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KODAI MATH. J.
**42** (2019), 130–159

# THE GAMMA FILTRATIONS OF $K$-THEORY OF COMPLETE FLAG VARIETIES

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**Abstract**

Let $G$ be a compact Lie group and $T$ its maximal torus. In this paper, we try to compute $gr_{\gamma}^{*}(G/T)$ the graded ring associated with the gamma filtration of the complex $K$-theory $K^{0}(G/T)$. We use the Chow rings of corresponding versal flag varieties.

## 1. Introduction

Let $p$ be a prime number. Let $K_{top}^{*}(X)$ be the complex $K$-theory localized at $p$ for a topological space $X$. There are two typical filtrations for $K_{top}^{0}(X)$, the topological filtration defined by Atiyah [2] and the $\gamma$-filtration defined by Grothendieck [5]. Let us write by $gr_{top}^{*}(X)$ and $gr_{\gamma}^{*}(X)$ the associated graded rings for these filtrations. (Also see Chapter 15 in the book [33] by Totaro.) Let $G$ and $T$ be a connected compact Lie group and its maximal torus. Then $gr_{top}^{*}(G/T) \cong H^{*}(G/T)_{(p)}$. However when $H^{*}(G)$ has $p$-torsion, it is not isomorphic to $gr_{\gamma}^{*}(G/T)$. In this paper, we try to compute $gr_{\gamma}^{*}(G/T)$.

To study the above (topological) $\gamma$-filtration, we use the corresponding algebraic $\gamma$-filtration. Given a field $k$ with $ch(k) = 0$, let $G_k$ and $T_k$ be a split reductive group and a split maximal torus over the field $k$, corresponding to $G$ and $T$. Let $B_k$ be the Borel subgroup containing $T_k$. Let $\mathbf{G}$ be a $G_k$-torsor. Then $\mathbf{F} = \mathbf{G}/B_k$ is a twisted form of the flag variety $G_k/B_k$. Hence we can consider the $\gamma$-filtration and its graded ring $gr_{\gamma}^{*}(\mathbf{F})$ for algebraic $K$-theory $K_{alg}^{0}(\mathbf{F})$ over $k$.

Let us write $\bar{\mathbf{F}} = \mathbf{F} \otimes \bar{k}$ for the algebraic closure $\bar{k}$ of $k$. It is well known from Chevalley that the restriction $res_K : K_{alg}^{0}(\mathbf{F}) \to K_{alg}^{0}(\bar{\mathbf{F}})$ is surjective when $G$ is simply connected. Panin [20] shows that $K_{alg}^{0}(\mathbf{F})$ is torsion free, which implies $res_K$ is injective. Hence when $G$ is simply connected, we see

$$K_{alg}^{0}(\mathbf{F}) \cong K_{alg}^{0}(\bar{\mathbf{F}}) \cong K_{top}^{0}(G/T).$$

2010 *Mathematics Subject Classification*. 57T15, 20G15, 14C15.
*Key words and phrases*. Gamma filtration, Chow ring, versal torsor, complete flag variety.
Received February 5, 2018; revised July 9, 2018.

Moreover we take $k$ such that there is a $G_k$-torsor $\mathbf{G}$ which is isomorphic to a versal $G_k$-torsor (for the definition of a versal $G_k$-torsor, see §3 below or see [3], [32], [14], [8]). Then $\mathbf{F} = \mathbf{G}/B_k$ is thought as the *most* twisted complete flag variety, and we say it the versal flag variety. In particular, its Chow ring $CH^*(\mathbf{F})$ is generated by Chern classes ([14], [8]). Hence we have;

Theorem 1.1. *Let $G$ be a compact simply connected Lie group. Let $\mathbf{F} = \mathbf{G}/B_k$ be the versal flag variety. Then*

$$gr_\gamma^*(G/T) \cong gr_\gamma^*(\mathbf{F}) \cong CH^*(\mathbf{F})/I \quad \textit{for some ideal } I \subset CH^*(\mathbf{F}).$$

*Remark.* Karpenko conjectures that the above $I = 0$ ([8], [9]).

Note that $gr_\gamma^*(G/T)$ is the associated ring of the topological $K$-theory $K_{top}^0(G/T)$ but $gr_\gamma^*(\mathbf{F})$ is that of the algebraic $K$-theory $K_{alg}^0(\mathbf{F})$. Hence the purely topological object $gr_\gamma^*(G/T)$ can be computed by a purely algebraic geometric object, the Chow ring of a twisted flag variety $\mathbf{F}$.

Let $BT$ be the classifying space of $T$. We consider the fiber sequence $G \to G/T \xrightarrow{i} BT$. The filtration defined from $K_{top}^*(i^{-1}(BT^j))$ of $K_{top}^*(G/T)$ for the $j$-th skeleton $BT^j$ of $BT$ gives the following the (modified Atiyah-Hirzebruch) spectral sequence

$$E_2^{*,*'} \cong H^*(BT; K_{top}^{*'}(G)) \Rightarrow K_{top}^*(G/T).$$

From the preceding theorem, we can see;

Corollary 1.2. *We have $E_\infty^{*,0} \cong gr_\gamma^*(G/T)$.*

However, the above spectral sequence itself seems to be difficult to compute. In this paper, the proof of Theorem 1.1 is also given by the computation of each simple Lie group. For example, in the following cases, $gr_\gamma(G/T)/p \cong gr_\gamma(\mathbf{F})/p$ can be computed (see also [44]).

Theorem 1.3. *Let $(G,p)$ be the following simply connected simple group with $p$ torsion in $H^*(G)$. Let $(G,p) = (Spin(n),2)$ for $7 \le n \le 10$, or an exceptional Lie group except for $(E_7,2)$ and $(E_8,2,3)$. Then there are elements $b_s$ in $S(t) = \mathbf{Z}[t_1,\dots,t_\ell] \cong H^*(BT)$ for $1 \le s \le \ell$ such that $gr_\gamma(G/T) \cong CH^*(\mathbf{F})$, and*

$$gr_\gamma(G/T)/p \cong S(t)/(p, b_ib_j, b_k \mid 1 \le i,j \le 2(p-1) < k \le \ell).$$

Theorem 1.4. *Let $(G,p) = (Spin(11),2)$. Then we can take $c_1$, $c_i'$ in $S(t) = H^*(BT) \cong \mathbf{Z}[t_1,\dots,t_5]$ for $2 \le i \le 5$ with $|c_1| = 2$, $|c_i'| = 2i$ such that*

$$gr_\gamma(G/T)/2 \cong S(t)/(2, c_i'c_j', c_1^8c_i', c_1^{16} \mid 2 \le i \le j \le 5,\ (i,j) \ne (2,4)).$$

*Remark.* Quite recently, Karpenko proves ([9]) that for $G = Spin(11)$ and $G = Spin(12)$, we have $gr_\gamma(\mathbf{F}) \cong CH^*(\mathbf{F})$. Hence the above ring is isomorphic to $CH^*(\mathbf{F})/2$.

The plan of this paper is the following. In §2, we recall and prepare the topological arguments for $H^*(G/T)$, $K(1)^*(G/T)$ and $BP^*(G/T)$. In §3, we recall the decomposition of the motive of a versal flag variety and recall the torsion index. In §4, §5, we study the graded rings of the $K$-theory (and the Morava $K(1)$-theory). In §6, §7, we study the cases $SO(m)$ and $Spin(m)$ for $p=2$. In §8–§11, we study the cases that $(G,p)$ are in Theorem 1.3 (e.g., $(E_8,5)$), $(E_8,3)$, $(E_7,2)$ and $(E_8,2)$ respectively.

The author thanks the referee who corrected errors in the first version of this paper.

## 2. Lie groups $G$ and the flag manifolds $G/T$

Let $G$ be a connected compact Lie group. By Borel, its $mod(p)$ cohomology is (for $p$ odd)

$$(2.1)\qquad H^*(G;\mathbf{Z}/p)\cong P(y)/p\otimes\Lambda(x_1,\dots,x_\ell),\quad \ell=rank(G)$$
$$with\ \ P(y)=\mathbf{Z}_{(p)}[y_1,\dots,y_s]/(y_1^{p^{r_1}},\dots,y_s^{p^{r_s}})$$

where the degree $|y_i|$ of $y_i$ is even and $|x_j|$ is odd. When $p=2$, a graded ring $grH^*(G;\mathbf{Z}/2)$ is isomorphic to the right hand side ring, e.g. $x_j^2=y_{i_j}$ for some $y_{i_j}$. In this paper, $H^*(G;\mathbf{Z}/2)$ means this $grH^*(G;\mathbf{Z}/2)$ so that (2.1) is satisfied also for $p=2$.

Let $T$ be the maximal torus of $G$ and $BT$ be the classifying space of $T$. We consider the fibering ([29], [16]) $G\xrightarrow{\pi}G/T\xrightarrow{i}BT$ and the induced spectral sequence

$$E_2^{*,*'}=H^*(BT;H^{*'}(G;\mathbf{Z}/p))\Rightarrow H^*(G/T;\mathbf{Z}/p).$$

The cohomology of the classifying space of the torus is given by $H^*(BT)\cong S(t)=\mathbf{Z}[t_1,\dots,t_\ell]$ with $|t_i|=2$, where $t_i=pr_i^*(c_1)$ is the 1-st Chern class induced from $T=S^1\times\cdots\times S^1\xrightarrow{pr_i}S^1\subset U(1)$ for the $i$-th projection $pr_i$. Note that $\ell=rank(G)$ is also the number of the odd degree generators $x_i$ in $H^*(G;\mathbf{Z}/p)$.

It is well known that $y_i$ are permanent cycles and that there is a regular sequence ([29], [16]) $(\bar{b}_1,\dots,\bar{b}_\ell)$ in $H^*(BT)/(p)$ such that $d_{|x_i|+1}(x_i)=\bar{b}_i$ (this $\bar{b}_i$ is called the transgressive element). Thus we get

$$E_\infty^{*,*'}\cong grH^*(G/T;\mathbf{Z}/p)\cong P(y)/p\otimes S(t)/(\bar{b}_1,\dots,\bar{b}_\ell).$$

Moreover we know that $G/T$ is a complex manifold such that $H^*(G/T)$ is torsion free, and

$$(2.2)\qquad H^*(G/T)_{(p)}\cong\mathbf{Z}_{(p)}[y_1,\dots,y_s]\otimes S(t)/(f_1,\dots,f_s,b_1,\dots,b_\ell)$$

where $b_i=\bar{b}\ mod(p)$ and $f_i=y_i^{p^{r_i}}\ mod(t_1,\dots,t_\ell)$.

Let $BP^*(-)$ be the Brown-Peterson theory with the coefficients ring $BP^*\cong\mathbf{Z}_{(p)}[v_1,v_2,\dots]$, $|v_i|=-2(p^i-1)$ ([6], [24]). Since $H^*(G/T)$ is torsion free, the

Atiyah-Hirzebruch spectral sequence (AHss) collapses. Hence we also know

(2.3) $$BP^*(G/T) \cong BP^*[y_1, \ldots, y_s] \otimes S(t)/(\tilde{f}_1, \ldots, \tilde{f}_s, \tilde{b}_1, \ldots, \tilde{b}_\ell)$$

where $\tilde{b}_i = b_i \, mod(BP^{<0})$ and $\tilde{f}_i = f_i \, mod(BP^{<0})$.

Recall the Morava $K$-theory $K(n)^*(X)$ with the coefficient ring $K(n)^* = \mathbf{Z}/p[v_n, v_n^{-1}]$. Similarly we can define connected or integral Morava K-theories with

$$k(n)^* = \mathbf{Z}/p[v_n], \quad \tilde{k}(n)^* = \mathbf{Z}_{(p)}[v_n], \quad \tilde{K}(n)^* = \mathbf{Z}_{(p)}[v_n, v_n^{-1}].$$

It is known (as the Conner-Floyd type theorem)

$$\tilde{K}(1)^*(X) \cong BP^*(X) \otimes_{BP^*} \tilde{K}(1)^*.$$

The above fact does not hold for $K(n)$-theory with $n \geq 2$.

Here we consider the connected Morava K-theory $k(n)^*(X)$ (such that $K(n)^*(X) \cong k(n)^*(X)[v_n^{-1}]$) and the Thom natural homomorphism $\rho : k(n)^*(X) \to H^*(X; \mathbf{Z}/p)$. Recall that there is an exact sequence (Sullivan exact sequence [24], [40])

$$\cdots \to k(n)^{*+2(p^n-1)}(X) \xrightarrow{v_n} k(n)^*(X) \xrightarrow{\rho} H^*(X; \mathbf{Z}/p) \xrightarrow{\delta} \cdots$$

such that $\rho \cdot \delta(X) = Q_n(X)$. Here the Milnor $Q_i$ operation

$$Q_i : H^*(X; \mathbf{Z}/p) \to H^{*+2p^i-1}(X; \mathbf{Z}/p)$$

is defined by $Q_0 = \beta$ and $Q_{i+1} = P^{p^i} Q_i - Q_i P^{p^i}$ for the Bockstein operation $\beta$ and the reduced power operation $P^j$.

We consider the Serre spectral sequence

$$E_2^{*,*'} \cong H^*(B; H^*(F; \mathbf{Z}/p)) \Rightarrow H^*(E; \mathbf{Z}/p).$$

induced from the fibering $F \xrightarrow{i} E \xrightarrow{\pi} B$ with $H^*(B) \cong H^{even}(B)$. By using the Sullivan exact sequence, we can prove;

Lemma 2.1 (Lemma 4.3 in [40]). *In the spectral sequence $E_r^{*,*'}$ above, suppose that there is $x \in H^*(F; \mathbf{Z}/p)$ such that*

($*$) $$y = Q_n(x) \neq 0 \quad \text{and} \quad b = d_{|x|+1}(x) \neq 0 \in E_{|x|+1}^{*,0}.$$

*Moreover suppose that $E_{|x|+1}^{0,|x|} \cong \mathbf{Z}/p\{x\} \cong \mathbf{Z}/p$. Then there are $y' \in k(n)^*(E)$ and $b' \in k(n)^*(B)$ such that $i^*(y') = y$, $\rho(b') = b$ and that*

($**$) $$v_n y' = \lambda \pi^*(b') \quad \text{in } k(n)^*(E), \quad \text{for } \lambda \neq 0 \in \mathbf{Z}/p.$$

*Conversely if ($**$) holds in $k(n)^*(E)$ for $y = i^*(y') \neq 0$ and $b = \rho(b') \neq 0$, then there is $x \in H^*(F; \mathbf{Z}/p)$ such that ($*$) holds.*

*Remark* (Remark 4.8 in [40]). The above lemma also holds letting $k(0)^*(X) = H^*(X; \mathbf{Z}_{(p)})$ and $v_0 = p$. This fact is well known (Lemma 2.1 in [29]).

Corollary 2.2. *Let $b \neq 0$ be the transgressive image of $x$, i.e. $d_{|x|+1}(x) = b \in H^*(G/T)/p$. Then there is a lift $b \in BP^*(BT) \cong BP^* \otimes S(t)$ of $b \in S(t)/p$ such that*

$$b = \sum_{i=0} v_i y(i) \in BP^*(G/T)/I_\infty^2 \quad (i.e.,\ b = v_i y(i) \in k(i)^*(G/T)/(v_i^2))$$

*where $y(i) \in H^*(G/T; \mathbf{Z}/p)$ with $\pi^* y(i) = Q_i x$.*

## 3. Versal flag varieties

Let $G_k$ be the split reductive algebraic group corresponding to $G$, and $T_k$ be the split maximal torus corresponding to $T$. Let $B_k$ be the Borel subgroup with $T_k \subset B_k$. Note that $G_k/B_k$ is cellular, and $CH^*(G_k/T_k) \cong CH^*(G_k/B_k)$. Hence we have

$$CH^*(G_k/B_k) \cong H^*(G/T) \quad and \quad CH^*(BB_k) \cong H^*(BT).$$

Let us write by $\Omega^*(X)$ the $BP$-version of the algebraic cobordism defined by Levine-Morel ([12], [13], [40], [42]) such that

$$\Omega^*(X) = MGL^{2*,*}(X)_{(p)} \otimes_{MU^*_{(p)}} BP^*, \quad \Omega^*(X) \otimes_{BP^*} \mathbf{Z}_{(p)} \cong CH^*(X)_{(p)}$$

where $MGL^{*,*'}(X)$ is the algebraic cobordism theory defined by Voevodsky with $MGL^{2*,*}(pt.) \cong MU^*$ the complex cobordism ring. There is a natural (realization) map $\Omega^*(X) \to BP^*(X(\mathbf{C}))$. In particular, we have $\Omega^*(G_k/B_k) \cong BP^*(G/T)$. Let $I_n = (p, v_1, \dots, v_{n-1})$ and $I_\infty = (p, v_1, \dots)$ be the (prime invariant) ideals in $BP^*$. We also note

$$\Omega^*(G_k/B_k)/I_\infty \cong BP^*(G/T)/I_\infty \cong H^*(G/T)/p.$$

Let $\mathbf{G}$ be a nontrivial $G_k$-torsor. We can construct a twisted form of $G_k/B_k$ by $(\mathbf{G} \times G_k/B_k)/G_k \cong \mathbf{G}/B_k$. We will study the twisted flag variety $\mathbf{F} = \mathbf{G}/B_k$.

By extending the arguments by Vishik [34] for quadrics to that for flag varieties, Petrov, Semenov and Zainoulline define the $J$-invariant of $\mathbf{G}$. Recall the expression (2.1) in §2

$$(*) \qquad H^*(G; \mathbf{Z}/p) \cong \mathbf{Z}/p[y_1, \dots, y_s]/(y_1^{p^{r_1}}, \dots, y_s^{p^{r_s}}) \otimes \Lambda(x_1, \dots, x_\ell).$$

Roughly speaking (for the detailed definition, see [23]), the $J$-invariant is defined as $J_p(\mathbf{G}) = (j_1, \dots, j_s)$ if $j_i$ is the minimal integer such that

$$y_i^{p^{j_i}} \in Im(res_{CH}) \quad mod(y_1, \dots, y_{i-1}, t_1, \dots, t_\ell)$$

for $res_{CH} : CH^*(\mathbf{F}) \to CH^*(\bar{\mathbf{F}})$. Here we take $|y_1| \leq |y_2| \leq \cdots$ in $(*)$. Hence $0 \leq j_i \leq r_i$ and $J_p(\mathbf{G}) = (0, \dots, 0)$ if and only if $\mathbf{G}$ split by an extension of the index coprime to $p$. One of the main results in [23] is

Theorem 3.1 (Theorem 5.13 in [23] and Theorem 4.3 in [28]). *Let $\mathbf{G}$ be a $G_k$-torsor over $k$, $\mathbf{F} = \mathbf{G}/B_k$ and $J_p(\mathbf{G}) = (j_1, \dots, j_s)$. Then there is a $p$-localized* *motive* $R(\mathbf{G})$ *such that the motive* $M(\mathbf{F})$ *of the variety* $\mathbf{F}$ *is decomposed as motives* (*in the category of Chow motives*)

$$M(\mathbf{F})_{(p)} \cong \oplus_u R(\mathbf{G}) \otimes \mathbf{T}^{\otimes u}.$$

*Here* $\mathbf{T}^{\otimes u}$ *are Tate motives with* $CH^*(\oplus_u \mathbf{T}^{\otimes u})/p \cong P'(y) \otimes S(t)/(b)$ *where*

$$P'(y) = \mathbf{Z}/p[y_1^{p^{j_1}}, \dots, y_s^{p^{j_s}}]/(y_1^{p^{r_1}}, \dots, y_s^{p^{r_s}}) \subset P(y)/p,$$
$$S(t)/(b) = S(t)/(b_1, \dots, b_\ell).$$

*The* $mod(p)$ *Chow group of* $\bar{R}(\mathbf{G}) = R(\mathbf{G}) \otimes \bar{k}$ *is given by*

$$CH^*(\bar{R}(\mathbf{G}))/p \cong \mathbf{Z}/p[y_1, \dots, y_s]/(y_1^{p^{j_1}}, \dots, y_s^{p^{j_s}}).$$

*Hence we have* $CH^*(\bar{\mathbf{F}})/p \cong CH^*(\bar{R}(\mathbf{G})) \otimes P'(y) \otimes S(t)/(b)$ *and*

$$CH^*(\mathbf{F})/p \cong CH^*(R(\mathbf{G})) \otimes P'(y) \otimes S(t)/(b).$$

*Remark.* In this paper, a map $A \to B$ (resp. $A \cong B$) for rings $A$, $B$ means a ring map (resp. a ring isomorphism). However $CH^*(R(\mathbf{G}))$ does not have a canonical ring structure. Hence a map $A \to CH^*(R(\mathbf{G}))$ (resp. $A \cong CH^*(R(\mathbf{G}))$) means only a (graded) additive map (resp. additive isomorphism). Hence all (except for the first) isomorphisms in the above theorem are those of only graded $\mathbf{Z}/p$-modules.

Let us consider an embedding of $G_k$ into the general linear group $GL_N$ for some $N$. This makes $GL_N$ a $G_k$-torsor over the quotient variety $S = GL_N/G_k$. Let $F$ be the function field $k(S)$ and define the *versal* $G_k$*-torsor* $E$ to be the $G_k$-torsor over $F$ given by the generic fiber of $GL_N \to S$. (For details, see [3], [32], [14], [8].)

$$\begin{array}{ccc} E & \longrightarrow & GL_N \\ \downarrow & & \downarrow \\ Spec(k(S)) & \longrightarrow & S = GL_N/G_k \end{array}$$

The corresponding flag variety $E/B_k$ is called the *versal* complete flag variety, which is considered as the most complicated twisted complete flag variety (for given $G_k$). It is known that the Chow ring $CH^*(E/B_k)$ is not dependent to the choice of generic $G_k$-torsors $E$ (Remark 2.3 in [8]).

Karpenko and Merkurjev showed the following result for a versal flag variety.

Theorem 3.2 (Karpenko Lemma 2.1 in [8]). *Let* $h^*(X)$ *be an oriented cohomology theory* (*e.g.*, $CH^*(X)$, $\Omega^*(X)$). *Let* $\mathbf{G}/B_k$ *be a versal flag variety. Then the natural map* $h^*(BB_k) \to h^*(\mathbf{G}/B_k)$ *is surjective.*

Corollary 3.3. *If* $\mathbf{G}$ *is versal, then* $CH^*(\mathbf{F}) = CH^*(\mathbf{G}/B_k)$ *is multiplicatively generated by elements* $t_i$ *in* $S(t)$.

Corollary 3.4. *If* $\mathbf{G}$ *is versal, then* $J(\mathbf{G}) = (r_1, \dots, r_s)$, *i.e.* $r_i = j_i$. *Hence* $P'(y) = \mathbf{Z}/p$ *in Theorem* 3.1.

*Proof.* If $j_i < r_i$, then $0 \neq y_i^{p^{j_i}} \in res(CH^*(\mathbf{F}) \to CH^*(\bar{\mathbf{F}}))$, which is in the image from $S(t)$ by the preceding theorem. This contradicts to $CH^*(\bar{\mathbf{F}})/p \cong P(y)/p \otimes S(t)/(b)$ and $0 \neq y_i^{p^{j_i}} \in P(y)/p$. □

Hence we have surjections for a versal variety $\mathbf{F}$

$$CH^*(BB_k) \twoheadrightarrow CH^*(\mathbf{F}) \stackrel{pr.}{\twoheadrightarrow} CH^*(R(\mathbf{G})).$$

We study in [44] what elements in $CH^*(BB_k)$ generate $CH^*(R(\mathbf{G}))$. By giving the filtration on $S(t)$ by $b_i$, we can write

$$grS(t)/p \cong A \otimes S(t)/(b_1, \dots, b_\ell) \quad for \ A = \mathbf{Z}/p[b_1, \dots, b_\ell].$$

In particular, we have maps $A \stackrel{i_A}{\to} CH^*(\mathbf{F})/p \to CH^*(R(\mathbf{G}))/p$. We also see that the above composition map is surjective (see also Lemma 3.7 below).

Lemma 3.5. *Suppose that there are* $f_1(b), \dots, f_s(b) \in A$ *such that* $CH^*(R(\mathbf{G}))/p \cong A/(f_1(b), \dots, f_s(b))$ *additively. Moreover if* $f_i(b) = 0$ *for* $1 \leq i \leq s$ *also in* $CH^*(\mathbf{F})/p$, *then we have the ring isomorphism*

$$CH^*(\mathbf{F})/p \cong S(t)/(p, f_1(b), \dots, f_s(b)).$$

*Proof.* Using $(f_1, \dots, f_s) \subset (b) = (b_1, \dots, b_\ell)$, we have additively

$$S(t)/(f_1, \dots, f_s) \cong (A \otimes S(t)/(b))/(f_1, \dots, f_s) \cong A/(f_1, \dots, f_s) \otimes S(t)/(b)$$
$$\cong CH^*(R(\mathbf{G}))/p \otimes S(t)/(b) \cong CH^*(\mathbf{F})/p.$$

Of course there is a ring surjective map $S(t)/(f_1, \dots, f_s) \to CH^*(\mathbf{F})/p$, this map must be isomorphic. □

Let $dim(G/T) = 2d$. Then the torsion index is defined as

$$t(G) = |H^{2d}(G/T; \mathbf{Z})/i^*H^{2d}(BT; \mathbf{Z})| \quad where \ i : G/T \to BT.$$

Let $n(\mathbf{G})$ be the greatest common divisor of the degrees of all finite field extension $k'$ of $k$ such that $\mathbf{G}$ becomes trivial over $k'$. Then by Grothendieck [5], it is known that $n(\mathbf{G})$ divides $t(G)$. Moreover, $\mathbf{G}$ is versal, then $n(\mathbf{G}) = t(G)$ ([32], [14], [8]). Note that $t(G_1 \times G_2) = t(G_1)t(G_2)$. It is well known that if $H^*(G)$ has a $p$-torsion, then $p$ divides the torsion index $t(G)$. Torsion index for simply connected compact Lie groups are completely determined by Totaro [31], [32].

For $N > 0$, let us write $A_N = \mathbf{Z}/p\{b_{i_1} \cdots b_{i_k} \mid |b_{i_1}| + \cdots + |b_{i_k}| \leq N\}$.

Lemma 3.6. *Let* $b \in Ker(pr)$ *for* $pr : A_{2d} \twoheadrightarrow CH^*(R(\mathbf{G}))$. *Then*

$$b = \sum b'u' \quad with \ b' \in A_{2d}, \ u' \in S(t)^+/(p, b), \ i.e., \ |u'| > 0.$$

Let us write

$$y_{top} = \Pi_{i=1}^{s} y_i^{p^{r_i}-1} \quad (reps.\ t_{top})$$

the generator of the highest degree in $P(y)$ (resp. $S(t)/(b)$) so that $f = y_{top}t_{top}$ is the fundamental class in $H^{2d}(G/T)$.

LEMMA 3.7. *The following map is surjective*

$$A_N \twoheadrightarrow CH^*(R(\mathbf{G}))/p \quad where\ N = |y_{top}|.$$

*Proof.* In the preceding lemma, $A_N \otimes u$ for $|u| > 0$ maps zero in $CH^*(R(\mathbf{G}))/p$. Since each element in $S(t)$ is written by an element in $A_N \otimes S(t)/(b)$, we have the lemma. □

COROLLARY 3.8. *If $b_i \neq 0$ in $CH^*(X)/p$, then so in $CH^*(R(\mathbf{G}_k))/p$.*

*Proof.* Let $pr(b_i) = 0$. From Lemma 3.6, $b_i = \sum b'u'$ for $|u'| > 0$, and hence $b' \in Ideal(b_1, \dots, b_{i-1})$. This contradict to that $(b_1, \dots, b_\ell)$ is regular. □

Now we consider the torsion index $t(G)$.

LEMMA 3.9 ([44]). *Let $\tilde{b} = b_{i_1} \cdots b_{i_k}$ in $S(t)$ such that in $H^*(G/T)_{(p)}$*

$$\tilde{b} = p^s\Big(y_{top} + \sum yt\Big), \quad |t| > 0$$

*for some $y \in P(y)$ and $t \in S(t)^+$. Then $t(G)_{(p)} \leq p^s$.*

## 4. Filtrations of $K$-theories

We first recall the topological filtration defined by Atiyah. Let $X$ be a topological space. Let $K_{top}^*(X)$ be the complex $K$-theory so that $K_{top}^0(X)$ is the Grothendieck group generated by complex bundles over $X$. Let $X^i$ be an $i$-dimensional skeleton of $X$. Define the topological filtration of $K_{top}^*(X)$ by $F_{top}^i(X) = Ker(K_{top}^*(X) \to K_{top}^*(X^i))$ and the associated graded algebra $gr_{top}^i(X) = F_{top}^i(X)/F_{top}^{i+1}(X)$. Consider the AHss (Atiyah-Hirzebruch spectral sequence)

$$E_2^{*,*'}(X) \cong H^*(X; K_{top}^{*'}) \Rightarrow K^*(X)_{top}.$$

By the construction of the spectral sequence, we have

LEMMA 4.1 (Atiyah [2]). $gr_{top}^*(X) \cong E_\infty^{*,0}(X)$.

Note that the above isomorphism

$$E_\infty^{2*,0} \cong gr^{2*}(K_{top}^*(X)) \to gr^{2*}(K_{top}^0(X)) \cong gr_{top}^{2*}(X)$$

is given by $x \mapsto B^*x$ for the Bott periodicity, i.e., $K^*_{top} \cong \mathbf{Z}[B, B^{-1}]$ and $deg(B) = (-2, -1)$.

Next we consider the geometric filtration. Let $X$ be a smooth algebraic variety over a subfield $k$ of $\mathbf{C}$. Let $K^*_{alg}(X)$ be the algebraic $K$-theory so that $K^0_{alg}(X)$ is the Grothendieck group generated by $k$-vector bundles over $X$. It is also isomorphic to the Grothendieck group generated by coherent sheaves over $X$ (we assumed $X$ smooth). This $K$-theory can be written by the motivic $K$-theory $AK^{*,*'}(X)$ ([36], [37], [41]), i.e.,

$$K^i_{alg}(X) \cong \oplus_j AK^{2j-i,j}(X).$$

In particular $K^0_{alg}(X) \cong AK^{2*,*}(X) = \oplus_j AK^{2j,j}(X)$.

The geometric filtration ([5], Chapter 15 in [33]) is defined as

$$F^{2i}_{geo}(X) = \{[O_V] \mid codim_X V \geq i\}$$

(and $F^{2i-1}_{geo}(X) = F^{2i}_{geo}(X)$) where $O_V$ is the structural sheaf of closed subvariety $V$ of $X$. Here we recall the motivic AHss ([41], [42])

$$AE_2^{*,*',*''}(X) \cong H^{*,*'}(X; K^{*''}) \Rightarrow AK^{*,*'}(X)$$

where $K^{2*} = AK^{2*,*}(pt.)$. Note that

$$AE_2^{2*,*,*''}(X) \cong H^{2*,*}(X; K^{*''}) \cong CH^*(X) \otimes K^{*''}.$$

Hence $AE_\infty^{2*,*,0}(X)$ is a quotient of $CH^*(X)$ by dimensional reason of degree of differential $d_r$ (i.e., $d_r AE_r^{2*,*,*''}(X) = 0$ for smooth $X$). Thus we have

Lemma 4.2 ([33], Lemma 6.2 in [43]). *We have*

$$gr^{2*}_{geo}(X) \cong AE_\infty^{2*,*,0}(X) \cong CH^*(X)/I$$

*where* $I = \bigcup_r Im(d_r)$.

Lemma 4.3 ([33], [43]). *Let* $t_{\mathbf{C}} : K^0_{alg}(X) \to K^0_{top}(X(\mathbf{C}))$ *be the realization map. Then* $F^i_{geo}(X) \subset (t^*_{\mathbf{C}})^{-1} F^i_{top}(X(\mathbf{C}))$.

At last, we consider the gamma filtration. Let $\lambda^i(x)$ be the exterior power of the vector bundle $x \in K^0_{alg}(X)$ and $\lambda_t(x) = \sum \lambda^i(x)t^i \in K^*_{alg}(X)[t]$. Let us denote

$$\lambda_{t/(1-t)}(x) = \gamma_t(x) = \sum \gamma^i(x)t^i \quad (i.e.\ \gamma^n(x) = \lambda^n(x+n-1)).$$

Hence $\gamma^i(x) \in K^0_{alg}(X)$ if so is $x$. The gamma filtration is defined as

$$F^{2i}_\gamma(X) = \{\gamma^{i_1}(x_1) \cdot \ldots \cdot \gamma^{i_m}(x_m) \mid i_1 + \cdots + i_m \geq i, x_j \in K^0_{alg}(X)\}.$$

Then we can see $F^i_\gamma(X) \subset F^i_{geo}(X)$. (Similarly we can define $F_\gamma(X)$ for a topological space $X$.) In Proposition 12.5 in [At], Atiyah proved $F^i_\gamma(X) \subset F^i_{top}(X)$ in $K_{top}(X)$. Moreover Atiyah's arguments work also in $K^0_{alg}(X)$ and this fact is well known ([4], [7], [33], [43]). Let $\varepsilon : K^0_{alg}(X) \to \mathbf{Z}$ be the augmentation map and $c_i(x) \in H^{2i,i}(X)$ the Chern class. Let $q : CH^*(X) \cong E_2^{2*,*,0} \twoheadrightarrow E_\infty^{2*,*,0}$ be the quotient map. Then (see p. 63 in [2]) we have

$$q(c_n(x)) = [\gamma^n(x - \varepsilon(x))].$$

Lemma 4.4 ([2], [33], [43]). *The condition $F_\gamma^{2*}(X) = F_{top}^{2*}(X)$ (resp. $F_\gamma^{2*}(X) = F_{geo}^{2*}(X)$) is equivalent to that $E_\infty^{2*,0}(X)$ (resp. $AE_\infty^{2*,*,0}(X)$) is (multiplicatively) generated by Chern classes in $H^{2*}(X)$ (resp. $CH^*(X)$).*

By Conner-Floyd type theorem ([24], [6]), it is well known

$$AK^{*,*'}(X) \cong (MGL^{*,*'}(X) \otimes_{MU^*} \mathbf{Z}) \otimes \mathbf{Z}[B, B^{-1}]$$

where the $MU^*$-module structure of $\mathbf{Z}$ is given by the Todd genus, and $B$ is the Bott periodicity. Since the Todd genus of $v_1$ (resp. $v_i$, $i > 1$) is 1 (resp. 0), we can write

$$AK^{*,*'}(X) \cong ABP^{*,*'}(X) \otimes_{BP^*} \mathbf{Z}[B, B^{-1}] \quad \textit{with } B^{p-1} = v_1.$$

Recall that $A\tilde{K}(1)^{*,*'}(X)$ is the algebraic Morava $K$-theory with $\tilde{K}(1)^* = \mathbf{Z}_{(p)}[v_1, v_1^{-1}]$. By the Landweber exact functor theorem (see [41]), we have $A\tilde{K}(1)^{*,*'}(X) \cong ABP^{*,*'}(X) \otimes_{BP^*} \tilde{K}(1)^*$. Thus we have

Lemma 4.5 ([43]). *There is a natural isomorphism*

$$A\tilde{K}^{*,*'}(X) \cong A\tilde{K}(1)^{*,*'}(X) \otimes_{\tilde{K}(1)^*} \mathbf{Z}[B, B^{-1}] \quad \textit{with } v_1 = B^{p-1}.$$

Lemma 4.6 ([43]). *Let $E(AK)_r$ (resp. $E(A\tilde{K}(1))_r$) be the AHss converging to $AK^{*,*'}(X)$ (resp. $A\tilde{K}(1)^{*,*'}(X)$). Then*

$$E(AK)_r^{*,*',*''} \cong E(A\tilde{K}(1))_r^{*,*',*''} \otimes_{\tilde{K}(1)^*} \mathbf{Z}_{(p)}[B, B^{-1}].$$

*In particular, $gr_{geo}^*(X) \cong E(AK)_\infty^{2*,*,0} \cong E(A\tilde{K}(1))_\infty^{2*,*,0}$.*

From the above lemmas, it is sufficient to consider the Morava $K$-theory $A\tilde{K}(1)^{*,*'}(X)$ when we want to study $AK^{*,*'}(X)$. For ease of notations, let us write simply

$$K^{2*}(X) = A\tilde{K}(1)^{2*,*}(X), \quad \textit{so that } K^*(pt.) \cong \mathbf{Z}_{(p)}[v_1, v_1^{-1}],$$
$$k^{2*}(X) = A\tilde{k}(1)^{2*,*}(X), \quad \textit{so that } k^*(pt.) \cong \mathbf{Z}_{(p)}[v_1].$$

Hereafter of this paper, we only consider this Morava $K$-theory $K^{2*}(X)$ instead of $AK^{2*,*}(X)$ or $K_{alg}^0(X)$.

## 5. The restriction map for the $K$-theory

We consider the restriction maps that

$$res_K : K^*(X) \to K^*(\bar{X}).$$

By Panin [20], it is known that $K^0_{alg}(\mathbf{F})$ is torsion free for each twisted flag varieties $\mathbf{F} = \mathbf{G}/B_k$. The following lemma is almost immediate from this Panin's result.

LEMMA 5.1. *Let* $\mathbf{F}$ *be a (twisted) flag variety. Then the restriction map* $res_K : K^*(\mathbf{F}) \to K^*(\bar{\mathbf{F}})$ *is injective.*

*Proof.* Recall that $res_{CH\otimes\mathbf{Q}} : CH^*(\mathbf{F})\otimes\mathbf{Q} \to CH^*(\bar{\mathbf{F}})\otimes\mathbf{Q}$ is isomorphic. Hence by the AHss, we see $K^*(\mathbf{F})\otimes\mathbf{Q} \cong K^*(\bar{\mathbf{F}})\otimes\mathbf{Q}$. Since $K^*(\mathbf{F})$ and $K^*(\bar{\mathbf{F}})$ are torsion free, the lemma is immediate. □

For each simply connected Lie group $G$, it is well known that $res_K$ is surjective from Chevalley. Thus we have

THEOREM 5.2 (Chevalley, Panin). *When $G$ is simply connected, $res_K$ is an isomorphism.*

Hence we have Theorem 1.1 in the introduction. However we will see it directly and explicitly for each simple Lie group. Moreover we will try to compute explicitly $gr_\gamma(X)$ for each simply connected simple Lie group.

LEMMA 5.3. *Let* $\mathbf{F}$ *be a versal complete flag variety. The restriction map* $res_K$ *is isomorphic if and only if for each generator* $y_i \in P(y)$ *in* $CH^*(\bar{\mathbf{F}})$, *there is* $c(i) \in K^*(BT)$ *such that* $c(i) = v_1^{s_i} y_i$, *for* $s_i \geq 0$ *in* $K^*(\bar{\mathbf{F}})/p$.

*Proof.* Consider the restriction map

$$res_k : k^*(R(\mathbf{G})) \to k^*(\bar{R}(\mathbf{G})) \cong k^* \otimes P(y)$$

where $k^* = \mathbf{Z}_{(p)}[v_1]$. Suppose that $res_K = res_k[v_1^{-1}]$ is surjective. Then since $\mathbf{F}$ is versal, there is $c(i) \in K^*(BB_k)$ with $c(i) = v_1^{s_i} y_i$ for some $s_i \in \mathbf{Z}$. Since $res_k(c(i)) \subset k^* \otimes P(y)$, we see $s_i \geq 0$. The converse is immediate. □

COROLLARY 5.4. *For each generator* $y_i \in P(y)$, *if there is* $x_{k(i)} \in \Lambda(x_1, \dots, x_\ell) \subset H^*(G; \mathbf{Z}/p)$ *such that* $Q_1 x_{k(i)} = y_i$, *then* $res_K$ *for a versal flag* $\mathbf{F}$ *is isomorphic.*

*Proof.* Let $d_r(x_k) = b_k \in H^*(BT)/p$. Then by Corollary 2.2, we have

$$b_k = v_1 y_i \quad mod(v_1^2) \quad in\ k(1)^*(G/T).$$

By induction, we can take generators $y_i$ satisfy the above lemma. □

COROLLARY 5.5. *Let* $G = G_1 \times G_2$, *and* $\mathbf{F} = \mathbf{G}/B_k$, $\mathbf{F}_i = \mathbf{G}_{i,k}/B_{i,k}$ *for* $i = 1, 2$. *Suppose* $\mathbf{F}$, $\mathbf{F}_i$ *are versal. If* $res_K$ *are isomorphic for* $K^*(\mathbf{F}_i)$ *then so is for* $K^*(\mathbf{F})$.

*Proof.* Since $\mathbf{F}$ is versal, we see $CH^*(\mathbf{F}_1) \otimes CH^*(\mathbf{F}_2) \to CH^*(\mathbf{F})$ is surjective, because $CH^*(\mathbf{F})$ generated by

$$CH^*(B(B_{1,k} \times B_{2,k})) \cong CH^*(BB_{1,k}) \otimes CH^*(BB_{2,k}).$$

Let $P_i(y) \subset H^*(G_i/T_i)$ be the polynomial parts corresponding to that in $H^*(G_i; \mathbf{Z}/p)$. Then $P(y) \cong P_1(y) \otimes P_2(y)$. By the assumption, $res_K$ are isomorphic for $\mathbf{F}_i$. Hence from the above lemma, for $y_i \in P_i(y)$, we have $v_1^{s_i} y_i = c(i) \in \Omega^*(BB_i)$. So for $y_1 y_2 \in P(y)$, we see

$$y_1 y_2 = v_1^{s_1+s_2} c(1)c(2) \quad \textit{with } c(1)c(2) \in \Omega^*(BB_k) \cong BP^*(BT). \qquad \square$$

*Proof of Theorem* 1.1 *without using Chevalley's theorem.* Each simply connect compact Lie group is a product of simple Lie groups. For each simple group except for $E_7$, $E_8$ with $p = 2$, we will show the existence of $x_k$ with $Q_1 x_k = y_i$ for each generator $y_i \in P(y)$ in §7–§9. (For example $Q_1(x_1) = y_1$ for all simply connected simple Lie groups.) For groups $E_7$, $E_8$ and $p = 2$, we will see that we can take $b_i = v_1^s y_i \, mod(2)$, for $s = 1$ or 2 in §10, §11. $\square$

Recall the fiber sequence $G \to G/T \xrightarrow{i} BT$. We consider the filtration $F^j$ of $K^*(G/T)$ defined by

$$F^j = Ker(K^*_{top}(G/T) \to K^*_{top}(i^{-1}(BT^j))$$

for the $j$-th skeleton $BT^j$ of $BT$. This filtration gives the following (modified) AHss

$$E_2^{*,*'} \cong H^*(BT; K^{*'}(G)) \Rightarrow K^*(G/T).$$

Corollary 5.6. *For the above spectral sequence $E_r^{*,*'}$, we have the isomorphism*

$$E_\infty^{*,*'} \cong K^{*'} \otimes gr_\gamma^*(G/T) \quad (e.g., \ E_\infty^{*,0} \cong gr_\gamma^*(G/T)).$$

*Proof.* When $\mathbf{F}$ is versal, $CH^*(\mathbf{F})$ is generated by elements in $CH^*(BT)$. Since

$$K^*(\mathbf{F}) \cong K^*(\bar{\mathbf{F}}) \cong K^*(G/T),$$

from Theorem 5.2, $K^*(G/T)$ is generated by elements in $K^*(BT) \cong K^* \otimes S(t)$. Hence we see

$$E_\infty^{*,*'} \cong K^{*'} \otimes S(t)/J \quad \textit{for some } J.$$

This means that $E_\infty^{*,0} \cong S(t)/J$ is a graded ring of $K^0(G/T)$.

By Atiyah (p. 63 in [2]), if there is a filtration and the associated spectral sequence such that the $E_\infty$ is generated by Chern classes, then it is isomorphic to $K^* \otimes gr_\gamma(G/T)$. $\square$

However, the study of the above spectral sequence seems not so easy. We will use the following technical lemma, to seek $gr_\gamma(G/T)$.

Lemma 5.7. *Suppose that there is a filtration of* $K^*(R(\mathbf{G}))$ *and* $\mathbf{Z}_{(p)}$*-module* $B$ *with maps* $\mathbf{Z}_{(p)}[b_1,\dots,b_\ell] \twoheadrightarrow B \to gr_{geo}(R(\mathbf{G}))$ *such that*

$$(1) \qquad grK^*(R(\mathbf{G})) \cong K^* \otimes B, \quad \text{and} \quad B/p \subset gr^*_{geo}(R(\mathbf{G}))/p.$$

*Moreover suppose that for any torsion element* $b \in B$ *with* $b \neq 0 \in B/p$, *if there exist* $r,s > 0$ *such that in* $K^*(R(\mathbf{G}))$

$$(2) \qquad p^r b = v_1^s b' \quad \text{for some } b' \in CH^*(R(\mathbf{G}_k)),$$

*then* $b' \in B$. *With these suppositions, we have* $B \cong gr^*_{geo}(R(\mathbf{G}))$.

*Proof.* Suppose that $b' \in gr_{geo}(R(\mathbf{G}))$ but $b' \notin B$. Since $B$ generates $K^*(R(\mathbf{G}))$ as a $K^*$-module, there is $\tilde{b} \in B$ such that $b' = v_1^{-s}\tilde{b}$ for $s > 0$ (otherwise $b' \in B$) in $K^*(R(\mathbf{G}))$. Let us write $\tilde{b} = p^r b$ in $K^*(R(\mathbf{G}))$. Then (2) is satisfied for these $b$, $b'$ since we see $r > 0$, otherwise $b = 0 \in CH^*(R(\mathbf{G}))$. Hence $b' \in B$, and it is a contradiction.

Since $p^r b = 0 \in CH^*(R(\mathbf{G}))$, we only consider a torsion element for $b$. □

To seek $gr_\gamma(R(\mathbf{G}))$, we first find some graded algebra $grK^*(R(\mathbf{G})) \cong K^* \otimes B$ such that each element $0 \neq \tilde{b} \in B$ is represented by an element (possible zero) in $CH^*(R(\mathbf{G}))$. Since $\tilde{b} \neq 0 \in K^*(R(\mathbf{G}))$, there are $r,s \geq 0$ such that

$$p^r\tilde{b} = v_1^s b', \quad \text{with } b' \neq 0 \in CH^*(R(\mathbf{G})) \cong k^*(R(\mathbf{G})) \otimes_{k^*} \mathbf{Z}_{(p)}.$$

Take new $B$ by replacing $p^r\tilde{b}$ by $b'$ in $B$. Continue these arguments, we may approximate $gr_\gamma(R(\mathbf{G}))$ by $B$.

## 6. The orthogonal group $SO(m)$ and $p = 2$

We consider the orthogonal groups $G = SO(m)$ and $p = 2$ in this section. The mod 2-cohomology is written as (see for example [17], [19])

$$grH^*(SO(m);\mathbf{Z}/2) \cong \Lambda(x_1, x_2, \dots, x_{m-1})$$

where $|x_i| = i$, and the multiplications are given by $x_s^2 = x_{2s}$. (Note that the suffix means its degree.)

For ease of argument, we only consider the case $m = 2\ell + 1$ (the case $m = 2\ell + 2$ works similarly) so that

$$H^*(G;\mathbf{Z}/2) \cong P(y) \otimes \Lambda(x_1, x_3, \dots, x_{2\ell-1})$$

$$grP(y)/2 \cong \Lambda(y_2, \dots, y_{2\ell}), \quad \text{letting } y_{2i} = x_{2i} \ (\text{hence } y_{4i} = y_{2i}^2).$$

The Steenrod operation is given as $Sq^k(x_i) = \binom{i}{k}(x_{i+k})$. The $Q_i$-operations are given by Nishimoto [19]

$$Q_n x_{2i-1} = y_{2i+2^{n+1}-2}, \quad Q_n y_{2i} = 0.$$

It is well known that the transgression $d_{2i}(x_{2i-1}) = c_i$ is the $i$-th elementary symmetric function on $S(t)$. Moreover we see that $Q_0(x_{2i-1}) = y_{2i}$ in $H^*(G; \mathbf{Z}/2)$. In fact, the cohomology $H^*(G/T)$ is computed completely by Toda-Watanabe [30]

THEOREM 6.1 ([30]). *There are* $y_{2i} \in H^*(G/T)$ *for* $1 \le i \le \ell$ *such that* $\pi^*(y_{2i}) = y_{2i}$ *for* $\pi : G \to G/T$, *and that we have an isomorphism*

$$H^*(G/T) \cong \mathbf{Z}[t_i, y_{2i}]/(c_i - 2y_{2i}, J_{2i})$$

*where* $J_{2i} = 1/4(\sum_{j=0}^{2i}(-1)^j c_j c_{2i-j})$ *letting* $y_{2j} = 0$ *for* $j > \ell$.

By using Nishimoto's result for $Q_i$-operation, from Corollary 2.2, we have

COROLLARY 6.2. *In* $BP^*(G/T)/I_\infty^2$, *we have*

$$c_i = 2y_{2i} + \sum v_n(y(2i + 2^{n+1} - 2))$$

*for some* $y(j)$ *with* $\pi^*(y(j)) = y_j$.

We have $c_i^2 = 0$ in $CH^*(\mathbf{F})/2$ from the natural inclusion $SO(2\ell+1) \to Sp(2\ell+1)$ (see [21], [44]) for the symplectic group $Sp(2\ell+1)$. Thus we have

THEOREM 6.3 ([21], [44]). *Let* $(G, p) = (SO(2\ell+1), 2)$ *and* $\mathbf{F} = \mathbf{G}/B_k$ *be versal. Then* $CH^*(\mathbf{F})$ *is torsion free, and*

$$CH^*(\mathbf{F})/2 \cong S(t)/(2, c_1^2, \dots, c_\ell^2), \quad CH^*(R(\mathbf{G}))/2 \cong \Lambda(c_1, \dots, c_\ell).$$

COROLLARY 6.4. *We have* $gr_\gamma(\mathbf{F}) \cong gr_{geo}(\mathbf{F}) \cong CH^*(\mathbf{F})$.

*Proof.* It is known that the image $Im(d_r)$ of the differentials of AHss are generated by torsion elements. Hence the ideal $I = \bigcup_r Im(d_r) = 0$. From Lemma 4.2, we see $gr_{geo}(\mathbf{F}) \cong CH^*(\mathbf{F})/I \cong CH^*(\mathbf{F})$. □

From Corollary 2.2, we see $c_i = 2y_i + v_1 y_{2+2}\, mod(v_1^2)$. Here we consider the $mod(2)$ theory version $res_{K/2} : K^*(\mathbf{F})/2 \to K^*(\bar{\mathbf{F}})/2$.

LEMMA 6.5. *We have* $Im(res_{K/2}) \cong K^* \otimes \Lambda(y_{2i} \mid 2i \ge 4)$. *Hence* $res_K$ *is not surjective.*

*Proof.* In $mod(2)$, we have $c_i = v_1 y_{2i+2}$. Hence for all $i \ge 2$, we have $y_{2i+2} \in Im(res_{K/2})$. Suppose $y_2 y \in Im(res_{K/2})$ for $0 \ne y \in \Lambda(y_{2i} \mid i \ge 2)$. Then from Theorem 6.3, there is $c \in \Lambda(c_i)$ such that $c = v_1^s y_2 y$. Since $c_i = v_1 y_{2i+2}$, we can write $c = v_1^t y'$ for $y' \in \Lambda(y_{2i} \mid i \ge 2)$. Hence $v_1^t(y' - v_1^{s-t} y_2 y) = 0$ in $K^*(\bar{\mathbf{F}})/2 = K^*/2 \otimes \Lambda(y_{2i})$, which is $K^*/2$-free. and this is a contradiction. □

In the next section, we will compute the case $(G', p) = (Spin(2\ell+1), 2)$. It is well known that $G/T \cong G'/T'$ for the maximal torus $T'$ of the spin group. Hence we see

Lemma 6.6. *Let* $\mathbf{F} = \mathbf{G}/B_k$ *and* $\mathbf{F}' = \mathbf{G}'/B_k'$ *be versal. Then*

$$gr_\gamma(\mathbf{F}) \not\cong gr_\gamma(\overline{\mathbf{F}}) \cong gr_\gamma(\overline{\mathbf{F}}') \cong gr_\gamma(\mathbf{F}').$$

*Proof*. The last two isomorphisms follow from the facts that $\mathbf{F}'$ is versal and $res_K$ is isomorphic for spin groups (see Lemma 7.3 below). □

Note that $c_2c_3 \neq 0$ in $gr_\gamma(\mathbf{F})$ but it is zero in $gr_\gamma^*(\mathbf{F}')$ from Theorem 7.8 for $Spin(n)$ when $n \leq 11$. Hence the restriction map $res_{gr_\gamma} : gr_\gamma^*(\mathbf{F}) \to gr_\gamma(\overline{\mathbf{F}})$ is not injective, while $res_K$ is injective.

## 7. The spin group $Spin(2\ell+1)$ and $p = 2$

Throughout this section, let $p = 2$, $G = SO(2\ell+1)$ and $G' = Spin(2\ell+1)$. By definition, we have the 2 covering $\pi : G' \to G$. It is well known that $\pi^* : H^*(G/T) \cong H^*(G'/T')$.

Let $2^t \leq \ell < 2^{t+1}$, i.e. $t = [log_2 \ell]$. The mod 2 cohomology is

$$H^*(G'; \mathbf{Z}/2) \cong H^*(G; \mathbf{Z}/2)/(x_1, y_1) \otimes \Lambda(z)$$
$$\cong P(y)' \otimes \Lambda(x_3, x_5, \dots, x_{2\ell-1}) \otimes \Lambda(z), \quad |z| = 2^{t+2} - 1$$

where $P(y) \cong \mathbf{Z}/2[y_2]/(y_2^{2^{t+1}}) \otimes P(y)'$. (Here $d_{2^{t+2}}(z) = y^{2^{t+1}}$ for $0 \neq y \in H^2(B\mathbf{Z}/2; \mathbf{Z}/2)$ in the spectral sequence induced from the fibering $G' \to G \to B\mathbf{Z}/2$.) Hence

$$grP(y)' \cong \otimes_{2i \neq 2^j} \Lambda(y_{2i}) \cong \Lambda(y_6, y_{10}, y_{12}, \dots, y_{2\ell}).$$

The $Q_i$ operation for $z$ is given by Nishimoto [19]

$$Q_0(z) = \sum_{i+j=2^{t+1}, i<j} y_{2i}y_{2j}, \quad Q_n(z) = \sum_{i+j=2^{t+1}+2^{n+1}-2, i<j} y_{2i}y_{2j} \quad \textit{for } n \geq 1.$$

We know that

$$grH^*(G/T)/2 \cong P(y)' \otimes \mathbf{Z}[y_2]/(y_2^{2^{t+1}}) \otimes S(t)/(2, c_1, c_2, \dots, c_\ell)$$
$$grH^*(G'/T')/2 \cong P(y)' \otimes S(t')/(2, c_2', \dots, c_\ell', c_1^{2^{t+1}}).$$

Here $c_i' = \pi^*(c_i)$ and $d_{2^{t+2}}(z) = c_1^{2^{t+1}}$ in the spectral sequence converging $H^*(G'/T')$. These are isomorphic, in particular, we have

Lemma 7.1. *The element* $\pi^*(y_2) = c_1 \in S(t')$ *and* $\pi^*(t_j) = c_1 + t_j$ *for* $1 \leq j \leq \ell$.

Take $k$ such that $\mathbf{G}$ is a versal $G_k$-torsor so that $\mathbf{G}'_k$ is also a versal $G'_k$-torsor. Let us write $\mathbf{F} = \mathbf{G}/B_k$ and $\mathbf{F}' = \mathbf{G}'/B'_k$. Then

$$CH^*(\bar{R}(\mathbf{G}'))/2 \cong P(y)'/2, \quad and \quad CH^*(\bar{R}(\mathbf{G}))/2 \cong P(y)/2.$$

The Chow ring $CH^*(R(\mathbf{G}'))/2$ is not computed yet (for general $\ell$), while we have the following lemmas.

Lemma 7.2. *We have a surjection*

$$\Lambda(c'_2, \dots, c'_\ell, c_1^{2^{t+1}}) \twoheadrightarrow CH^*(R(\mathbf{G}'))/2.$$

Lemma 7.3. *The restriction* $res_K : K^*(\mathbf{F}') \to K^*(\bar{\mathbf{F}}')$ *is isomorphic.*

*Proof.* Recall the relation

$$c'_i = 2y_{2i} + v_1 y_{2i+2} \quad mod(v_1^2) \quad in \ K^*(\bar{\mathbf{F}}').$$

We consider the $mod(2)$ restriction map $res_{K/2}$. Since $c'_i = v_1 y_{2i+2}\, mod(2)$, we see $y_{2i+2} \in Im(res_{K/2})$. So $y_{2i'} \in Im(res_{K/2})$ for $2i' \geq 6$, e.g., for all $y_{2i} \in P(y)' = \Lambda(y_{2i} \mid i \neq 2^j)$. Hence $res_{K/2}$ is surjective. Thus $res_K$ itself surjective. □

Corollary 7.4. *Let* $G'' = Spin(2\ell + 2)$ *and* $\mathbf{F}'' = \mathbf{G}''/B_k$. *Then* $res_K : K^*(\mathbf{F}'') \to K^*(\bar{\mathbf{F}}'')$ *is isomorphic.*

*Proof.* This is immediate from $P(y)' \cong P(y)''$. □

Corollary 7.5. *We have* $K^*(R(\mathbf{G}'))/2 \cong K^*/2 \otimes \Lambda(c'_i \mid i \neq 2^j - 1)$.

*Proof.* Recall $c'_{2^j-1} = 2(y_{2^j-1}) + v_1 y_{2^j}$ where $y_{2^j} = c_1^{2^j} = 0 \in K^*(\bar{R}(\mathbf{G}))/2$. So $c'_{2^j-1} = 0 \in K^*(R(\mathbf{G}))/2$, since $res_K$ is injective. We have the maps

$$K^*/2 \otimes \Lambda(c'_{i-1} \mid i \neq 2^j) \twoheadrightarrow K^*(R(\mathbf{G}))/2$$
$$\xrightarrow{isom.} K^*(\bar{R}(\mathbf{G}))/2 \cong K^*/2 \otimes \Lambda(y_{2i} \mid i \neq 2^j)$$

by $c'_{i-1} \mapsto v_1 y_i$. This map is isomorphic from the above lemma. Hence we have the corollary. □

Hence $c'_{2^j-1}$ is not a module generator in $K^*(R(\mathbf{G}))$. So we can write

$$2^r b = v_1^s c'_{2^j-1} \quad r, s \geq 1,$$

so that $b$ is torsion element in $CH^*(R(\mathbf{G}))$. In fact we have

Lemma 7.6. *Let* $k < t$ *i.e.,* $2^{k+1} < \ell$. *Then in* $K^*(R(\mathbf{G}))$, *we have*

$$2^{2^k} c'_{2^k} + \sum_{i=1}^{2^k-1} (-1)^i 2^{2^k-i} v_1^i c'_{2^k+i} = v_1^{2^k} c'_{2^{k+1}-1}.$$

*Proof.* Let us write $c''_{2^k} = c'_{2^k} - 2y_{2(2^k)} = v_1 y_{2(2^k+1)}$. Then

$$v_1 y_{2(2^k+2)} = c'_{2^k+1} - 2y_{2(2^k+1)} = c'_{2^k+1} - 2v_1^{-1}c''_{2k}.$$

By induction on $i$, we have

$$v_1 y_{2(2^k+i+1)} = (-1)^i 2^i v_1^{-i} c''_{2^k} + \sum_{0<j<i} (-1)^j 2^j v_1^{-j} c'_{2(2^k+j)}.$$

When $i = 2^k - 1$, we see $y_{2(2^{k+1})}$ is in $K^*(\mathbf{F})$ (it is zero in $K^*(R(\mathbf{G}))$ from Lemma 3.6). Hence we have the equation in the lemma. □

In general, $gr_\gamma(R(\mathbf{G}))$ seems complicated. Here we only note the following proposition. Let us write $\mathbf{Z}_{(p)}$-free module

$$\Lambda_{\mathbf{Z}}(a_1, \dots, a_n) = \mathbf{Z}_{(p)}\{a_{i_1} \cdots a_{i_s} \mid 1 \le i_1 < \cdots < i_s \le n\}.$$

PROPOSITION 7.7. *Let us write*

$$B_1 = \Lambda_{\mathbf{Z}}(c'_i \mid i \ne 2^j - 1,\ 2 \le i \le \ell - 1), \quad B_2 = \Lambda_{\mathbf{Z}}(c'_i \mid i \ne 2^j,\ 3 \le i \le \ell).$$

*Then there are additive injections for* $i = 1, 2$ *and the surjection*

$$B_i/2 \subset gr_\gamma^*(R(\mathbf{G}))/2 \twoheadrightarrow (B_1/2 + B_2/2)$$

*for degree* $* < |c_1^{2^{t+1}}| = 2^{t+2}$.

*Proof.* First note that when $* < 2^{t+2}$, we have the surjection $\Lambda(c'_2, \dots, c'_\ell) \twoheadrightarrow CH^*(R(\mathbf{G}'))/2$. Recall $k^*(X)$ is the $K$-theory with $k^*(pt) \cong \mathbf{Z}_{(p)}[v_1]$. We consider the map

$$\begin{aligned} \rho : k^*/2 \otimes B_1 &\to k^*(R(\mathbf{G}))/2 \to k^*(\bar{R}(\mathbf{G}))/2 \\ &\subset K^*/2 \otimes \Lambda(y_{2i} \mid i \ne 2^j) \cong K^*(\bar{R}(\mathbf{G}))/2 \end{aligned}$$

by $c'_i \mapsto v_1 y_{2i+2}$. We show

$$Im(\rho) = k^*/2\{v_1^s y_{2i_1} \cdots y_{2i_s} \mid i_k \ne 2^j,\ 1 \le i_1 < \cdots < i_s \le \ell - 1\}.$$

In fact, $Im(\rho)$ contains the right hand side module since $c'_i \mapsto v_1 y_{2i+2}$. Suppose that $\rho(b_I) = v_1^k y_{2i_1} \cdots y_{2i_s}$ for $b_I \in B_1$. Then $b_I$ contains $c'_{i_1-1} \cdots c'_{i_s-1}$, which must maps to $v_1 y_{i_1} \cdots v_1 y_{i_s}$. Hence $k \ge s$. Therefore each generator $c'_I = c'_{i_1} \cdots c'_{i_s} \in \Lambda(c'_i \mid i \ne 2^j - 1)$ is also a $k^*$-module generator of $Im(\rho)$. Hence it is nonzero in $CH^*(R(\mathbf{G}))/2$ from $k^*(X) \otimes_{k^*} \mathbf{Z}_{(2)} \cong CH^*(X)$. Thus $B_1/2 \subset CH^*(R(\mathbf{G}))/2$.

Next we consider for $B_2$. We consider the map

$$\begin{aligned} \rho_{\mathbf{Q}} : k^* \otimes B_2 &\to k^*(R(\mathbf{G})) \to k^*(\bar{R}(\mathbf{G})) \twoheadrightarrow k^*(\bar{R}(\mathbf{G})) \otimes_{k^*} \mathbf{Q} \\ &\cong CH^*(\bar{R}(\mathbf{G})) \otimes \mathbf{Q} \cong \Lambda_{\mathbf{Z}}(y_{2i} \mid i \ne 2^j) \otimes \mathbf{Q}. \end{aligned}$$

by $c_i' \mapsto 2y_{2i}$. By the arguments similar to the case $B_1$, we see

$$Im(\rho_{\mathbf{Q}}) = k^*\{2^s y_{2i_1} \cdots y_{2i_s} \mid i_k \neq 2^j,\ 1 \leq i_1 < \cdots < i_s \leq \ell\}.$$

Therefore each generator $c_I' = c_{i_1}' \cdots c_{i_s}' \in \Lambda(c_i' \mid i \neq 2^j)$ is also $k^*$-module generator of $Im(\rho_{\mathbf{Q}})$. Hence $B_2/2 \subset CH^*(R(\mathbf{G}))/2$. □

*Remark.* Note that $c_{2^k}' c_{2^j-1}'$ contains $2v_1 y_{2^{k+1}+2} y_{2^{j+1}-2}$, while this element is also contained in $c_{2^k+1}' c_{2^j-2}'$.

Now we consider examples. For groups $Spin(7)$, $Spin(9)$, the graded ring $gr_\gamma(G')/2$ are given in the next section (in fact these groups are of type $(I)$). We consider here the group $G' = Spin(11)$. The cohomology is written as

$$H^*(G'; \mathbf{Z}/2) \cong \mathbf{Z}/2[y_6, y_{10}]/(y_6^2, y_{10}^2) \otimes \Lambda(x_3, x_5, x_7, x_9, z_{15}).$$

By Nishimoto, we know $Q_0(z_{15}) = y_6 y_{10}$. It implies $2y_6 y_{10} = d_{16}(z_{15}) = c_1^8$. Since $y_{top}' = y_6 y_{10}$, we have $t(G') = 2$.

THEOREM 7.8. *For $(G', p) = (Spin(11), 2)$, we have the isomorphisms*

$$gr_\gamma(R(\mathbf{G}))/2 \cong \mathbf{Z}/2\{1, c_2', c_3', c_4', c_5', c_2'c_4', c_1^8\},$$
$$gr_\gamma(\mathbf{F})/2 \cong S(t)/(2, c_i'c_j', c_i'c_1^8, c_1^{16} \mid 2 \leq i \leq j \leq 5,\ (i,j) \neq (2,4)).$$

*Proof.* Consider the restriction map $res_K$

$$K^*(R(\mathbf{G})) \cong K^*\{1, c_2', c_4', c_2'c_4'\} \xrightarrow{\cong} K^*\{1, y_6, y_{10}, y_6 y_{10}\} \cong K^*(\bar{R}(\mathbf{G}))$$

by $res_K(c_2') = v_1 y_6$, $res_K(c_4') = v_1 y_{10}$. Note $c_2'c_4' \neq 0 \in CH^*(R(\mathbf{G}))/2$, from the preceding proposition. (In fact, $v_1 y_6 y_{10} \notin Im(res_K)$.)

Next using $res_K(c_3') = 2y_6$, $res_K(c_5') = 2y_{10}$, and $res_K(c_1^8) = 2y_6 y_{10}$, we have

$$\begin{aligned} gr_2 K^*(R(\mathbf{G})) &= K^*(R(\mathbf{G}))/2 \oplus 2K^*(R(\mathbf{G})) \\ &\cong K^*/2\{1, c_2', c_4', c_2'c_4'\} \oplus K^*\{2, c_3', c_5', c_1^8\}. \end{aligned}$$

From this and Lemma 5.7, we show the first isomorphism. (Note generators $c_2', \ldots, c_1^8$ are all nonzero in $CH^*(R(\mathbf{G}))/2$ by Corollary 3.8.)

Let us write $y_6^2 = y_6 t \in CH^*(\bar{\mathbf{F}})/2$ for $t \in S(t)$. Then

$$c_2'c_3' = 2v_1 y_6^2 = 2v_1 y_6 t = v_1 c_3' t \quad in\ K^*(\mathbf{F}).$$

Hence we see $c_2'c_3' = 0$ in $CH^*(\mathbf{F})/I \cong gr_\gamma^*(\mathbf{F})$. We also note

$$c_3'c_5' = 4y_6 y_{10} = 2c_1^8 \quad in\ K^*(\bar{\mathbf{F}}).$$

By arguments similar to the proof of Lemma 3.5, we can show the second isomorphism. □

*Remark.* Quite recently, Karpenko proves ([9]) that for $G = Spin(11)$, we have $gr_\gamma(\mathbf{F}) \cong CH^*(\mathbf{F})$. Hence the above ring is isomorphic to $CH^*(\mathbf{F})/2$.

## 8. Groups of type $(I)$ (e.g., $(G,p) = (E_8, 5)$)

When $G$ is simply connected and $P(y)$ is generated by just one generator, we say that $G$ is of type $(I)$. Except for $(E_7, p = 2)$ and $(E_8, p = 2, 3)$, all exceptional (simple) Lie groups are of type $(I)$. The groups $Spin(m)$ for $7 \leq m \leq 10$ are also of type $(I)$. Note that in these cases, it is known $rank(G) = \ell \geq 2p - 2$.

*Example.* The case $(G,p) = (E_8, 5)$ is of type $I$, we have [17]

$$H^*(E_8; \mathbf{Z}/5) \cong \mathbf{Z}/5[y_{12}]/(y_{12}^5) \otimes \Lambda(z_3, z_{11}, z_{15}, z_{23}, z_{27}, z_{35}, z_{39}, z_{47})$$

where suffix means its degree. The cohomology operations are given

$$\beta(z_{11}) = y_{12}, \quad \beta(z_{23}) = y_{12}^2, \quad \beta(z_{35}) = y_{12}^3, \quad \beta(z_{47}) = y_{12}^4,$$
$$P^1 z_3 = z_{11}, \quad P^1 z_{15} = z_{23}, \quad P^1 z_{27} = z_{35}, \quad P^1 z_{39} = z_{47}.$$

Similarly, for each group $(G,p)$ of type $(I)$, we can write

$$H^*(G; \mathbf{Z}/p) \cong \mathbf{Z}/p[y]/(y^p) \otimes \Lambda(x_1, \dots, x_\ell) \quad |y| = 2(p+1).$$

The cohomology operations are given as

$$\beta : x_{2i} \mapsto y^i, \quad P^1 : x_{2i-1} \mapsto x_{2i} \quad \textit{for } 1 \leq i \leq p-1.$$

Hence we have $Q_1(x_{2i-1}) = Q_0(x_{2i}) = y^i$ for $1 \leq i \leq p-1$.

Theorem 8.1. *Let $G$ be of type $(I)$. Then $res_K$ is isomorphic, and*

$$gr_\gamma^*(R(\mathbf{G})) \cong (\mathbf{Z}/p\{b_1, b_3, \dots, b_{2p-3}\} \oplus \mathbf{Z}_{(p)}\{1, b_2, \dots, b_{2p-2}\}).$$

*Moreover $gr_\gamma(G/T)/p \cong gr_\gamma(\mathbf{F})/p$ is isomorphic to*

$$S(t)/(p, b_i b_j, b_k \,|\, 1 \leq i, j \leq 2p-2 < k \leq \ell).$$

*Proof.* From $Q_1 x_1 = y$, we see $b_1 = v_1 y \, mod(v_1^2)$ in $k(1)^*(G/T)$. From Corollary 5.4, we see $res_K$ is isomorphic, namely the map

$$K^*(R(\mathbf{G})) \cong K^*[b_1]/(b_1^p) \to K^*[y]/(y^p) \cong K^*(\bar{R}(\mathbf{G}))$$

is isomorphic by $res_K(b_1) = v_1 y$.

From Corollary 2.2, the facts $Q_1(x_{2i-1}) = Q_0(x_{2i}) = y^i$ for $1 \leq i \leq p-1$ imply in $K^*(G/T)$ that we can take

$$b_{2i-1} = v_1 y^i, \quad b_{2i} = p y^i.$$

Hence we have, in $K^*(R(\mathbf{G})) \cong K^*(\bar{R}(\mathbf{G}))$,

$$b_1^i = v_1^i y^i = v_1^{i-1} b_{2i-1}, \quad p b_1^i = p v_1^i y^i = v_1^i b_{2i}.$$

Thus we get the isomorphism, using $v_1^{-1} \in K^*$,

$$gr_p K^*(R(\mathbf{G})) = K^*(R(\mathbf{G}))/p \oplus pK^*(R(\mathbf{G})) \cong K^* \otimes B$$
$$\textit{where } B = (\mathbf{Z}/p\{1, b_1, b_3, \dots, b_{2p-3}\} \oplus \mathbf{Z}_{(p)}\{p, b_2, b_4, \dots, b_{2p-2}\}).$$

Note that $b_i$ are nonzero in $CH^*(R(\mathbf{G}))/p$, it gives $gr_\gamma(R(\mathbf{G}))/p \cong B/p$ from Lemma 5.7. Moreover we have the first isomorphism of this theorem.

The above $B/p$ is rewritten using $A = \mathbf{Z}/p[b_1, \dots, b_\ell]$

$$B/p \cong A/(b_ib_j, b_k \mid 1 \le i, j \le 2p-2 < k).$$

From arguments similar to the proof of Lemma 3.5, we have the theorem. □

Here we recall the (original) Rost motive $R_a$ (we write it by $R_n$) defined from a nonzero pure symbol $a$ in $K^M_{n+1}(k)/p$ ([25], [26], [37], [38]). We have isomorphisms (with $|y| = 2(p^n-1)/(p-1)$)

$$CH^*(\bar{R}_n) \cong \mathbf{Z}[y]/(y^p), \quad \Omega^*(\bar{R}_n) \cong BP^*[y]/(y^p).$$

Theorem 8.2 ([35], [42], [15]). *The restriction $res_\Omega : \Omega^*(R_n) \to \Omega^*(\bar{R}_n)$ is injective. Recall $I_n = (p, \dots, v_{n-1}) \subset BP^*$. Then*

$$Im(res_\Omega) \cong BP^*\{1\} \oplus I_n \otimes (\mathbf{Z}_{(p)}[y]^+/(y^p)) \subset BP^*[y]/(y^p).$$

*Hence writing $v_jy^i = c_j(y^i)$, we have*

$$CH^*(R_n)/p \cong \mathbf{Z}/p\{1, c_j(y^i) \mid 0 \le j \le n-1,\ 1 \le i \le p-1\}.$$

*Example.* In particular, we have the isomorphism

$$CH^*(R_2)/p \cong \mathbf{Z}_{(p)}\{1, c_0(y), \dots, c_0(y^{p-1})\} \oplus \mathbf{Z}/p\{c_1(y), \dots, c_1(y^{p-1})\}.$$

By Petrov-Semenov-Zainoulline ([23]), it is known when $G$ is of type $(I)$, we can see $J(\mathbf{G}) = (1)$ and $R(\mathbf{G}) \cong R_2$.

Corollary 8.3. *We have $gr_\gamma(\mathbf{F}) \cong CH^*(\mathbf{F})_{(p)}$.*

*Proof.* We have the additive isomorphism

$$\begin{aligned} gr_\gamma(R(\mathbf{G})) &\cong B = \mathbf{Z}/p\{b_1, \dots, b_{2p-3}\} \oplus \mathbf{Z}_{(p)}\{1, b_2, \dots, b_{2p-2}\} \\ &\cong \mathbf{Z}/p\{c_1(y), \dots, c_1(y^{p-1})\} \oplus \mathbf{Z}_{(p)}\{1, c_0(y), \dots, c_0(y^{p-1})\}, \end{aligned}$$

which is isomorphic to $CH^*(R_2)$. The fact that $gr_\gamma(\mathbf{F}) \cong CH^*(\mathbf{F})/I$ for some ideal $I$ implies $I = 0$ and the theorem. □

The above $I$ is nonzero for $R_n$ when $n \ge 3$.

Lemma 8.4. *There is an isomorphism for $n \ge 3$,*

$$gr^{2*}_{geo}(R_n) \cong CH^*(R_n)/I \quad \text{with } I = \mathbf{Z}/p\{c_2, \dots, c_{n-1}\}[y]/(y^{p-1}).$$

*Proof.* First, note $v_j = 0$ in $K^* = \mathbf{Z}_{(p)}[v_1, v_1^{-1}]$ for $j \neq 0, 1$. We have

$$v_1 c_j(y) = v_1 v_j y = v_j c_1(y) \quad in\ \Omega^*(\bar{R}_n).$$

Since $res_\Omega$ is injective, we see $v_1 c_j(y) = v_j c_1(y) = 0$ in $K^*(R_n)/p$. □

## 9. The case $G = E_8$ and $p = 3$

Throughout this section, let $(G, p) = (E_8, 3)$ and $\mathbf{G}$ be versal. The cohomology $H^*(G; \mathbf{Z}/3)$ is isomorphic to ([17])

$$\mathbf{Z}/3[y_8, y_{20}]/(y_8^3, y_{20}^3) \otimes \Lambda(z_3, z_7, z_{15}, z_{19}, z_{27}, z_{35}, z_{39}, z_{47}).$$

Here the suffix means its degree, e.g., $|z_i| = i$. By Kono-Mimura [11] the actions of cohomology operations are also known

Theorem 9.1 ([11]). *We have $P^3 y_8 = y_{20}$, and*

$$\beta\colon\ z_7 \mapsto y_8, \quad z_{15} \mapsto y_8^2, \quad z_{19} \mapsto y_{20}, \quad z_{27} \mapsto y_8 y_{20}, \quad z_{35} \mapsto y_8^2 y_{20},$$
$$z_{39} \mapsto y_{20}^2, \quad z_{47} \mapsto y_8 y_{20}^2,$$
$$P^1\colon\ z_3 \mapsto z_7, \quad z_{15} \mapsto z_{19}, \quad z_{35} \mapsto z_{39}$$
$$P^3\colon\ z_7 \mapsto z_{19}, \quad z_{15} \mapsto z_{27} \mapsto -z_{39}, \quad z_{35} \mapsto z_{47}.$$

We use notations $y = y_8$, $y' = y_{20}$, and $x_1 = z_3, \dots, x_8 = z_{47}$. Then we can rewrite the isomorphisms

$$H^*(G; \mathbf{Z}/3) \cong \mathbf{Z}/3[y, y']/(y^3, (y')^3) \otimes \Lambda(x_1, \dots, x_8),$$
$$grH^*(G/T; \mathbf{Z}/3) \cong \mathbf{Z}/3[y, y']/(y^3, (y')^3) \otimes S(t)/(b_1, , \dots, b_8).$$

From Corollary 2.2, we have

Corollary 9.2 ([44]). *We can take $b_1 \in BP^*(BT)$ such that*

$$v_1 y + v_2 y' = b_1 \quad in\ BP^*(G/T)/I_\infty^2.$$

From the preceding theorem, we know that all $y^i (y')^j$ except for $(i, j) = (0, 0)$ and $(2, 2)$ are $\beta$-image. Hence we have

Corollary 9.3 ([44]). *For all nonzero monomials $u \in P(y)^+/3$ except for $(yy')^2$, it holds $3u \in S(t)$. In fact, in $BP^*(G/T)/I_\infty^2$*

$$b_1 = v_1 y + v_2 y', \quad b_2 = 3y, \quad b_3 = 3y^2 + v_1 y' \quad b_4 = 3y',$$
$$b_5 = 3yy', \quad b_6 = 3y^2 y' + v_1 (y')^2, \quad b_7 = 3(y')^2, \quad b_8 = 3y(y')^2.$$

Lemma 9.4. *The restriction map $res_K$ is isomorphic.*

*Proof.* From $Q_1x_1 = y$ and $Q_1x_3 = y'$, we see $b_1 = v_1y$, $b_3 = 3y^2 + v_1y' \ mod(v_1^2)$ in $k^*(G/T)$. From Corollary 5.4, we have the lemma. □

In $K^*(R(\mathbf{G}))$, let $\bar{b}_3 = b_3 - 3(v_1^{-1}b_1)^2$ so that we have $\bar{b}_3 = v_1y'$. We have the following graded algebra, while $\bar{b}_3 \notin CH^*(R(\mathbf{G}))$.

Lemma 9.5. *We have*

$$gr_3(K^*(R(\mathbf{G}))) = K^*(R(\mathbf{G}))/3 \oplus 3K^*(R(\mathbf{G})) \cong K^* \otimes (P(\bar{b}) \oplus \bar{B}_2) \quad where$$

$$\begin{cases} P(\bar{b}) = \mathbf{Z}_{(3)}[b_1, \bar{b}_3]/(b_1^3, \bar{b}_3^3), \\ \bar{B}_2 = \mathbf{Z}_{(3)}\{3, b_2, b_4, b_5, b_6', b_7, b_8\} \oplus \mathbf{Z}_{(3)}\{b_1b_2, b_1b_8\} \quad with \ b_6' = v_1b_6 - b_3^2. \end{cases}$$

*Proof.* By definition of $\bar{b}_3$, we see that the restriction map

$$K^*(R(\mathbf{G})) \cong K^*[b_1, \bar{b}_3]/(b_1^3, \bar{b}_3^3) \to K^*[y, y']/(y^3, (y')^3) \cong K^*(\bar{R}(\mathbf{G}))$$

is isomorphic by $res_K(b_1) = v_1y$ and $res_K(\bar{b}_3) = v_1y'$. Here we note that

$$b_2 = 3y = 3v_1^{-1}b_1, \quad b_4 = 3y' = 3v_1^{-1}\bar{b}_3, \quad b_5 = 3v_1^{-2}b_1\bar{b}_3$$
$$b_6 = 3v_1^{-3}b_1^2\bar{b}_3 + v_1^{-1}\bar{b}_3^2, \quad b_7 = 3v_1^{-2}\bar{b}_3^2, \quad b_8 = 3v_1^{-3}b_1\bar{b}_3^2.$$

Moreover we have $3b_1^2\bar{b}_3^2 = 3v_1^4(yy')^2 = v_1^3b_1b_8$.

Therefore we have the isomorphism

$$K^* \otimes (\mathbf{Z}\{b_2, b_4, b_5, b_7, b_8\} \oplus \mathbf{Z}\{b_6' = v_1b_6 - b_3^2\} \oplus \mathbf{Z}\{b_1b_8\})$$
$$\cong \tilde{K}^* \otimes (3\mathbf{Z}\{b_1, \bar{b}_3, b_1\bar{b}_3, \bar{b}_3^2, b_1\bar{b}_3^2\} \oplus 3\mathbf{Z}\{b_1^2\bar{b}_3\} \oplus 3\mathbf{Z}\{b_1^2\bar{b}_3^2\}).$$

Here we used $b_6' = 3v_1^{-2}b_1^2\bar{b}_3 \ mod(9)$, since $b_3 = \bar{b}_3 \ mod(3)$.

For the element $3b_1^2$, we note $3b_1^2 = 3v_1^2y^2 = v_1b_1b_2$. Thus we have the graded ring $gr_3(K_3^*(R(\mathbf{G})))/3$. □

Lemma 9.6. *We have the injection* (*of graded modules*)

$$\mathbf{Z}/3\{b_2, b_4, b_5, b_6, b_7, b_8\} \oplus \mathbf{Z}/3\{b_1b_2, b_1b_8, b_2^2, b_2b_8\} \subset CH^*(R(\mathbf{G}))/3.$$

*Proof.* From Corollary 3.8, we see

$$\mathbf{Z}/3\{b_1, b_2, \dots, b_8\} \subset CH^*(R(\mathbf{G}))/3.$$

We see that $t(E_8)_{(3)} = 9$ from $res_k(b_2b_8) = 9(yy')^2$ ([31]). We also see $res_k(b_1b_8) = 3v_1(yy')^2$. The fact $t(E_8) \neq 3$ implies that $b_2b_8$, $b_1b_8$ are nonzero in $CH^*(R(\mathbf{G}))/3$.

For the element $b_2^2$, we consider the restriction $res_k(b_2^2) = 9y^2$. Since $3y^2 \notin Im(res_k)$, we see $b_2^2 \neq 0 \in CH^*(R(\mathbf{G}))/3$. Since $3b_1b_2 = v_1b_2^2$, we also see $b_1b_2 \neq 0$ in $CH^*(R(\mathbf{G}))/3$. □

We will consider a filtration of $gr_3(K^*(R(\mathbf{G}))$ and its associated ring

$$gr'(K^*(R(\mathbf{G}))) = gr(gr_3(K^*(R(\mathbf{G})))),$$

which is isomorphic to $K^* \otimes B$ for some $B \subset CH^*(R(\mathbf{G}))$, to apply Lemma 5.3.

Since $b_3 = \bar{b}_3 \ mod(3)$, we see $P(\bar{b})/3 \cong P(b)/3$. So we can replace $\bar{P}(\bar{b})/3$ by $P(b)/3$ in $gr_3(K^*(R(\mathbf{G}))$. Using $3b_1b_2 = v_1b_2^2$ and $3b_1b_8 = v_1b_2b_8$, we can write

$$gr_3(K^*\{b_1b_2, b_1b_8\}) \cong K^*/3\{b_1b_2, b_1b_8\} \oplus K^*\{b_2^2, b_2b_8\}.$$

Since $b_3^2 + b_6' = v_1b_6$, we can replace $b_6'$ by $b_6$. Thus we get the another graded ring $gr'(K^*(R(\mathbf{G})))$.

Proposition 9.7. *There is a filtration whose associated graded ring is*

$$gr'K^*(R(\mathbf{G})) \cong K^* \otimes (B_1 \oplus B_2) \quad \textit{where}$$
$$\begin{cases} B_1 = P(b)/(3) \oplus \mathbf{Z}/3\{b_1b_2, b_1b_8\}, \quad P(b) = \mathbf{Z}_{(3)}[b_1, b_3]/(b_1^3, b_3^3), \\ B_2 = \mathbf{Z}_{(3)}\{3, b_2, b_4, b_5, b_6, b_7, b_8\} \oplus \mathbf{Z}_{(3)}\{b_2^2, b_2b_8\}. \end{cases}$$

If it would hold $(B_1 \oplus B_2)/3 \subset CH^*(R(\mathbf{G}))/3$, then we have $gr_\gamma(R(\mathbf{G}))/3 \cong (B_1 \oplus B_2)/3$ from Lemma 5.7. However the above supposition is not correct.

Corollary 9.8. *There is a submodule $A \subset P(b)/3$ such that*

$$gr_\gamma(R(\mathbf{G}))/3 \cong (B_1/A) \oplus A' \oplus (B_2/3)$$

*where there is a surjection $s : A \twoheadrightarrow A'$ as non-graded modules.*

*Proof.* For $0 \neq a \in P(b)$, if $a = 0$ in $CH^*(R(\mathbf{G}))$, then there is $a' \in CH^*(R(\mathbf{G}))$ such that $a = v_1^k a'$ in $k^*(R(\mathbf{G}))$ for $k > 0$. Let $s(a) = a'$. Then we have the corollary applying Lemma 5.7. □

For ease of arguments, we write $d(x) = 1/4|x|$, e.g.,

$$d(v_1) = -1, \quad d(b_1) = 1, \quad d(b_2) = 2, \quad d(b_3) = 4, \quad d(b_4) = 5$$
$$d(b_5) = 7, \quad d(b_6) = 9, \quad d(b_7) = 10, \quad d(b_8) = 12.$$

We easily see that $b_1$, $b_3$, $b_1^2$, $b_3^2$, $b_1b_3$ are nonzero in $CH^*(R(\mathbf{G}))/3$. Hence

$$A \subset \mathbf{Z}/3\{b_1^2b_3, b_1b_3^2, b_1^2b_3^2\}.$$

Lemma 9.9. *We see $b_1^2b_3 \notin A$.*

*Proof.* In $k^*(\bar{R}(\mathbf{G}))$, we have

$$b_1^2b_3 = (v_1y)^2(3y^2 + v_1y') = v_1^3(y^2y') \quad mod(I_\infty^4),$$

by using facts that $y^3 \in I_\infty$ from $y^3 = 0 \in CH^*(\bar{R}(\mathbf{G}))/3$. Suppose $b_1^2 b_3 = v_1 b$ in $k^*(R(\mathbf{G}))$ for $b \in A(b) = \mathbf{Z}[b_1, \dots, b_\ell]$. Then

$$(*) \qquad b = (v_1)^2 (y^2 y') \quad mod(I_\infty^3), \quad d(b) = 7.$$

If $b = b_k \; mod(b_i b_j)$, then by dimensional reason,

$$b_k = b_5 = 3yy' + \lambda v_1^2 (y^2 y'), \quad \lambda \in \mathbf{Z}.$$

This case, of course, does not satisfy $(*)$.

Let $b = \sum b(1)b(2)$ for $b(i) \in A(b)^+$. Since $b(i) \in I_\infty$, we see that $b(j) = 3\lambda y(i) + v_1 y(i')$ for $y(i') \neq 0 \in P(y)$. Each $b(i)$ must contain $v_1$, and it is one of $b_1$, $b_3$, $b_6$. By dimensional reason (such as $d(b) = 7$, $d(b_6) = 9$), these cases do not happen. We see (more easily) that the cases $b_1^2 b_3 = v_1^s b$ do not happen for $s \geq 2$. □

Lemma 9.10. *We have* $b_1 b_3^2 \in A$.

*Proof.* We have

$$b_1 b_3^2 = v_1 y (3y^2 + v_1 y')^2 = v_1^3 y (y')^2 \quad mod(I_\infty^4)$$

in $k^*(\bar{R}(\mathbf{G}))$, using $y^3 \in I_\infty$. Similarly we have

$$b_1 b_6 = v_1 y (3y^2 y' + v_1 (y')^2) = v_1^2 y (y')^2 \quad mod(I_\infty^3).$$

Let us write $b = b_1 b_3^2 - v_1 b_1 b_6$. Then $b \in k^* \otimes P(b)$ and $b \in I_\infty^4$. We also note $d(b) = 9$. We will prove $b \in I_\infty^2 \; Im(res_k)$. Then $b_1 b_3^2 \in A$ and $s(b_1 b_3^2) = (b_1 b_6) \in A'$ in Corollary 9.8.

First note that $b$ does not contain (as a sum of) $v_1^4 y^i (y')^j$ for $(i,j) \neq (2,2)$, because $d(b) = 9$ and $d(y^i (y')^j) \leq d(y(y')^2) = 12$. If $b$ contains $3\mu y^i (y')^j$ for $(i,j) \neq (2,2)$ and $\mu \in I_\infty^3$, then take off the element $\mu b_k$ from $b$ where $b_k = 3y_1^i (y')^j + \cdots$. Hence we only need to consider $b = \mu' (yy')^2$ and $\mu' \in I_\infty^4$. Since $d((yy')^2) = 14$, we can write, moreover,

$$b = \mu'' v_1^5 (yy')^2, \quad \mu'' \in \mathbf{Z}_{(3)}.$$

Next we see

$$(*) \qquad b' = b - \mu'' v_1^2 b_1^2 b_6 = \mu'' v_1^4 (3y^4 y') \quad mod(I_\infty^6) \quad (in\ k^*(\bar{R}(\mathbf{G})).$$

Here $y^4(y') \in I_\infty$ and hence $b' \in I_\infty^6$.

If $b' = 9v_1^4 (yy')^2$, then $b' \in Im(res_k)$. If $b' = 3v_1^5 (yy')^2$, then $b' = v_1^4 b_1 b_8 \; mod(I_\infty^7)$. Moreover $b' \in Ideal(v_1^6)$ means just zero in $k^*(R(\mathbf{G}))$, because $d(b) + 6 = 15 > d(y^2 (y')^2) = 14$. Thus we can take $b \in I_\infty^2 \; Im(res_k)$. □

Lemma 9.11. *We have*

$$A = \mathbf{Z}/3\{b_1 b_3^2, b_1^2 b_3^2\}, \quad and \quad A' = \mathbf{Z}/3\{b_1 b_6, b_1^2 b_6\}.$$

*Proof.* From the above lemma, we can take $b_1 b_2^2 = v_1 b_1 b_6$ in $k^*(\bar{R}(\mathbf{G}))$. Hence $b_1^2 b_2^2 = v_1 b_1^2 b_6$. So it is sufficient to prove that $b_1^2 b_6 \neq 0 \in CH^*(R(\mathbf{G}))/3$.

Suppose that $b_1^2 b_6 = v_1 b$. Then

$$b = v_1^2 y^2 (y')^2 \quad mod(I_\infty^3).$$

However, we can prove that such $b$ (with $d(b) = 12$) does not exist by arguments similar to the proof of Lemma 9.9. $\square$

THEOREM 9.12. *Let* $(G, p) = (E_8, 3)$ *and* $\mathbf{G}$ *be versal. For* $B_1$, $B_2$ *in Proposition* 9.7, *we have*

$$gr_\gamma(R(\mathbf{G}))/3 \cong B_1/(3, b_1 b_3^2, b_1^2 b_3^2) \oplus \mathbf{Z}/3\{b_1 b_6, b_1^2 b_6\} \oplus B_2/3.$$

## 10. The case $G = E_7$ and $p = 2$

Throughout this section, let $(G, p) = (E_7, 2)$ and $\mathbf{G}$ be versal.

THEOREM 10.1. *The cohomology* $grH^*(G; \mathbf{Z}/2)$ *is given*

$$\mathbf{Z}/2[y_6, y_{10}, y_{18}]/(y_6^2, y_{10}^2, y_{18}^2), \otimes \Lambda(z_3, z_5, z_9, z_{15}, z_{17}, z_{23}, z_{27}).$$

Hence we can rewrite

$$grH^*(G; \mathbf{Z}/2) \cong \mathbf{Z}/2[y_1, y_2, y_3]/(y_1^2, y_2^2, y_3^2) \otimes \Lambda(x_1, \dots, x_7).$$

LEMMA 10.2. *The cohomology operations act as*

$$x_1 = z_3 \xrightarrow{Sq^2} x_2 = z_5 \xrightarrow{Sq^4} x_3 = z_9 \xrightarrow{Sq^8} x_4 = z_{17}$$
$$x_5 = z_{15} \xrightarrow{Sq^8} x_6 = z_{23} \xrightarrow{Sq^4} x_7 = z_{27}$$
$$x_5 = z_{15} \xrightarrow{Sq^2} x_4 = z_{17}$$

*The Bockstein acts* $Sq^1(x_{i+1}) = y_i$ *for* $1 \le i \le 3$, *and*

$$Sq^1\colon\ x_5 = z_{15} \mapsto y_1 y_2, \quad x_6 = z_{23} \mapsto y_1 y_3, \quad x_7 = z_{27} \mapsto y_2 y_3.$$

LEMMA 10.3 ([44]). *In* $H^*(G/T)/(4)$, *for all monomials* $u \in P(y)^+/2$, *except for* $y_{top} = y_1 y_2 y_3$, *the elements* $2u$ *are written as elements in* $H^*(BT)$. *Moreover, in* $BP^*(G/T)/I_\infty^2$, *there are* $b_i \in BP^*(BT)$ *such that*

$$b_2 = 2y_1, \quad b_3 = 2y_2, \quad b_4 = 2y_3, \quad b_6 = 2y_1 y_3, \quad b_7 = 2y_2 y_3,$$
$$b_1 = v_1 y_1 + v_2 y_2 + v_3 y_3, \quad b_5 = 2y_1 y_2 + v_1 y_3.$$

*Proof.* The last two equations are given by $Q_1(x_1) = y_1$, $Q_2(x_1) = y_2$,

$$Q_3(x_1) = y_3, \quad Q_1(x_5) = y_3, \quad Q_0(x_5) = y_1 y_2. \qquad \square$$

LEMMA 10.4 ([31]). *We have* $t(E_7)_{(2)} = 2^2$.

*Proof.* We get the result from $b_2 b_7 = (2y_1)(2y_2 y_3) = 2^2 y_{top}$. $\square$

COROLLARY 10.5. *Elements $b_1b_6$, $b_1b_7$, $b_2b_7$ are all nonzero in $CH^*(R(\mathbf{G}))/2$.*

*Proof.* Note that $|y_1y_2y_3| = 34$. In $\Omega^*(\bar{\mathbf{F}})/I_\infty^3$, we see

$$b_1b_5 = 2v_3y_{top}, \quad b_1b_6 = 2v_2y_{top}, \quad b_1b_7 = 2v_1y_{top}.$$

These elements are $\Omega^*$-module generators in $Im(res_k^{\bar{k}}(\Omega^*(\mathbf{F})) \to \Omega(\bar{\mathbf{F}}))$ because $2y_1y_2y_3 \notin Im(res_k^{\bar{k}})$ from the fact $t(\mathbf{G}) = 2^2$. □

By Chevalley's theorem, $res_K$ is surjective. Hence in $K^*(\bar{R}(\mathbf{G}))/2$,

$$v_1^s y_2 = b \in \mathbf{Z}/2[b_1, \dots, b_\ell] \quad \textit{some } s \in \mathbf{Z}.$$

Since $y_2 \notin Im(res_{K/2})$, we see $s \geq 1$. The facts that $|b_i| \geq 6$ and $|y_2| = 10$ imply $s = 2$.

LEMMA 10.6. *We can take $b_2 = 2y_1 + v_1^2y_2$ in $k^*(G/T)/(v_1^3)$.*

COROLLARY 10.7. *The restriction map $res_K$ is isomorphic, i.e.,*

$$K^*(R(\mathbf{G})) \cong K^* \otimes \Lambda_{\mathbf{Z}}(b_1, b_2, b_5).$$

*Proof.* Recall that $K^*(\bar{R}(\mathbf{G})) \cong K^* \otimes P(y) \cong K^* \otimes \Lambda_{\mathbf{Z}}(y_1, y_2, y_3)$. The corollary follows from the $mod(2)$ restriction map $res_{K/2}$, which is given as $b_1 \mapsto v_1y_1$, $b_2 \mapsto v_1^2y_2$, $b_5 \mapsto v_1y_3$. □

Here we give a direct proof (without using Chevalley's theorem) of the above lemma. For this, we use $k(1)^*$ theory (with the coefficients ring $k(1)^* = k^*/p = \mathbf{Z}/p[v_1]$) of the Eilenberg-MacLane space $K(\mathbf{Z}, 3)$. It is well known for $p$ odd (e.g., see [1])

$$H^*(K(\mathbf{Z},3); \mathbf{Z}/p) \cong \mathbf{Z}/p[y_1, y_2, \dots] \otimes \Lambda(x_1, x_2, \dots)$$

where $P^{p^{n-1}} \cdots P^1(x_1) = x_n$ and $\beta x_{n+1} = y_n$. When $p = 2$, some graded ring $grH^*(K(\mathbf{Z},3); \mathbf{Z}/2)$ is isomorphic to the right hand side of the above isomorphism.

LEMMA 10.8 (Theorem 3.4.4. (2) in [39]). *We can take $v_1^py_2 = 0$ in $grk(1)^*(K(\mathbf{Z},3))$.*

*Proof.* We consider the AHss

$$E_2^{*,*'} \cong H^*(K(\mathbf{Z},3); \mathbf{Z}/p) \otimes k(1)^* \Rightarrow k(1)^*(K(\mathbf{Z},3)).$$

It is known that all $y_n = Q_n(x_1)$ are permanent cycles. The first nonzero differential $d_{2p-1} = v_1 \otimes Q_1$, is given as

$$x_1 \mapsto v_1y_1, \quad x_2 \mapsto 0, \quad x_3 \mapsto v_1y_1^p, \quad x_4 \mapsto v_1y_2^p, \dots$$

(e.g., $Q_1(x_{i+2}) = y_i^p$). Hence $x_2$ and $y_1, y_2, \dots$ exist in $E_{2p}^{*,0}$.

On the other hand, from Anderson-Hodgkin [1], it is well known

$$k(1)^*(K(\mathbf{Z};n))[v_1^{-1}] = K(1)^*(K(\mathbf{Z};n)) \cong \mathbf{Z}/p \quad \text{for all } n \geq 2.$$

Hence there is $s \geq 0$ such that $v_1^s y_2 \in Im(d_r)$ for some $r$. By dimensional reason, we have $d_{2(p-1)p+1}(x_2) = v_1^p y_2$. □

*Proof of Lemma* 10.6. The element $x_1 \in H^3(E_7;\mathbf{Z})$ represents the map $i : E_7 \to K(\mathbf{Z};3)$ such that $i^*(x_1) = x_1$ and $i^*(y_i) = y_i$. Hence $v_1^2 y_2 = 0$ in $grk(1)^*(E_7)$.

On the other hand,

$$k(1)^*(E_7) \cong k(1)^*(E_7/T)/(Ideal(S(t)^+)).$$

Hence we can write $v_1^2 y_2 = ty_1 + t'$ for $t, t' \in S(t)^+$. Since $|v_1^2 y_2| = |y_1| = 6$, we see $t = 0$. Thus we get $v_1^2 y_2 \in k(1)^*(BT)$. □

Proposition 10.9. *There is a filtration whose associated graded ring is*

$$grK^*(R(\mathbf{G})) \cong K^* \otimes (B_1 \oplus B_2) \quad \textit{where}$$

$$\begin{cases} B_1 = P(b)/2 \oplus \mathbf{Z}/2\{b_1b_7\}, & \textit{where } P(b) = \Lambda_{\mathbf{Z}}(b_1, b_2, b_5) \\ B_2 = \mathbf{Z}\{2, 2b_1, b_3, b_4, b_1b_3, b_6, b_7, b_2b_7\} \end{cases}$$

*Proof.* Recall that

$$K^*(R(\mathbf{G})) \cong K^* \otimes P(b) \cong K^* \otimes P(y) \cong K^*(\bar{R}(\mathbf{G})).$$

Let us write $\bar{b}_2 = b_2 - 2v_1^{-1}b_1$ and $\bar{b}_5 = b_5 - v_1^{-1}b_1b_3$ so that $\bar{b}_2 = v_1^2 y_2$, $\bar{b}_5 = v_1 y_3$ in $K^*(\bar{R}(\mathbf{G}))$. Let us write $P(\bar{b}) = \Lambda(b_1, \bar{b}_2, \bar{b}_5)$.

We consider $gr_2K^*(R(\mathbf{G})) = K^*(R(\mathbf{G}))/2 \oplus 2K^*(R(\mathbf{G}))$. We have

$$\begin{aligned} K^*(R(\mathbf{G})) &\supset 2K^* \otimes P(b) = 2K^* \otimes P(\bar{b}) \\ &= 2K^*\{1, b_1, \bar{b}_2, \dots, \bar{b}_2\bar{b}_3, b_1\bar{b}_2\bar{b}_3\} = 2K^*\{1, y_1, y_2, \dots, y_1y_2y_3\} \\ &= K^*\{2, 2b_1, b_3, b_4, b_1b_3, b_6, b_7, b_1b_7\}. \end{aligned}$$

Here we used $2v_1y_1 = 2b_1$, $2y_2 = b_3$, $2y_3 = b_4$, $2v_1y_1y_2 = b_1b_3$, and

$$2y_1y_3 = b_6, \quad 2y_2y_3 = b_7, \quad 2v_1y_1y_2y_3 = b_1b_7.$$

The equation $2b_1b_7 = 4v_1y_1y_2y_3 = v_1b_2b_7$ implies that

$$grK^*(R(\mathbf{G})) = gr_2K^*(R(\mathbf{G}))/(2b_1b_7) \oplus \mathbf{Z}\{b_2b_7\},$$

which gives the graded ring in this lemma. □

We can not get $gr_\gamma(R(\mathbf{G}))$ here, while we can see the following proposition.

Proposition 10.10. *Let* $(G, p) = (E_7, 2)$ *and* $\mathbf{G}$ *be versal. Then there is an injection* (*of graded modules*)

$$B_1/(b_2b_5, b_1b_2b_5) \oplus B_2/(2, 2b_1) \subset gr_\gamma(R(\mathbf{G}))/2.$$

*Proof.* Here we only give a proof that $b_1b_2$ in $B_1$ is non zero in $CH^*(R(\mathbf{G}))/2$. The other cases (e.g. $b_1b_5$) are proved similarly. We see

$$b_1b_2 = v_1y_1(2y_1 + v_1^2y_2) = v_1^3y_1y_2 \quad mod(2v_1I_\infty, I_\infty^4).$$

Suppose $b_1b_2 = v_1b$. Then we see $b = v_1^2y_1y_2 \; mod(2I_\infty, v_1^3)$. We can show that this does not happen, as the proof of Lemma 9.9. (For example, if $b = b_k \; mod(b_ib_j)$, then $b = b_3$ and this is a contradiction since $b_3$ contains $2y_2$.) □

## 11. The case $G = E_8$ and $p = 2$

It is known ([17]) that

$$grH^*(E_8; \mathbf{Z}/2) \cong \mathbf{Z}/2[y_1, y_2, y_3, y_4]/(y_1^8, y_2^4, y_3^2, y_4^2) \otimes \Lambda(x_1, \ldots, x_8)$$

so that $grH^*(E_7, \mathbf{Z}/2) \cong grH^*(E_8, \mathbf{Z}/2)/(y_1^2, y_2^2, y_4, x_8)$.

Here $Sq^2(x_7) = x_8$, $\beta(x_8) = y_4$. Hence

$$Q_1(x_7) = Q_0(x_8) = y_4, \quad |y_4| = 30.$$

PROPOSITION 11.1. *Let $(G, p) = (E_8, 2)$ and $\mathbf{G}$ be the versal $G_k$-torsor. The restriction map $res_K$ is isomorphic. In fact, for the transgressive elements $b_1$, $b_2$, $b_5$, $b_7$ in $K^*(\mathbf{G}/B_k)/2$, we have*

$$K^*(R(\mathbf{G}))/2 \cong K^*/2[b_1, b_2, b_5, b_7]/(b_1^8, b_2^4, b_5^2, b_7^2).$$

*Proof.* From the case $E_7$, we have $b_2 = v_1^2y_2$ in $K^*(\bar{R}(\mathbf{G}))/2$. We also see $b_1 = v_1y_1$, $b_5 = v_1y_3$, $b_7 = v_1y_4$ in $K^*(\bar{R}(\mathbf{G}))/2$. □

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