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VOJTA'S CONJECTURE, SINGULARITIES AND MULTIPLIER-TYPE IDEALS

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Abstract

We formulate a generalization of Vojta's conjecture in terms of log pairs and variants of multiplier ideals.

1. Introduction

Vojta's famous conjecture in Diophantine geometry was originally stated for a smooth projective variety X over a number field and a simple normal crossing divisor D on X. This conjecture is important, since it provides a general framework unifying various deep arithmetic results and conjectures; for instance, in dimension one, the conjecture incorporates celebrated finiteness results of Roth, Siegel and Faltings. In [8], Vojta generalized it further, replacing D with an arbitrary proper closed subscheme $Z \subset X$. This generalized conjecture says that if A is a big divisor on X and $\varepsilon > 0$ is a real number, then all rational points x in an open dense subset of X satisfies the inequality

$$h_{K_X}(x) + m_Z(x) - m_{\mathscr{I}^-(X,Z)}(x) \le \varepsilon h_A(x) + O(1).$$

Here $\mathscr{I}^-(X,Z)$ is the multiplier ideal sheaf of the pair $(X,(1-\delta)Z)$ for $0 < \delta \ll 1$.

In this paper, we further generalize Vojta's conjecture from the viewpoint of the minimal model program, allowing X to have (not necessarily normal) **Q**-Gorenstein singularities (Conjecture 5.2). In this generalization, the inequality becomes

$$h_{K_X+Z}(x) - N_{\mathscr{H}(X,Z)}(x) - m_{\mathscr{I}^-(X,Z)}(x) \le \varepsilon h_A(x) + O(1).$$

Here $\mathscr{H}(X,Z)$ is a new variant of the multiplier ideal sheaf which we will introduce. It turns out that the new conjecture is equivalent to the original one. In the last inequality, the first term $h_{K_X+Z}(x)$ is clearly the contribution of the

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canonical "divisor" $K_X + Z$. The second term $N_{\mathscr{H}(X,Z)}(x)$ and the third term $m_{\mathscr{I}^-(X,Z)}(x)$ should be viewed as the contribution of singularities of the log pair (X,Z).

When X is normal, the ideal sheaf $\mathscr{H}(X,Z)$ is defined as $f_*\mathcal{O}_Y(\lfloor K_{Y/X} - f^*Z\rfloor)$ for a log resolution $f : Y \to X$ with $\lfloor\cdot\rfloor$ denoting the round down, while the usual multiplier ideal uses the round up. The author finds it interesting that Diophantine geometry leads us to this new sheaf. The sheaf would be of independent interest and should be studied further from the purely algebro-geometric viewpoint.

Our generalization of Vojta's conjecture has some interesting consequences as the original conjecture does. We can derive a slight generalization of a conjecture of Lang and Vojta on the non-density of integral points (Proposition 6.3). Another consequence, which is more geometric, roughly says that given a log pair (X,Z), most of the closed subsets $Y \subset X$ having dense rational points intersect the locus of bad singularities of (X,Z) (Proposition 6.4).

This paper is a substantially changed and shortened version of the preprint [9]. The outline of the paper is as follows. In Section 2 we recall basic facts on singularities of log pairs. Section 3 is devoted to define the sheaves $\mathscr{I}^-(X,Z)$ and $\mathscr{H}(X,Z)$ and to prove their basic properties. In Section 4, we recall Weil functions, height functions, counting functions and proximity functions. Section 5 is the main part of the paper. Here we formulate the generalization of Vojta's conjecture and prove that it is equivalent to the original conjecture. In Section 6, we derive two consequences of our generalization of Vojta's conjecture.

Throughout the paper, we fix a number field k. A *variety* means a separated integral scheme of finite type over k. We suppose that every morphism of varieties is a morphism of k-schemes. From Section 4 on, we fix a finite set S of places of k which includes all the infinite places.

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2. Singularities of log pairs

DEFINITION 2.1. A variety X is said to be $\mathbf{Q}$-*Gorenstein* if
(1) X satisfies Serre's condition S_2,
(2) X is Gorenstein in codimension one, and
(3) a canonical divisor K_X is $\mathbf{Q}$-Cartier.

For instance, a variety having only local complete intersection singularities is Gorenstein, hence $\mathbf{Q}$-Gorenstein. From the first two conditions, which are automatic if X is normal, a canonical divisor K_X exists, is unique up to linear equivalence and is Cartier in codimension one (see [1, Def. 1.6]). Therefore the last condition makes sense and is equivalent to that for some $m \in \mathbf{Z}_{>0}$, the reflexive power $\omega_X^{[m]} := (\omega_X^{\otimes m})^{**}$ of the canonical sheaf ω_X is invertible.

Definition 2.2. A **Q**-*subscheme* of a variety X is a formal linear combination $Z = \sum_{i=1}^{n} c_i Z_i$ of proper closed subschemes $Z_i \subsetneq X$ with $c_i \in \mathbf{Q}$. The *support* of Z, denoted by $\mathrm{Supp}(Z)$, is defined to be the closed subset $\bigcup_{c_i \neq 0} Z_i$. We say that a **Q**-subscheme $\sum_{i=1}^{n} c_i Z_i$ is effective if $c_i \geq 0$ for every i. A *log pair* is the pair (X, Z) of a **Q**-Gorenstein variety X and an effective **Q**-subscheme Z of X.

Remark 2.3. One may introduce the following equivalence relation on **Q**-subschemes: two **Q**-subschemes $\sum_i c_i Z_i$ and $\sum_j c'_j Z'_j$ of the same variety X are *equivalent* if there exists a positive integer r such that rc_i and rc'_j are all integers and the genuine subschemes $\sum_i rc_i Z_i$ and $\sum_j rc'_j Z'_j$ are identical, where we mean by $rc_i Z_i$ the closed subscheme defined by the rc_i-th power of the defining ideal sheaf of Z_i and by $\sum_i rc_i Z_i$ the closed subscheme defined by the product of the ideal sheafs of $rc_i Z_i$ and we define $\sum_j rc'_j Z'_j$ similarly. Replacing the given **Q**-subscheme Z with an equivalent one does not change classes of singularities to which the log pair (X, Z) belong or does not change multiplier-type ideals discussed in the next section (cf. Remark 2.7).

Remark 2.4. If X is a normal **Q**-Gorenstein variety and D is an effective **Q**-Cartier **Q**-Weil divisor, then D is written as bE with E an effective Cartier divisor and $b \in \mathbf{Q}_{\geq 0}$; this allows us to regard the pair (X, D) as a log pair in the sense defined above. The log pair defined in this way is unique modulo the equivalence relation in the last remark.

Definition 2.5. A *resolution* of a variety X is a projective birational morphism $f : Y \to X$ such that Y is smooth over k. Let $(X, Z = \sum_{i=1}^{n} c_i Z_i)$ be a log pair. A *log resolution* of (X, Z) is a resolution $f : Y \to X$ of X such that

(1) for every i, the scheme-theoretic inverse image $f^{-1}(Z_i)$ is a Cartier divisor (that is, if $\mathscr{I}_{Z_i}$ is the defining ideal sheaf of Z_i, then the pull-back $f^{-1}\mathscr{I}_{Z_i}$ as an ideal sheaf is locally principal),
(2) if we denote by $\mathrm{Exc}(f)$ the exceptional set of f, then the closed subset $\mathrm{Exc}(f) \cup \bigcup_{i=1}^{n} f^{-1}(Z_i)_{\mathrm{red}}$ of Y is a simple normal crossing divisor.

From Hironaka's theorem, every variety has a resolution and every log pair has a log resolution.

Definition 2.6. For a **Q**-Gorenstein variety X and a resolution $f : Y \to X$ of X, the *relative canonical divisor* $K_{Y/X}$ *of* Y *over* X is defined as a **Q**-Weil divisor of Y supported in $\mathrm{Exc}(f)$ as follows. If m is a positive integer such that $\omega_X^{[m]}$ is invertible, then the image of the natural morphism $f^*\omega_X^{[m]} \to \omega_Y^{\otimes m} \otimes_{\mathscr{O}_Y} K(Y)$ is written as $\omega_Y^{\otimes m}(\Delta)$ for some (**Z**-)divisor Δ. We define

$$K_{Y/X} := -\frac{1}{m}\Delta.$$

For a log resolution $f : Y \to X$ of a log pair (X, Z), we define the *relative canonical divisor* $K_{Y/(X,Z)}$ *of* Y *over* (X, Z) as the $\mathbf{Q}$-Weil divisor $K_{Y/X} - f^*Z$. Here, if we write $Z = \sum_{i=1}^{l} c_i Z_i$, then we define the *pull-back* f^*Z as the $\mathbf{Q}$-Weil divisor

Remark 2.7. If Z' is another $\mathbf{Q}$-subscheme equivalent to Z and $f : Y \to X$ is a log resolution of both (X, Z) and (X, Z'), then the $\mathbf{Q}$-Weil divisors f^*Z and f^*Z' are identical, and so are $K_{Y/(X,Z)}$ and $K_{Y/(X,Z')}$.

Definition 2.8. Let (X, Z) be a log pair, let $f : Y \to X$ be a log resolution of it and let us write

$$K_{Y/(X,D)} = \sum_F a_F \cdot F,$$

F running over all prime divisors of Y. We say that (X, Z) is *strongly canonical*[1] (resp. *Kawamata log terminal*, *log canonical*) if $a_F \geq 0$ (resp. $a_F > -1$, $a_F \geq -1$) for every F.

As is well-known, these notions are independent of the choice of a log resolution. These are also local; a log pair (X, Z) is strongly canonical (resp. Kawamata log terminal, log canonical) if and only if every point $x \in X$ has an open neighborhood $U \subset X$ such that $(U, Z|_U)$ is so. This is just because for an open subset $U \subset X$ and a log resolution $f : Y \to X$ of (X, Z), we have $K_{f^{-1}(U)/(U, Z|_U)} = K_{Y/(X,Z)}|_{f^{-1}(U)}$. We define the *non-sc locus* (resp. *non-klt locus*, *non-lc locus*) of (X, Z) to be the smallest closed subset $W \subset X$ such that $(X \backslash W, Z|_{X \backslash W})$ is strongly canonical (resp. Kawamata log terminal, log canonical). We write it as $\mathrm{NonSC}(X, Z)$ (resp. $\mathrm{NonKLT}(X, Z)$, $\mathrm{NonLC}(X, Z)$). Clearly we have

$$\mathrm{NonLC}(X, Z) \subset \mathrm{NonKLT}(X, Z) \subset \mathrm{NonSC}(X, Z). \tag{2.1}$$

3. Multiplier-type ideal sheaves

In this section, we define two variants $\mathscr{I}^-$ and $\mathscr{H}$ of multiplier ideals, which will be necessary to formulate our generalization of Vojta's conjecture. We denote by (X, Z) a log pair throughout the section.

When X is normal, the multiplier ideal sheaf $\mathscr{I}(X, Z)$ is usually defined to be $f_*\mathcal{O}_Y(\lceil K_{Y/(X,Z)} \rceil)$ for a log resolution $f : Y \to X$ of (X, Z) (see [4, Def. 9.3.56]). Here $\lceil \cdot \rceil$ denotes the round up of a $\mathbf{Q}$-Weil divisor, that is, for a $\mathbf{Q}$-Weil divisor $E = \sum_i c_i E_i$ with $c_i \in \mathbf{Q}$ and E_i prime divisors, we define $\lceil E \rceil :=$

[1] A log pair (X, Z) is said to be *canonical* if $a_F \geq 0$ for every *exceptional* prime divisor F of f (see [1, Def. 2.8]).

$\sum_i \lceil c_i \rceil E_i$. The round down $\lfloor E \rfloor$ is similarly defined and will be used below. When X is not normal, $f_* \mathcal{O}_Y(\lceil K_{Y/(X,Z)} \rceil)$ is no longer an ideal sheaf. To handle this trouble, we replace f_* with $f_\clubsuit$ defined as follows: for a resolution $f : Y \to X$ and a divisor E of Y (with $\mathbf{Z}$-coefficients), we define $f_\clubsuit \mathcal{O}_Y(E)$ as the largest ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ such that the ideal pull-back $f^{-1}\mathcal{I}$ is contained in $\mathcal{O}_Y(E)$ as an $\mathcal{O}_Y$-submodule of the function field constant sheaf. We now define the *multiplier ideal sheaf* $\mathcal{I}(X,Z)$ of a log pair (X,Z) to be $f_\clubsuit \mathcal{O}_Y(\lceil K_{Y/(X,Z)} \rceil)$ for a log resolution $f : Y \to X$ of (X,Z).

Lemma 3.1. *The ideal sheaf $\mathcal{I}(X,Z)$ defined above is independent of the choice of a log resolution f.*

Proof. This is well-known, when X is smooth. In the general case, the lemma follows from the following two facts. The first one is that if $g : Y' \to Y$ and $f : Y \to X$ are projective birational morphisms such that Y and Y' are smooth, then for a divisor E on Z, we have $f_\clubsuit(g_* \mathcal{O}_{Y'}(E)) = (f \circ g)_\clubsuit \mathcal{O}_{Y'}(E)$. The second one is that if f and $f \circ g$ as above are log resolutions of (X,Z), then $g_* \mathcal{O}_{Y'}(\lceil K_{Y'/(X,Z)} \rceil) = \mathcal{O}_Y(\lceil K_{Y/(X,Z)} \rceil)$, which follows from [4, Lem. 9.2.19]. $\square$

It is easy to see that there exists $\delta_0 > 0$ such that for every $\delta \in \mathbf{Q}$ with $0 < \delta \le \delta_0$, we have $\mathcal{I}(X,(1-\delta)Z) = \mathcal{I}(X,(1-\delta_0)Z)$. We define

$$\mathcal{I}^-(X,Z) := \mathcal{I}(X,(1-\delta)Z) \quad (0 < \delta \ll 1).$$

Proposition 3.2 (cf. [4, Def. 9.3.9]). *We have*

$$\mathrm{NonLC}(X,Z) \subset \mathrm{Supp}(\mathcal{O}_X/\mathcal{I}^-(X,Z)) \subset \mathrm{NonKLT}(X,Z). \tag{3.1}$$

Moreover, if $(X \backslash \mathrm{Supp}(Z), 0)$ *is Kawamata log terminal, then*

$$\mathrm{NonLC}(X,Z) = \mathrm{Supp}(\mathcal{O}_X/\mathcal{I}^-(X,Z)). \tag{3.2}$$

Proof. Let $f : Y \to X$ be a log resolution of (X,Z). For $0 < \delta \ll 1$ and a prime divisor F of Y, we have

$$\mathrm{mult}_F(K_{Y/(X,Z)}) < -1 \Rightarrow \mathrm{mult}_F(\lceil K_{Y/(X,Z)} + \delta f^*Z \rceil) < 0 \Rightarrow \mathrm{mult}_F(K_{Y/(X,Z)}) \le -1.$$

Here $\mathrm{mult}_F(E)$ denotes the multiplicity of F in the divisor E. This shows the first assertion.

To show the second assertion, it suffices to show that if $(X \backslash \mathrm{Supp}(Z), 0)$ is Kawamata log terminal, then

$$\mathrm{mult}_F(K_{Y/(X,Z)}) \ge -1 \Rightarrow \mathrm{mult}_F(\lceil K_{Y/(X,Z)} + \delta f^*Z \rceil) \ge 0.$$

If $\mathrm{mult}_F(K_{Y/(X,Z)}) > -1$, then we obviously have $\mathrm{mult}_F(\lceil K_{Y/(X,Z)} + \delta f^*Z \rceil) \ge 0$. If $\mathrm{mult}_F(K_{Y/(X,Z)}) = -1$, then, since $(X \backslash \mathrm{Supp}(Z), 0)$ is Kawamata log terminal, F is contained in $\mathrm{Supp}(f^*Z)$. Hence $\mathrm{mult}_F(K_{Y/(X,Z)} + \delta f^*Z) > -1$ and $\mathrm{mult}_F(\lceil K_{Y/(X,Z)} + \delta f^*Z \rceil) \ge 0$. $\square$

LEMMA 3.3. *Let $f : Y \to X$ be a projective birational morphism of smooth varieties and E a (not necessarily effective) $\mathbf{Q}$-Weil divisor on X. Then*

$$f_*\mathcal{O}_Y(\lfloor K_{Y/X} + f^*E \rfloor) = \mathcal{O}_X(\lfloor E \rfloor).$$

Proof. First suppose that $\lfloor E \rfloor = 0$. To show the lemma in this case, it suffices to show that $\lfloor K_{Y/X} + f^*E \rfloor$ is an effective divisor supported in $\mathrm{Exc}(f)$. Since $K_{Y/X}$ and E are effective, so is $\lfloor K_{Y/X} + f^*E \rfloor$. On the locus where f is an isomorphism, the two divisors $\lfloor K_{Y/X} + f^*E \rfloor$ and $\lfloor E \rfloor$ coincide, the latter being zero by the assumption. This proves the lemma in this case.

For the general case, we write $\{E\} := E - \lfloor E \rfloor$. Obviously, $\lfloor \{E\} \rfloor = 0$. From the projection formula and the case considered above, we have

$$\begin{aligned} f_*\mathcal{O}_Y(\lfloor K_{Y/X} + f^*E \rfloor) &= f_*\mathcal{O}_Y(\lfloor K_{Y/X} + f^*\{E\} + f^*\lfloor E \rfloor \rfloor) \\ &= f_*\mathcal{O}_Y(\lfloor K_{Y/X} + f^*\{E\} \rfloor + f^*\lfloor E \rfloor) \\ &= f_*(\mathcal{O}_Y(\lfloor K_{Y/X} + f^*\{E\} \rfloor) \otimes_{\mathcal{O}_Y} f^*\mathcal{O}_X(\lfloor E \rfloor)) \\ &= \mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(\lfloor E \rfloor) \\ &= \mathcal{O}_X(\lfloor E \rfloor). \end{aligned}$$ □

PROPOSITION 3.4. *For a log pair (X, Z), the ideal sheaf $f_\clubsuit\mathcal{O}_Y(\lfloor K_{Y/(X,Z)} \rfloor)$ is independent of a log resolution $f : Y \to X$ of (X, Z).*

Proof. Let $f : Y \to X$ and $f' : Y' \to X$ be log resolutions of (X, Z). Without loss of generality, we may suppose that f' factors as $Y' \xrightarrow{g} Y \xrightarrow{f} X$. Then

$$K_{Y'/(X,Z)} = K_{Y'/Y} + g^*K_{Y/(X,Z)}.$$

From the above lemma,

$$\begin{aligned} f'_\clubsuit\mathcal{O}_{Y'}(\lfloor K_{Y'/(X,Z)} \rfloor) &= f_\clubsuit(g_*\mathcal{O}_{Y'}(\lfloor K_{Y'/Y} + g^*K_{Y/(X,Z)} \rfloor)) \\ &= f_\clubsuit\mathcal{O}_Y(\lfloor K_{Y/(X,Z)} \rfloor). \end{aligned}$$ □

For a log pair (X, Z), taking a log resolution $f : Y \to X$ of (X, Z), we define an ideal sheaf $\mathcal{H}(X, Z)$ on X as

$$\mathcal{H}(X, Z) := f_\clubsuit\mathcal{O}_Y(\lfloor K_{Y/(X,Z)} \rfloor).$$

Remark 3.5. As far as the definition of $\mathcal{H}(X, Z)$ is concerned, we do not need the "simple normal crossing" assumption in the definition of log resolutions.

PROPOSITION 3.6. *For a log pair (X, Z), we have*

$$\mathrm{Supp}(\mathcal{O}_X / \mathcal{H}(X, Z)) = \mathrm{NonSC}(X, Z).$$

Proof. Let $f : Y \to X$ be a log resolution of (X, Z). If (X, Z) is strongly canonical, then $K_{Y/(X,Z)} \geq 0$ and $\lfloor K_{Y/(X,Z)} \rfloor \geq 0$. By the definition of $f_\clubsuit$, we have

$$\mathscr{H}(X, Z) = f_\clubsuit \mathscr{O}_Y(\lfloor K_{Y/(X,Z)} \rfloor) = \mathscr{O}_X.$$

This shows $\mathrm{Supp}(\mathscr{O}_X/\mathscr{H}(X, Z)) \subset \mathrm{NonSC}(X, Z)$.

If (X, Z) is not strongly canonical around $x \in X$, then there exists a prime divisor F on Y such that $x \in f(F)$ and $\mathrm{mult}_F(K_{Y/(X,Z)}) < 0$. Therefore,

$$\mathscr{O}_Y \not\subset \mathscr{O}_Y(\lfloor K_{Y/(X,Z)} \rfloor).$$

This remains true even if we replace X with any open neighborhood of x. This shows $\mathrm{Supp}(\mathscr{O}_X/\mathscr{H}(X, Z)) \supset \mathrm{NonSC}(X, Z)$. □

Corollary 3.7. *If (X, Z) is log canonical, then $\mathscr{H}(X, Z)$ is the defining ideal sheaf of* $\mathrm{NonSC}(X, Z)$ *regarded as the reduced closed subscheme, that is, $\mathscr{O}_X/\mathscr{H}(X, Z)$ is reduced.*

Proof. Let $\mathscr{N}$ be the defining ideal of $\mathrm{NonSC}(X, Z)$. From Proposition 3.6, we have $\mathscr{H} \subset \mathscr{N}$. To see the opposite inclusion, let $U \subset X$ be an open subset and $g \in \mathscr{N}(U)$. For a log resolution $f : Y \to X$ of (X, Z), f^*g vanishes along the closed set $f^{-1}(\mathrm{NonSC}(X, Z))$. The last set contains every prime divisor F on Y having a negative coefficient in $\lfloor K_{Y/(X,Z)} \rfloor$, which is equal to -1 since (X, Z) is log canonical. Therefore, $f^*g \in \mathscr{O}_Y(\lfloor K_{Y/(X,Z)} \rfloor)(f^{-1}U)$ and hence $g \in \mathscr{H}(U)$ and $\mathscr{N} \subset \mathscr{H}$. □

4. Weil functions

We denote by M_k the set of places of k. From now on, we fix a finite set $S \subset M_k$ containing all infinite places. Let X be a projective variety over k. To a proper closed subscheme $Z \subset X$, we associate a Weil function

$$\lambda_Z : X(k) \times M_k \to [0, +\infty],$$

following [5], which is unique up to addition of M_k-bounded functions. Weil functions have the following properties [5, Th. 2.1]:

(1) For a morphism $f : Y \to X$ of projective varieties and a proper closed subscheme $Z \subset X$, we have $\lambda_Z \circ f = \lambda_{f^{-1}Z}$.
(2) For $Z \subset Z' \subset X$, we have $\lambda_Z \leq \lambda_{Z'}$.
(3) For proper closed subschemes $Z, Z' \subset X$, we have $\lambda_{Z+Z'} = \lambda_Z + \lambda_{Z'}$. Here $Z + Z'$ is the closed subscheme defined by the product of the defining ideals of Z and Z'.

Here comparisons of Weil functions are made up to addition of M_k-bounded functions. When Z is an effective Cartier divisor locally defined by a regular function f, then a Weil function λ_Z of Z should be locally of the form

$$\lambda_Z(x, v) = -\log\|f(x)\|_v + \alpha(x)$$

for a locally M_k-bounded function α. Here, if p is the place of $\mathbf{Q}$ with $v|p$ and $|\cdot|_p$ denotes the p-adic absolute value, then the norm $\|\cdot\|_v$ is defined by $\|a\|_v := |N_{k_v/\mathbf{Q}_p}(a)|_p$.

Definition 4.1. For a proper closed subscheme $Z \subset X$, we define the *height function* h_Z, the *counting function* $N_Z = N_{Z,S}$ and the *proximity function* $m_Z = m_{Z,S}$ on $X(k)$ relative to λ and k by

$$h_Z(x) := \sum_{v \in M_k} \lambda_Z(x,v), \quad N_Z(x) := \sum_{v \in M_k \setminus S} \lambda_Z(x,v), \quad m_Z(x) := \sum_{w \in S} \lambda_Z(x,v).$$

These are functions on $X(k)$ with values in $[0,+\infty]$ and taking value $+\infty$ exactly on $Z(k)$. When $\mathscr{I}$ is the defining ideal sheaf of Z, we denote these functions also by $h_{\mathscr{I}}$, $N_{\mathscr{I}}$ and $m_{\mathscr{I}}$ respectively. If $Z = \sum_i c_i Z_i$ is a $\mathbf{Q}$-subscheme, then we define

$$h_Z := \sum_i c_i h_{Z_i}, \quad N_Z := \sum_i c_i N_{Z_i}, \quad m_Z := \sum_i c_i m_{Z_i}$$

as $\mathbf{R}$-valued functions on $(X \setminus \mathrm{Supp}(Z))(k)$.

For a (not necessarily effective) Cartier divisor D, the height function h_D on $(X \setminus \mathrm{Supp}(D))(k)$ defined as above extends to an $\mathbf{R}$-valued function on the whole set $X(k)$ and defines a unique function h_D up to addition of bounded functions. The function class h_D modulo bounded functions depends only on the linear equivalence class of D. Furthermore, we can easily generalize this to $\mathbf{Q}$-Cartier $\mathbf{Q}$-Weil divisors; if D is a $\mathbf{Q}$-Cartier $\mathbf{Q}$-Weil divisor and if n is a positive integer such that nD is Cartier, then a height function h_D is defined as $\frac{1}{n} h_{nD}$. In particular, for a $\mathbf{Q}$-Gorenstein projective variety X, we can define a height function h_{K_X} of a canonical divisor K_X up to addition of bounded functions. For a log pair (X,Z), we define the height function $h_{K_{(X,Z)}}$ of $K_{(X,Z)} = K_X + Z$ to be $h_{K_X} + h_Z$.

5. Vojta's conjecture for log pairs

The original form of Vojta's conjecture is as follows:

Conjecture 5.1 ([6]). *Let X be a smooth projective variety, A a big divisor on X, D a reduced simple normal crossing divisor on X and ε a positive real number. Then there exists a proper closed subset $W \subset X$ such that for all $x \in (X \setminus W)(k)$, we have*

$$h_{K_X}(x) + m_D(x) \le \varepsilon h_A(x) + O(1).$$

Using log pairs and multiplier-type ideals introduced in Section 3, we formulate a generalization of this conjecture as follows:

Conjecture 5.2. *Let (X,Z) be a log pair with X projective, A a big divisor on X and ε a positive real number. Then there exists a proper closed subset $W \subset X$ such that for all $x \in (X \backslash W)(k)$, we have*

$$h_{K_{(X,Z)}}(x) - N_{\mathscr{H}(X,Z)}(x) - m_{\mathscr{I}^-(X,Z)}(x) \leq \varepsilon h_A(x) + O(1). \tag{5.1}$$

We can view the left hand side of this inequality as follows. The main term is $h_{K_{(X,Z)}}(x)$ and the other two terms are correction terms arising from singularities of (X,Z). Indeed, from Proposition 3.6, the term $N_{\mathscr{H}(X,Z)}(x)$ can be thought of as the contribution of the non-sc locus $\mathrm{NonSC}(X,Z)$. From Proposition 3.2, $m_{\mathscr{I}^-(X,Z)}(x)$ can be thought of as the contribution of $\mathrm{NonKLT}(X,Z)$ (or $\mathrm{NonLC}(X,Z)$ if $(X \backslash \mathrm{Supp}(Z), 0)$ is Kawamata log terminal).

Example 5.3. Let X be a smooth projective variety and D a reduced simple normal crossing divisor on X. Then (X,D) is log canonical. From Propositions 3.2 and 3.6 and Corollary 3.7, the left hand side of (5.1) is equal to

$$h_{K_{(X,D)}}(x) - N_D(x) = h_{K_X} + m_D.$$

Thus Conjecture 5.2 is the same as Conjecture 5.1 in this case.

Although Conjecture 5.2 deals with a more general setting than Conjecture 5.1 does, they are in fact equivalent:

Proposition 5.4. *Let (X,Z) be a log pair with X projective and $f : Y \to X$ a log resolution of (X,Z). Suppose that Conjecture* 5.1 *holds for Y and the reduced simple normal crossing divisor*

$$\lceil K_{Y/(X,Z)} + \delta f^* Z \rceil - \lfloor K_{Y/(X,Z)} \rfloor$$

for $0 < \delta \ll 1$. Then Conjecture 5.2 *holds for (X,Z). In particular, Conjectures* 5.1 *and* 5.2 *are equivalent.*

Proof. The proof is similar to the one of Vojta's similar result [8, Prop. 4.3]. By definition,

$$f^{-1}\mathscr{H}(X,Z) \subset \mathscr{O}_Y(\lfloor K_{Y/(X,Z)} \rfloor)$$

and for $0 < \delta \ll 1$,

$$f^{-1}\mathscr{I}^-(X,Z) \subset \mathscr{O}_Y(\lceil K_{Y/(X,Z)} + \delta f^* Z \rceil).$$

These imply

$$N_{\mathscr{H}(X,Z)} \circ f \geq N_{-\lfloor K_{Y/(X,Z)} \rfloor},$$
$$m_{\mathscr{I}^-(X,Z)} \circ f \geq m_{-\lceil K_{Y/(X,Z)} + \delta f^* Z \rceil}.$$

We have

$$\begin{aligned}
&(h_{K_{(X,Z)}} - N_{\mathscr{H}(X,Z)} - m_{\mathscr{I}^-(X,Z)}) \circ f \\
&\quad \le h_{K_Y} - h_{K_{Y/(X,Z)}} - N_{-\lfloor K_{Y/(X,Z)} \rfloor} - m_{-\lceil K_{Y/(X,Z)} + \delta f^*Z \rceil} \\
&\quad \le h_{K_Y} + h_{-\lfloor K_{Y/(X,Z)} \rfloor} - N_{-\lfloor K_{Y/(X,Z)} \rfloor} - m_{-\lceil K_{Y/(X,Z)} + \delta f^*Z \rceil} \\
&\quad = h_{K_Y} + m_{\lceil K_{Y/(X,Z)} + \delta f^*Z \rceil - \lfloor K_{Y/(X,Z)} \rfloor}.
\end{aligned}$$

The proposition follows from the fact that the pullback f^*A of a big divisor A on X is big. □

Remark 5.5. If X is smooth and $Z \subsetneq X$ is a genuine closed subscheme, then $m_{\mathscr{I}^-(X,Z)}$ is the same correction term as the one used in [8, Conj. 4.2]. In this case, for a log resolution $f : Y \to X$ of (X, Z), $K_{Y/(X,Z)} = K_{Y/X} - f^{-1}Z$ is a $\mathbf{Z}$-divisor. Since $K_{Y/X} \ge 0$,

$$\mathscr{H}(X,Z) = f_*\mathscr{O}_Y(K_{Y/X} - f^{-1}Z) \supset f_*\mathscr{O}_Y(-f^{-1}Z) \supset \mathscr{I}_Z,$$

where $\mathscr{I}_Z$ is the defining ideal sheaf of Z. Therefore,

$$\begin{aligned}
h_{K_{(X,Z)}} - N_{\mathscr{H}(X,Z)} - m_{\mathscr{I}^-(X,Z)} &\ge h_{K_X} + h_Z - N_Z - m_{\mathscr{I}^-(X,Z)} \\
&= h_{K_X} + m_Z - m_{\mathscr{I}^-(X,Z)}.
\end{aligned}$$

Example 5.6. There exists a normal projective rational surface X such that $(X, 0)$ is Kawamata log terminal and K_X is ample. For instance, consider the Fermat hypersurface $F = V(x_0^n + x_1^n + x_2^n + x_3^n) \subset \mathbf{P}_k^3$ for $n \ge 5$ and suppose that k contains a primitive n-th root ζ of 1. The quotient variety $X := F/\langle g \rangle$ for the action $g((x_0 : x_1 : x_2 : x_3)) = (\zeta x_0 : \zeta x_1 : x_2 : x_3)$ is such a surface. This example shows the necessity of the correction term $N_{\mathscr{H}(X,Z)}$ in Conjecture 5.2. Indeed, for such an X, if we set $Z = 0$ and $A = K_X$, then inequality (5.1) is written as

$$(1 - \varepsilon) h_{K_X}(x) - N_{\mathrm{NonSC}(X,0)}(x) \le O(1). \tag{5.2}$$

One can prove that X is rational (that is, birational to $\mathbf{P}_k^2$) [9, Sec. 7], hence the k-point set $X(k)$ is Zariski dense. If there were no correction term $N_{\mathrm{NonSC}(X,0)}$, this would contradict Northcott's theorem.

6. Log pairs of general type

DEFINITION 6.1. We say that (X, Z) is *of general type* if X is projective and for a resolution $f : Y \to X$, the $\mathbf{Q}$-divisor $f^*K_{(X,Z)}$ is big.

For a proper birational morphism $f : Y \to X$ of normal varieties and a $\mathbf{Q}$-Cartier $\mathbf{Q}$-Weil divisor D on X, D is big if and only if f^*D is so [3, Lem. 2.1.13]. Hence the above definition is independent of the choice of f. If (X, Z) is a log pair of general type, then for any big divisor A on X, there exist $\varepsilon > 0$

and a proper closed subset $W \subset X$ such that for $x \in (X \setminus W)(k)$, we have

$$\varepsilon h_A(x) \le h_{K_{(X,Z)}}(x) + O(1).$$

This follows from [7, Prop. 10.11].

Definition 6.2. For a variety U over k, a subset set $C \subset U(k)$ is said to be *S-integral* if for a projective compactification X of U, the counting function $N_{X \setminus U, S} = N_{X \setminus U}$ of $X \setminus U$ with the reduced scheme structure is bounded on C.

From the functoriality of Weil functions, the above definition is independent of the choice of the projective compactification.

Proposition 6.3. *Let (X, Z) be a log pair of general type. Suppose that (X, Z) is log canonical and $(X \setminus \mathrm{Supp}(Z), 0)$ is Kawamata log terminal. Suppose also that Conjecture 5.2 holds. Then no S-integral subset of $(X \setminus \mathrm{NonSC}(X, Z))(k)$ is Zariski dense.*

Proof. From Proposition 3.2, $\mathscr{I}^-(X, Z) = \mathscr{O}_X$. From Conjecture 5.2, for $0 < \varepsilon \ll \varepsilon' < 1$ and a big divisor A on X, there exists a proper closed subset $W \subset X$ such that for $x \in (X \setminus W)(k)$, we have

$$\begin{aligned}(1 - \varepsilon')h_{K_{(X,Z)}}(x) &\le h_{K_{(X,Z)}}(x) - \varepsilon h_A(x) + O(1) \\ &\le N_{\mathscr{H}(X,Z)}(x) + O(1).\end{aligned}$$

From Proposition 3.6, $N_{\mathscr{H}(X,Z)}$ is bounded on any S-integral subset of $(X \setminus \mathrm{NonSC}(X, Z))(k)$, and the assertion follows from Northcott's theorem. □

When X is smooth and Z is a reduced simple normal crossing divisor, then the assumption on singularities in the proposition is automatic and we have $\mathrm{NonSC}(X, Z) = \mathrm{Supp}(Z)$. In this case, the proposition is a well-known conjecture due to Lang and Vojta (see [2, p. 17, p. 223], [6, Prop. 4.1.2]).

Proposition 6.4. *Let (X, Z) be a log pair of general type. Suppose that Conjecture 5.2 holds. Then there exists a proper closed subset $W \subset X$ such that if $Y \subset X$ is a closed subset with $Y(k) \subset Y$ Zariski dense, then either $Y \subset W$ or $Y \cap \mathrm{NonSC}(X, Z) \neq \emptyset$.*

Proof. From Conjecture 5.2, Propositions 3.2 and 3.6, and (2.1), there exist a proper closed subset $W \subset X$ and a constant $c > 0$ such that for all $x \in (X \setminus W)(k)$, we have

$$\begin{aligned}\frac{1}{2}h_{K_{(X,Z)}}(x) &\le N_{\mathscr{H}(X,Z)}(x) + m_{\mathscr{I}^-(X,Z)}(x) + O(1) \\ &\le c h_{\mathrm{NonSC}(X,Z)}(x) + O(1).\end{aligned}$$

For a log resolution $f : \tilde{X} \to X$, since $f^*K_{(X,Z)}$ is big, it is $\mathbf{Q}$-linearly equivalent to $A+E$ such that A is an ample $\mathbf{Q}$-divisor and E is an effective $\mathbf{Q}$-divisor. Since h_E is bounded below on $\tilde{X}\backslash\mathrm{Supp}(E)$, from Northcott's theorem, for any $B>0$, there are only finitely many $x \in \tilde{X}\backslash\mathrm{Supp}(E)$ with $h_{f^*K_{(X,Z)}}(x) \le B$. Replacing W with $W \cup f(\mathrm{Supp}(E))$, we may suppose that for any $B>0$, there are only finitely many $x \in (X\backslash W)(k)$ with $h_{K_{(X,Z)}}(x) \le B$.

Let $Y \subset X$ be a closed subset such that $Y \not\subset W$ and $Y \cap \mathrm{NonSC}(X,Z) = \emptyset$. The last condition and the functoriality of Weil functions imply that $h_{\mathrm{NonSC}(X,Z)}|_{Y(k)}$ is equal to the height function h_0 of the zero divisor on Y, in particular, $h_{\mathrm{NonSC}(X,Z)}$ is bounded on $Y(k)$. Hence $h_{K_{(X,Z)}}$ is bounded on $(Y\backslash W)(k)$. It follows that $Y(k)$ is not Zariski dense in Y. □

Example 6.5. Consider a rational surface as in Example 5.6. Since it is rational, it contains infinitely many irreducible curves which are rational over k. If Conjecture 5.2 holds, then all but finitely many of these curves must pass through one of the non-canonical singular points of the surface.

A variety V over k is said to be *potentially dense* if for some finite extension L/k, $V(L)$ is Zariski dense in V. Lang [2, p. 17] conjectured that a variety over k of positive dimension is potentially dense if and only if the union of the images of all non-constant rational maps from abelian varieties to $V \otimes_k \bar{k}$ over $\bar{k}$ is Zariski dense. He also conjectured that a smooth projective variety of general type is not potentially dense. Combining these conjectures, we get the conjecture that for a smooth projective variety X of general type, the union of all potentially dense closed subsets $Y \subset X$ is not Zariski dense. Proposition 6.4 is an analogue of this conjecture. Indeed, if in the proposition we only assume Y to be potentially dense instead of $Y(k)$ being dense, then the proposition becomes a generalization of this conjecture. If we can take the closed subset W in Conjecture 5.2 to be stable under the base change to every finite extension L/k, then this stronger version of the proposition also follows.

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