ (E_1d)^2&=(E_2c)(E_1d)-2adE_1d-(2c+1)bE_1d, \\ (E_2d)^2&=-(E_1c)(E_2d)-(2c+1)aE_2d+2bdE_2d. \end{aligned}$$

From (4.17), (4.18) and (4.24), we get

(4.25) $$\begin{aligned} &2((E_2c)(E_1d)-(E_1c)(E_2d)) \\ &\quad =-F(4G+1)+(E_1c)^2+(E_2c)^2+(E_1d)^2+(E_2d)^2. \end{aligned}$$

Thus, from (4.21)$\sim$(4.25), we get

(4.26) $$\begin{aligned} \Delta G=2(&-(F+G)(4G+1)-2aE_1G+2bE_2G \\ &+(E_1c)^2+(E_2c)^2+(E_1d)^2+(E_2d)^2). \end{aligned}$$

LEMMA 3. *The following holds on M'.*

(4.27) $$\Delta(4G+1)^3=24(4G+1)(-(4G+1)^2G+6\|\text{grad}\, G\|^2).$$

Proof. By the definition of the function G, we get

$$E_iG=(2c+1)E_ic+2dE_id\,, \qquad 1\leqq i\leqq 2\,. \tag{4.28}$$

From (4.17), (4.18) and (4.28), we get

$$\begin{aligned}(4G+1)E_1c&=(2c+1)E_1G-2dE_2G+2bd(4G+1)\,,\\ (4G+1)E_2c&=2dE_1G+(2c+1)E_2G+2ad(4G+1)\,,\\ (4G+1)E_1d&=2dE_1G+(2c+1)E_2G-(2c+1)b(4G+1)\,,\\ (4G+1)E_2d&=-(2c+1)E_1G+2dE_2G-(2c+1)a(4G+1)\,.\end{aligned} \tag{4.29}$$

From (4.29), taking account of the definitions of the functions F and G, we get

$$\begin{aligned}&(4G+1)^2((E_1c)^2+(E_2c)^2+(E_1d)^2+(E_2d)^2)\\ &\quad=2(4G+1)((4G+1)^2F+\|\mathrm{grad}\,G\|^2\\ &\qquad+a(4G+1)E_1G-b(4G+1)E_2G)\,.\end{aligned} \tag{4.30}$$

Thus, from (4.26) and (4.30), we have finally (4.27). Q. E. D.

We are now in a position to prove Theorems A, B and C. First, we shall prove Theorem A. We denote by ν_p^k the k-th normal space and by σ_p^k the k-th fundamental form of the isometric immersion ψ at $p\in M'$. Then from (4.8) with (4.9), we see that ν_p^1 and ν_p^2 are generated respectively by $\{\sigma_p^2(E_1, E_1)=\sigma(E_1, E_1), \sigma_p^2(E_1, E_2)=\sigma(E_1, E_2)\}$ and $\{\sigma_p^3(E_1, E_1, E_1)=(c/4)N(E_1, \sigma(E_1, E_1))+(d/4)N(E_2, \sigma(E_1, E_1)), \sigma_p^3(E_2, E_1, E_1)=(d/4)N(E_1, \sigma(E_1, E_1))-((c+1)/4)N(E_2, \sigma(E_1, E_1))\}$, where $E_2=JE_1$.

If $G(p)\neq 0$, then it follows that $\dim\nu_p^1=2$, $\dim\nu_p^2=2$, and hence the degree of the immersion ψ is 3. So, we assume that $G=0$ on M'. Let p be any point of M' and define E by

$$\|(\nabla'_E\sigma)(E, E)\|=\operatorname*{Max}_{\substack{X\in T_p(M)\\ \|X\|=1}}\|(\nabla'_X\sigma)(X, X)\|\,.$$

Let $\{X_1, X_2=JX_1\}$ be an orthonormal frame field near p satisfying the condition (4.19) for the basis $\{E_1=E, E_2=JE\}$ at p. Then, we may easily see that $d=0$ (and hence $c^2+c=0$) at p. We may assume that $c=-1$ at p. We put

$$\zeta=-\frac{d}{4}N(X_1, \sigma(X_1, X_1))+\frac{c}{4}N(X_2, \sigma(X_1, X_1)) \qquad \text{near } p\,.$$

Then, taking account of (3.1), (3.7), (3.8), (4.2), (4.8), (4.9), (4.20) and (4.29), we get

$$\begin{aligned}\sigma_p^4(E_1, E_1, E_1, E_1)&=-(E_2d+a)\zeta_p\\ &=-2G(p)\zeta_p=0.\end{aligned} \tag{4.31}$$

Similarly, we get

$$\sigma_p^4(E_2, E_1, E_1, E_1)=0. \tag{4.32}$$

Thus, from (4.31) and (4.32), taking account of (4.12) and the symmetricity of σ_p^4, we have finally $\sigma_p^4=0$, and hence the degree of ψ is 3. This completes the proof of Theorem A. Next, we shall prove Theorem B. We assume that the Gaussian curvature K of M is constant and $K\neq 1$. From (4.1), we get $\|\sigma\|^2=2(1-K)$, and hence from (4.10) and (4.14)

$$F=0 \quad \text{on } M=M'. \tag{4.33}$$

Thus, from (4.33) and Lemma 4.2, we get

$$G=-\frac{3}{2}K \quad \text{on } M. \tag{4.34}$$

From (4.34) and Lemma 4.3, it follows that $G(4G+1)=0$. If $4G+1=0$, then, from (4.34), we have $K=1/6$, and otherwise, we have $K=0$. This completes the proof of Theorem B.

Lastly, we shall prove Theorem C. We suppose that M is compact and $M'\neq\emptyset$. Then $\|\sigma\|$ takes its maximum at some point $p\in M'$. Then, from (4.10), we have $F(p)=0$. Thus, from Lemmas 4.1 and 4.2, we have

$$0\geqq(\Delta\log\|\sigma\|)(p)\geqq-\frac{1}{2}+3K(p), \tag{4.35}$$

and hence $K(p)\leqq 1/6$.

Thus, if M is compact and $K>1/6$ on M, from (4.35), it follows that $M'=\emptyset$, and hence the first half of Theorem C is proved. The latter half of Theorem C is immediately followed by using Lemmas 4.1 and 4.2, and Green's theorem. From Lemmas 4.2 and 4.3, taking account of Green's theorem and Gauss-Bonnet theorem, we have the following

THEOREM D. *Assume that M is compact and $K<1$ on M. If the function G satisfies the inequality $-1/4\leqq G\leqq 0$ on M, then $G=0$ or $-1/4$ on M, and furthermore M is diffeomorphic to a 2-dimensional torus (resp. a 2-dimensional sphere) in the case where $G=0$ on M (resp. $G=-1/4$ on M).*

We remark that the equality $G=0$ (resp. $G=-1/4$) on M' is equivalent to

$$\Delta\log(1-K)=6K, \quad \text{on } M' \tag{4.36}$$

(resp. (4.37) $\Delta\log(1-K)=-1+6K$ on M')

5. Some examples

EXAMPLE 1. Let $M=\{x\in S^6;\ x=x_2e_2+x_4e_4+x_6e_6\}$, and ι be the inclusion map from M into S^6. Then, we may easily see that (M, ι) is a 2-dimensional almost complex and totally geodesic submanifold of S^6.

EXAMPLE 2. Let $M=S^2_{1/6}=\{(y_1, y_2, y_3)\in \boldsymbol{R}^3;\ y_1^2+y_2^2+y_3^2=6\}$ and ψ_0 be a C^∞ map from M into S^6 defined by

$$
\begin{aligned}
(5.1)\qquad & \psi_0(y_1, y_2, y_3) \\
&=\left(\frac{\sqrt{6}}{72}(2y_1^3-3y_1y_2^2-3y_1y_3^2)\right)e_1+\left(\frac{\sqrt{15}}{72}(3y_2^2y_3-y_3^3)\right)e_2 \\
&+\left(\frac{\sqrt{15}}{72}(y_2^3-2y_2y_3^2)\right)e_3+\left(\frac{1}{24}(4y_1^2y_2-y_2^3-y_2y_3^2)\right)e_4 \\
&+\left(\frac{1}{24}(4y_1^2y_3-y_2^2y_3-y_3^3)\right)e_5+\left(\frac{\sqrt{10}}{24}(y_1y_2^2-y_1y_3^2)\right)e_6 \\
&+\left(\frac{\sqrt{10}}{12}y_1y_2y_3\right)e_7, \qquad \text{for } (y_1, y_2, y_3)\in S^2_{1/6}.
\end{aligned}
$$

Then, we may easily check that $(S^2_{1/6}, \psi_0)$ is a 2-dimensional almost complex submanifold of S^6 and furthermore, any almost complex submanifold $(S^2_{1/6}, \psi)$ of S^6 is obtained by $\psi=\alpha\cdot\psi_0$ for some $\alpha\in G_2$.

EXAMPLE 3. Let $M=\boldsymbol{R}^2$ be a 2-dimensional Euclidean space with the canonical metric and ψ be a C^∞ map from $\boldsymbol{R}^2$ into S^6 defined by

$$
\begin{aligned}
(5.2)\qquad \psi(u, v)=&\sqrt{\frac{2}{3}}\left(\cos\sqrt{\frac{1}{2}}u\right)\left(\left(\sin\sqrt{\frac{3}{2}}v\right)a_1-\left(\cos\sqrt{\frac{3}{2}}v\right)b_1\right) \\
&+\sqrt{\frac{2}{3}}\left(\sin\sqrt{\frac{1}{2}}u\right)\left(\left(\sin\sqrt{\frac{3}{2}}v\right)a_2-\left(\cos\sqrt{\frac{3}{2}}v\right)b_2\right) \\
&+\left(\sqrt{\frac{1}{3}}\cos\sqrt{2}u\right)a_3+\left(\sqrt{\frac{1}{3}}\sin\sqrt{2}u\right)b_3,
\end{aligned}
$$

for $(u, v)\in\boldsymbol{R}^2$, where $a_i, b_i\in C_+=\boldsymbol{R}^7$ such that $(a_i, a_j)=\delta_{ij}$, $(a_i, b_j)=0$, $(b_i, b_j)=\delta_{ij}$, $1\leqq i, j\leqq 3$, and

$$
a_1\times b_1=-b_3,\quad a_3\times a_1=b_2,\quad a_3\times b_1=-a_2,
$$

$$
a_2\times b_2=b_3,\quad a_1\times a_2=b_1\times b_2=-a_3\times b_3.
$$

For example, $(a_1, a_2, a_3, b_1, b_2, b_3)=(e_3, -e_2, e_5, -e_7, e_6, e_4)$ satisfies the relations in (5.2). We may easily check that $(\boldsymbol{R}^2, \psi)$ is a 2-dimensional almost complex submanifold of S^6.

The above immersion ψ induces an immersion $\Psi: T^2=\boldsymbol{R}^2/\Gamma\rightarrow S^6$ in the natural way, where Γ denotes the lattice group in $\boldsymbol{R}^2$ generated by $\left\{2\sqrt{2}\pi(1, 0), 2\sqrt{\frac{2}{3}}\pi(0, 1)\right\}$.

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NIIGATA UNIVERSITY
NIIGATA, JAPAN