Y. WATANABE
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KÄHLERIAN METRICS GIVEN BY CERTAIN SMOOTH POTENTIAL FUNCTIONS

BY YOSHIYUKI WATANABE

§1. Introduction.

As is well known, every Kählerian metric $ds^2=2\Sigma g_{\alpha\bar{\beta}}dz_\alpha d\bar{z}_\beta$ is locally expressible in the form

$$g_{\alpha\bar{\beta}}=\frac{\partial^2\phi}{\partial z_\alpha\partial\bar{z}_\beta},$$

with respect to local complex coordinates $\{z_\alpha\}$, $\alpha=1, \cdots, n$, where $\phi(z, \bar{z})$ is a real valued function of $\{z_\alpha, \bar{z}_\alpha\}$.

We now consider a Kählerian metric $g_{\alpha\bar{\beta}}$ with ϕ such that $\phi=f(t)$, $t=\Sigma z_\alpha\bar{z}_\alpha$, where $t\to f(t)\in C^\infty(R)$. S. S. Eum [1] studied such a Kählerian metric with non-zero constant holomorphic curvature defined in the complex number space C^n, and showed it is Fubinian, i.e.,

$$f(t)=\frac{1}{k}\log(kt+b)+c\,, \tag{1.1}$$

where k ($\neq 0$), b (>0) and c are constant. A Kählerian manifold with constant holomorphic curvature is harmonic (cf. S. Tachibana [7]). For this reason, the present author [12] studied Kählerian manifolds, which are harmonic. On the other hand, P. F. Klembeck [2] has shown that the complex space C^n admits a complete Kählerian metric h with components

$$h_{\alpha\bar{\beta}}=\frac{\partial^2 f(t)}{\partial z_\alpha\partial\bar{z}_\beta},\quad f(t)=\int_0^\infty\frac{1}{r}\log(1+r)dr\,, \tag{1.2}$$

which has strictly positive curvature. The Kählerian manifold (C^n, h) is harmonic at the origin 0 of C^n (cf. §3). But the scalar curvature is not constant as we can directly compute. Therefore (C^n, h) is not harmonic, because a harmonic Riemannian manifold is Einsteinian (cf. [4]). Thus we are very interested in a Kählerian metric locally expressed in the form

$$g_{\alpha\bar{\beta}}=\frac{\partial^2 f(t)}{\partial z_\alpha\partial\bar{z}_\beta},\quad t\to f(t)\in C^\infty(R)\,, \tag{1.3}$$

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with respect to local complex coordinates $\{z_\alpha\}$ (we can see another interesting example in [8]).

In the present paper, we shall study Kählerian metrics, which are locally expressible in the form (1.3) and satisfy one of the following conditions:

$$R=\text{constant}, \tag{A}$$

where R is the scalar curvature.

$$R_{ij;k;l}-R_{ij;l;k}=0\,, \tag{B}$$

where R_{ij} is the Ricci tensor and (;) denotes the covariant differentiation.

The main theorem is

THEOREM 1. *A Kählerian manifold* (D^n, g) *given by* (1.3) *is flat or Fubinian if it satisfies the condition* (A) *where* D^n *is* C^n *or its star-shaped subdomain at the origin of* C^n

Preliminary facts will be given in § 2 following to Yano-Bochner's notation (cf. [14]). In § 3, we show that the Kählerian metric given by (1.2) is harmonic at the origin 0 of C^n. In § 4, we shall prove that a Kählerian manifold (C^n, g) given by (1.3) is locally flat or Fubinian if it satisfy the condition (B). The last section is devoted to prove Theorem 1.

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§ 2. Pleliminaries.

We agree to adopt the summation convention and the following ranges of indices throughout the paper:

$$1\leqq i,\ j,\ k,\ \cdots\ \leqq 2n,$$

$$1\leqq \alpha,\ \beta,\ \gamma,\ \cdots\ \leqq n.$$

Consider an n complex dimensional Kählerian manifold with metric

$$ds^2=\Sigma g_{jk}dz_j dz_k\,, \tag{2.1}$$

where $\{z_\alpha\}$ are local complex coordinates and $\bar{z}_\alpha=z_{\bar{\alpha}}$ (=conjugate of z_α). As the metric is Kählerian, g_{jk} satisfy the following conditions:

$$g_{\alpha\beta}=g_{\bar{\alpha}\bar{\beta}}=0\,, \tag{2.2}$$

and (2.1) becomes

(2.3) $$ds^2=2\Sigma g_{\alpha\bar{\beta}}dz_\alpha d\bar{z}_\beta\,.$$

g^{jk} satisfy the corresponding equations to (2.2). The Christoffel symbols $\Gamma^i{}_{jk}$ vanish except

(2.4) $$\Gamma^\alpha_{\beta\gamma}=g^{\alpha\bar{\varepsilon}}\frac{\partial g_{\beta\bar{\varepsilon}}}{\partial z_\gamma}\,,$$

and their conjugates. As to the curvature tensor $R^i{}_{jkl}$, only the components of the form $R^\alpha{}_{\beta\gamma\bar{\delta}}$ and $R^\alpha{}_{\beta\bar{\gamma}\delta}$ and their conjugate can different from zero, and it holds that

(2.5) $$R^\alpha{}_{\beta\gamma\bar{\delta}}=\frac{\partial\Gamma^\alpha{}_{\beta\gamma}}{\partial\bar{z}_\delta}\,,$$

from which

(2.6) $$R_{\beta\bar{\gamma}}=R^\alpha{}_{\beta\bar{\gamma}\alpha}=-\frac{\partial\Gamma^\alpha_{\beta\alpha}}{\partial\bar{z}_\gamma}\,,\qquad R_{\beta\gamma}=R_{\bar{\beta}\bar{\gamma}}=0\,.$$

The scalar curvature $R=g^{jk}R_{jk}$ is $2g^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}$. A Kählerian manifold is called a space of constant holomorphic curvature if its curvature tensor satisfies

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}}=\frac{R}{2n(n+1)}\,(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}}+g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}})\,.$$

Let C^n be the complex number space with complex coordinates $\{z_\alpha\}$, and D^n be C^n or its star-shaped subdomain centered at the origin of C^n. Now we are going to compute the curvature tensor, the Ricci tensor and the scalar curvature from the Kählerian metric given by (1.3).

First from (1.3), we have

(2.7) $$g_{\alpha\bar{\beta}}=f'\delta_{\alpha\beta}+f''\bar{z}_\alpha z_\beta\,,$$

where dashes mean differentiation with respect to t. As the metric is positive definite, the function we consider should satisfy

(2.8) $$f'>0\,,\quad f'+tf''>0\,,$$

on D^n, because $g^{\alpha\bar{\beta}}$ are given by

(2.9) $$g^{\alpha\bar{\beta}}=\frac{1}{f'}\left(\delta_{\alpha\beta}-\frac{f''}{f'+tf''}z_\alpha\bar{z}_\beta\right).$$

From (2.4), we have

$$\Gamma^\alpha_{\beta\gamma}=\frac{f''}{f'}\,(\bar{z}_\beta\delta_{\alpha\gamma}+\bar{z}_\gamma\delta_{\alpha\beta})+\sigma z_\alpha\bar{z}_\beta\bar{z}_\gamma\,,$$

where

(2.10) $$\sigma(t)=\frac{f'f'''-2f''^2}{f'(f'+tf'')}.$$

(2.5) and some computations give the following equations (cf. [1], [8]):

(2.11) $$R^{\alpha}{}_{\beta\gamma\bar{\delta}}=\frac{f'f'''-f''^2}{f'^2}z_{\bar{\delta}}(\bar{z}_{\beta}\delta_{\alpha\gamma}+\bar{z}_{\gamma}\delta_{\alpha\beta})+\frac{f''}{f'}(\delta_{\beta\delta}\delta_{\alpha\gamma}+\delta_{\gamma\delta}\delta_{\alpha\beta})$$
$$+\sigma' z_{\alpha}\bar{z}_{\beta}\bar{z}_{\gamma}z_{\delta}+\sigma z_{\alpha}(\bar{z}_{\beta}\delta_{\gamma\delta}+\bar{z}_{\gamma}\delta_{\beta\delta})$$

and

(2.12) $$R_{\beta\bar{\delta}}=\mu\delta_{\beta\delta}+\lambda\bar{z}_{\beta}z_{\delta},$$

where λ and μ are functions defined by

(2.13) $$\lambda=-\frac{(n+1)(f'f'''-f''^2)}{f'^2}-\sigma' t-\sigma$$

and

(2.14) $$\mu=-\frac{(n+1)f''}{f'}-\sigma t.$$

As a direct consequence of (2.13) and (2.14) we get

(2.15) $$\mu'=\lambda,$$

which is remarkable. The scalar curvature R is given by

(2.16) $$R=\frac{2}{f'}\left\{\lambda t+n\mu-\frac{t(\lambda t+\mu)f''}{f'+tf''}\right\}.$$

§ 3. A Kählerian manifold being harmonic at only one point.

Our purpose of this section is to show that the Kählerian manifold (C^n, h) with metric h given by (1.2) is harmonic at the origin O of C^n. M. Itoh [3] has shown that in such a (C^n, h) every geodesic parametrized by arc length emanating from O is described by

(3.1) $$z_{\alpha}=A^{\alpha}\sinh(s),\quad \Sigma A^{\alpha}\bar{A}^{\alpha}=1\ (A^{\alpha}=\text{constant})$$

where s is the geodesic distance measured from O with respect to h. Then from (3.1), we have

(3.2) $$\Sigma z_{\alpha}\bar{z}_{\alpha}=\sinh^2 s.$$

Differentiating (3.2) by z_{β}, we have

(3.3) $$\bar{z}_{\alpha}=2\sinh(s)\cosh(s)\frac{\partial s}{\partial z_{\alpha}}$$
$$=\sinh(2s)\frac{\partial s}{\partial z_{\alpha}},$$

from which

$$h^{\alpha\bar\beta}\frac{\partial s}{\partial z_\alpha}\frac{\partial s}{\partial \bar z_\beta}=\frac{1}{4}, \tag{3.4}$$

because of

$$h^{\alpha\bar\beta}=\frac{\sinh^2 s}{\log(\cosh^2 s)}\delta_{\alpha\beta}-\left\{\frac{1}{\log(\cosh^2 s)}-\coth^2 s\right\}z_\alpha \bar z_\beta .$$

Differentiating (3.3) by $\bar z_\beta$, we have

$$\delta_{\alpha\beta}=2\cosh(2s)\frac{\partial s}{\partial \bar z_\beta}\frac{\partial s}{\partial z_\alpha}+\sinh(2s)\frac{\partial^2 s}{\partial z_\alpha \partial \bar z_\beta}.$$

Multiplying this equation by $h^{\alpha\bar\beta}$ and taking account of (3.4), we obtain

$$\frac{2(n-1)\sinh^2 s}{\log(\cosh^2 s)}=\cosh(2s)+\sinh(2s)\Delta s ,$$

since $\Delta s=2h^{\alpha\bar\beta}\dfrac{\partial^2 s}{\partial z_\alpha \partial \bar z_\beta}$. Therefore it follows that

$$\Delta s=\frac{(n-1)\tanh(s)}{2\log(\cosh(s))}-\coth(2s). \tag{3.5}$$

Let (M, g) be an n-dimensional Riemannian manifold and O a point of M. We denote by s the geodesic distance measured from O to the point in a neighborhood of O. If Δs is a function of s only, then (M, g) is called to be harmonic at the point O. When (M, g) is harmonic at any point, it is called a harmonic Riemannian manifold (cf. [4]).

As the right hand side of (3.5) is a function of s only, (C^n, h) is harmonic at the origin of C^n. However, (C^n, h) is not harmonic, because it is not locally flat or locally Fubinian (See Theorem 1).

§ 4. A kählerian metric satisfying the condition (B).

Let D^n be C^n or its star-shaped subdomain centered at the origin O of C^n. In this section, let (D^n, g) be a Kählerian manifold with metric g given by (1.3). Suppose that (D^n, g) satisfies the condition (B)[1]:

$$R_{ij;k;l}-R_{ij;l;k}=0. \tag{4.1}$$

Then by the Ricci's formula, we have

$$R_{hj}R^h{}_{ikl}+R_{ih}R^h{}_{jkl}=0,$$

1) This condition was studied by K. Sekigawa [5], K. Sekigawa and H. Takagi [6], H. Takagi [9], S. Tanno and other authors.

2) The author is grateful to Prof. S. Tanaka with whom the author had several conversations on differential equations in this section.

from which

$$R_{\alpha j}R^{\alpha}{}_{ikl}+R_{\bar{\alpha}j}R^{\bar{\alpha}}{}_{ikl}+R_{i\alpha}R^{\alpha}{}_{jkl}+R_{i\bar{\alpha}}R^{\bar{\alpha}}{}_{jkl}=0\,.$$

Thus (4.1) is equivalent to

$$R_{\alpha\bar{\lambda}}R^{\alpha}{}_{\beta\gamma\bar{\delta}}+R_{\beta\bar{\alpha}}R^{\bar{\alpha}}{}_{\bar{\lambda}\gamma\bar{\delta}}=0 \text{ (conj.)}, \tag{4.2}$$

by virtue of (2.5) and (2.6).

Now substituting (2.11) and (2.12) into the left hand side of (4.2), we can see that it reduces to the following (cf. Y. Watanabe [13], p. 79).

$$\left\{\frac{\lambda f''}{f'}+t\lambda\sigma+\mu\sigma-\frac{\mu(f'f'''-f''^2)}{f'^2}\right\}(z_\alpha\bar{z}_\gamma\delta_{\beta\delta}-\bar{z}_\beta z_\delta\delta_{\alpha\gamma})=0\,. \tag{4.2$'$}$$

Now we assume that $n\geqq 2$. Then we have

$$\frac{\lambda f''}{f'}+t\lambda\sigma+\mu\sigma-\frac{\mu(f'f'''-f''^2)}{f'^2}=0\,, \tag{4.3}$$

taking account of $f(t)\in C^\infty(R)$. (4.3) gives

$$\lambda f'f''+t\lambda\sigma f'^2+\mu\sigma f'^2-\mu(f'f'''-2f''^2)-\mu f''^2=0\,,$$

which implies

$$\lambda f'f''+t\lambda\sigma f'^2-t\mu\sigma f'f''-\mu f''^2=0\,,$$

because of (2.10). Thus we obtain

$$(f''+t\sigma f')(\lambda f'-\mu f'')=0\,. \tag{4.4}$$

In the followings, we consider two cases, i. e., Case I where $f''+tf'\sigma=0$ and Case II where $\lambda f'-\mu f''=0$.

Case I: We assume that $f''+t\sigma f'=0$ in an open subdomain Δ_1^n of D^n. Taking account of (2.10), we have

$$\frac{tf'(f'f'''-2f''^2)}{f''(f'+tf'')}+f''=0\,,$$

from which

$$tf'f'''+f'f''-tf''^2=0\,. \tag{4.5}$$

Putting $u=f'$ in (4.5), we have

$$tuu''+uu'-tu'^2=0\,. \tag{4.6}$$

The general solution of (4.6) is given by

$$u=bt^a\,,$$

where a and b are integral constants. Therefore the general solution of (4.5) is given by

(4.7) $$f=\frac{b}{a+1}t^{a+1}+c\,,$$

where a, b and c are integral constants. From (2.8) it is easily seen that $a=0$ and $b>0$ where $\Delta_1^n \ni O$. In this case the corresponding Kählerian metric g is flat, i.e.,

(4.8) $$f=bt+c \qquad (b>0)$$

when $\Delta_1^n \ni O$.

Case II: Suppose that

(4.9) $$\lambda f'-\mu f''=0$$

holds in a subdomain Δ_2^n of D^n. First by (2.7) and (2.12) we see that the potential function satisfying (4.9) gives an Einsteinian metric. By (2.13) and (2.14), we have

$$f''\{(n+1)f''+t\sigma f'\}=(n+1)(f'f'''-f''^2)+t\sigma' f'^2+\sigma f'^2\,,$$

from which

(4.10) $$(n+1)(2f''^2-f'f''')=(t\sigma'+\sigma)f'^2-t\sigma f'f''\,.$$

Taking account of (2.10), we have

(4.11) $$tf'\sigma'+\{(n+2)f'+ntf''\}\sigma=0\,.$$

If Δ_2^n contains the origin O, putting $t=0$ in (4.11) we have

(4.12) $$\sigma(0)=0\,,$$

because of $f'(0)>0$.

If $t>0$, then multiplying (4.11) by $t^{n+1}(f')^{n-1}$, we have

$$t^{n+2}(f')^n\sigma'+(n+2)t^{n+1}(f')^n\sigma+nt^{n+2}(f')^{n-1}f''\sigma=0\,,$$

from which, integrating

(4.13) $$t^{n+2}(f')^n\sigma=c\,,$$

where c is an integral constant. Since the function f satisfies (4.9), by the argument of continuity of the left hand side of (4.12), we can conclude that $c=0$ if Δ_2^n contains the origin O. Thus we have

(4.14) $$\sigma(t)=0\,,$$

together with (4.12) if $\Delta_2^n \ni O$. Moreover since

$$\left(\frac{f''}{f'^2}\right)'=\frac{f'f'''-2f''^2}{f'^3}\,,$$

(4.14) has two solutions: One is $f_1=at+b$ where $a(>0)$ and b are constant and the other $f_2=\frac{1}{k}\log(kt+c)+d$ where $k(\neq 0)$ and $c(>0)$ and d are constants, that is, when $O\in\Delta_2^n$,

$$f_1=at+b \quad (a>0), \tag{4.14)'}$$

or

$$f_2=\frac{1}{k}\log(kt+c)+d \quad (k\neq 0,\ c>0). \tag{4.14)''}$$

Note that in the case of (4.14)″ the corresponding Kählerian manifold (Δ_2^n, g) is of constant holomorphic curvature k (see S. S. Eum [1]).

Finally the function f given by (4.8) does not satisfy (4.13) with $c\neq 0$ and can be smoothly connected only the solution $f=bt+c(b>0)$ of (4.14)′. Conversely the solution $f=\frac{1}{k}\log(kt+c)+d$ $(k\neq 0,\ c>0)$ of (4.14)″ does not satisfy (4.7).

Thus we obtain the following

THEOREM 2. *Let D^n be C^n $(n\geqq 2)$ or its star-shaped subdomain containing the origin O of C^n. Suppose that the Kählerian metric given by* (1.3) *satisfies the condition* (4.1). *Then it is flat or Fubinian.*

5. Proof of Main Theorem.

By assumption R is constant in (2.16). Multiplying (2.16) by $f'(f'+tf'')$, we have

$$\frac{R}{2}f'(f'+tf'')=t\lambda f'+nf'\mu+(n-1)tf''\mu\,. \tag{5.1}$$

Putting $t=0$ in (5.1), we have

$$\frac{R}{2}f'(0)-n\mu(0)=0\,, \tag{5.2}$$

because of $f'(0)>0$. For $t>0$, we multiply (5.1) by $t^{n-1}(f')^{n-2}$. Then taking account of (2.15), we have

$$\frac{R}{2}\{(f')^n t^{n-1}+t^n(f')^{n-1}f''\}$$
$$=t^n(f')^{n-1}\mu'+nt^{n-1}(f')^n\mu+(n-1)t^n(f')^{n-1}f''\mu\,,$$

from which

$$\frac{R}{2n}t^n(f')^n=t^n(f')^{n-1}\mu+C\,,$$

where C is an integral constant. Then we have

(5.3) $$t^n(f')^{n-1}\left\{\frac{R}{2n}f'-\mu\right\}=C.$$

But we can see that $C=0$, taking limit of the left hand side of (5.3) of t as t tends to 0. Therefore we have

(5.5) $$\frac{R}{2n}f'^2(f'+tf'')+(n+1)f'f''+tf'f'''+(n-1)tf''^2=0,$$

Putting $t=0$ in (5.5), we have

(5.6) $$f''(0)+\frac{R}{2n(n+1)}f'(0)^2=0.$$

Multiplying (5.5) by $(f')^{n-2}t^n$, we have

$$\frac{R}{2n}\{(f')^{n+1}t^n+(f')^n t^{n+1}f''\}+(n+1)(f')^{n-1}t^n f'' \\ +t^{n+1}(f')^{n-1}f'''+(n-1)t^n(f')^{n-2}f''^2=0,$$

from which

(5.7) $$t^{n+1}(f')^{n-1}\left\{f''+\frac{R}{2n(n+1)}f'^2\right\}=\tilde{C},$$

where $\tilde{C}$ is an integral constant. But we have

$$\tilde{C}=0$$

taking limit of the left hand side of (5.7) of t as t tends to 0. Thus we have

(5.8) $$f''+\frac{R}{2n(n+1)}f'^2=0.$$

together with (5.6).

Now if $R=0$ in (5.8), then $f''=0$, i.e., $f'=$constant. Hence it follows that the corresponding Kählerian metric is flat. If $R\neq 0$ in (5.8), then the corresponding Kählerian metric is Fubinian and the holomorphic curvature is $\frac{R}{2n(n+1)}$ (cf. (2.11)) because of an elementary calculation. This proves the theorem.

By Theorem 1, we have the following

COROLLARY 3. *Let (M, g) be a complete Kählerian manifold. Suppose that g is expressed in the form* (1.3) *and its scalar curvature R is non-positive constant. Then (M, g) is the unitary space C^n or the complex hyperbolic space H^n according as $R=0$ or $R<0$.*

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DEPARTMENT OF MATHEMATICS
TOYAMA UNIVERSITY
TOYAMA, JAPAN