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ON ENDOMORPHISMS OF HYPERSURFACES

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Abstract

For any prime $p \geq 5$, we show that generic hypersurface $X_p \subset \mathbf{P}^p$ defined over $\mathbf{Q}$ admits a non-trivial rational dominant self-map of degree > 1, defined over $\bar{\mathbf{Q}}$. A simple arithmetic application of this fact is also given.

1. Introduction

1.1. Let X be an algebraic variety (smooth, projective, over a field of characteristic 0). Then two groups of symmetries of X, namely $\mathrm{Aut}(X)$ (biregular automorphisms) and $\mathrm{Bir}(X)$ (birational ones), come for free. The question on whether these groups differ is classical and very important. For instance, the property $\mathrm{Aut}(X) = \mathrm{Bir}(X)$, signifying *birational* (*super*)*rigidity* of X, is an obstruction for X to be rational (such X had been studied in numerous papers including [16], [24], [9], [17] and [18]).

Typically one has $\mathrm{Aut}(X) = \{\mathrm{id}\}$ however. In this regard, it is natural to consider another (more general) class of symmetries of X, namely $\mathrm{End}(X)$, consisting of *rational dominant* endomorphisms of X. Then one may ask (what seems to be a folklore) whether $\mathrm{End}(X) = \mathrm{Bir}(X)$? (Note that the latter property is an obstruction for X to be unirational.) This question gets immediate answer ('yes') when X—with $\mathrm{Pic}(X) = \mathbf{Z}$ at least—is of general type (as the endomorphisms preserve the spaces $H^0(X, mK_X)$ for all $m \in \mathbf{Z}$). On the other hand, already in the Calabi-Yau case things are not that straightforward; although still one gets $\mathrm{End}(X) = \mathrm{Bir}(X)$ when X is a general K3 surface for example (see [7]).

Presently, we would like to treat rationally connected X, namely $X := X_N \subset \mathbf{P}^N$ being a hypersurface of degree N (see [24], [15], [6] for some other aspects of the geometry of these X_N). Recall that according to [10] every X_N, $N \geq 4$, is non-rational, having $\mathrm{Bir}(X_N) = \mathrm{Aut}(X_N)$.

Our main result is

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THEOREM 1.2. *For any prime $p \geq 5$, (Zariski) general $\mathbf{Q}$-hypersurface $X_p \subset \mathbf{P}^p$ admits an endomorphism f_{X_p} of degree > 1, defined over $\bar{\mathbf{Q}}$. More precisely, f_{X_p} has a field of definition—$\mathbf{k}$ below,—same for all X_p.*

Thus the "non-regular" geometry of $X = X_p$ is pretty much fruitful (compare with results in [2], [4], [23], accounting for *regular* self-maps of X). At the same time, the proof of Theorem 1.2 is not effective, and it would be interesting to describe (a part of) $\mathrm{End}(X)$ explicitly. For example, what is the $\mathrm{End}(X)$-action on the universal Chow group of X (cf. [25], [3])? It is also plausible to get rid of the degree/ground field/dimension assumption in the formulation of our result. Say, can one take *any* Fano manifold X in place of X_p, or at least any (composite) integer N instead of p?

1.3. In order to prove Theorem 1.2 we first relate X_p to those hypersurfaces that have lots of endomorphisms. The latter are $X_d \subset \mathbf{P}^N$, defined over a given field, that happen to be unirational over this field, provided that $N \gg d$ is sufficiently large (see **2.1** below for the precise statements).

Next we employ the *degree formula* from [26] (cf. [22], [19], [14]). Recall that this formula relates the connective K-theory classes of two algebraic varieties X and Y admitting some rational map $f : X \dashrightarrow Y$. However, in order to get something fruitful in this way (e.g. to show that f with $\deg f = 0$ does not exist) one has to consider X, Y, etc. to be defined over an algebraically *non-closed* field $\mathbf{k}$, and (roughly speaking) to have no points over $\mathbf{k}$. This is the reason for the degree (resp. dimension/ground field) restriction in Theorem 1.2.

Remark 1.4. One of the basic obstructions for applying our method to arbitrary degree d (Fano) hypersurfaces $X_d \subset \mathbf{P}^N$ is that those usually have points over the field of definition (cf. [5]). There may also be no such nice congruence relation as in Corollary 2.5 below. But still, once we find (sufficiently many?) endomorphisms of $X_p \subset \mathbf{P}^p$ which preserve some projective subspace $\Gamma \subset X_p$, one may hope (by projecting from Γ) to obtain non-trivial endomorphisms of X_d for some (all?) $d \leq p-1$.

The principal part of our arguments relies on (a part of) the main result in [8] which asserts the existence of $X = X_p = Y$ as indicated above. It is then not hard to derive Theorem 1.2 for the given X_p and the general case follows easily (see Section 3 for details). Again we indicate that the initial X_p is very special. (It is defined over a field $\mathbf{k}$ of cohomological dimension ≤ 1 and does not contain points over the extensions of $\mathbf{k}$ whose degrees are coprime with p.)

1.5. The next result was motivated by the paper [1]:

COROLLARY 1.6. *Let $X_p \subset \mathbf{P}^p$ be as in Theorem 1.2. Then there exists a possibly larger number field $K \supseteq \mathbf{Q}$ such that Zariski closure of the f_{X_p}-orbit of the set $X_p(K)$ of K-rational points on X_p has dimension ≥ 2.*

One may consider Corollary 1.6 as a generalization of [13, Theorem 1.4]. Yet, unfortunately, our conclusion is weaker and it would be interesting to establish potential density of the set $X_p(\mathbf{Q})$ in X_p (e.g. by refining the arguments of Section 4 below).

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2. Preliminaries

2.1. Fix some integers d and $r \geq 1$. Recall that every smooth hypersurface $X_d \subset \mathbf{P}^N$ of degree d contains a projective subspace $\Lambda \simeq \mathbf{P}^r$ when $N \gg 1$ (see [12], [20]). More precisely, X_d corresponds to (generic) point in the incidence subvariety

$$Z := \{(X_d, \Lambda) \mid \Lambda \subset X_d\} \subset \mathbf{P}^{\binom{N+d}{d}} \times \mathrm{Gr}(r+1, N+1),$$

with *dominant* projections $Z \to \mathbf{P}, \mathrm{Gr}$.

Further, if Λ is given over an arbitrary field $\mathbf{k}_0 \subset \mathbf{C}$ by equations $l_{N-r+1} = \cdots = l_N = 0$ for some linear forms l_i, then X_d can be chosen to be defined over $\mathbf{k} := \mathbf{k}_0(\sqrt{-1})$. Indeed, any X_d passing through Λ has the defining equation $\sum_{i=N-r+1}^{N} \phi_i l_i = 0$, for some (varying) forms ϕ_i of degree $d-1$. Hence, since the set of $\mathbf{k}$-points is obviously dense in $\mathbf{P}^{\binom{N+d-1}{d-1}}$ (w.r.t. the complex analytic topology), one can approximate ϕ_i by such $\mathbf{k}$-forms of degree $d-1$ that X_d remains smooth.

Thus, since one can always find generic $\Lambda \in \mathrm{Gr}(r, N)$ to be defined over $\mathbf{k}$, the above discussion provides a smooth hypersurface X_d and a projective subspace $\Lambda \subset X_d$, both over $\mathbf{k}$, for any given d, r and sufficiently large $N = N(d, r)$. From [12, Corollary 3.7] we obtain

Theorem 2.2 (Harris, Mazur, Pandharipande). *X_d is $\mathbf{k}$-unirational.*

2.3. Let X, Y be smooth projective geometrically irreducible $\mathbf{k}_0$-varieties of dimension $d = \dim X = \dim Y$. Assume that there is a rational $\mathbf{k}_0$-map $f : X \dashrightarrow Y$. The degree $\deg f$ equals 0 if f is non-dominant; otherwise $\deg f := [\mathbf{k}_0(X) : f^*\mathbf{k}_0(Y)]$.

We recall the next result from [26]:

THEOREM 2.4 (Zainoulline). *$\chi(\mathcal{O}_X) \cdot \tau_{d-1} \equiv \deg f \cdot \chi(\mathcal{O}_Y) \cdot \tau_{d-1} \mod n_Y$, where $\chi(\cdot)$ is the Euler characteristic,*

$$\tau_{d-1} := \prod_{p \text{ prime}} p^{[(d-1)/(p-1)]}$$

is the $(d-1)$-st Todd number, and n_Y is the g.c.d. of degrees of all closed points on Y.

An immediate consequence of Theorem 2.4 is

COROLLARY 2.5 (cf. [26, Example 6.4]). *In the previous notation, if $n_X = p$, $X = Y = X_p \subset \mathbf{P}^p$ (i.e. $d = p - 1$ and $N = p$) for some prime $p \geq 3$, and (more generally) f is a $\mathbf{k}$-map, then $\deg f > 0$. (Such X is called* incompressible *over $\mathbf{k}$.)*[1)]

Proof. Regard X as a hypersurface over $\mathbf{k}$. Then, since $[\mathbf{k} : \mathbf{k}_0] \leq 2$, we get $n_X = p$ again. The claim now follows from Theorem 2.4 (with $\mathbf{k}_0$ replaced by $\mathbf{k}$) and the fact that $\chi(\mathcal{O}_X) = 1 = (\tau_{p-2}, p)$. □

Existence of $X = X_p$ as in Corollary 2.5 is guaranteed by the following result (see [8, Theorem 8]):

PROPOSITION 2.6 (Colliot-Thélène). *For every prime $p \geq 5$, there is a smooth (over $\bar{\mathbf{Q}}$) hypersurface $X \subset \mathbf{P}^p$, given (over $\mathbf{Q}$) by the equation*

$$x_1^p + lx_2^p + \cdots + l^{p-1}x_p^p - \alpha x_0^p = 0$$

for some integers l, α, so that $n_X = p$.

Remark 2.7. Let t be a transcendental parameter, $X_d \subset \mathbf{P}^N$ some $\mathbf{Q}(t)$-hypersurface, and $d = N = p$. One may assume (e.g. by identifying X_d with an appropriate pencil of degree $d = p$ hypersurfaces) that this X_d specializes via $t \mapsto 0$ to X from Proposition 2.6. It follows then that generic such X_d also satisfies $n_{X_d} = p$. (Here "generic hypersurface" means a point in a Zariski open subset of the $\mathbf{Q}(t)$-variety $\mathbf{P}^{\binom{N+d}{d}}$ parameterizing all hypersurfaces of degree d in $\mathbf{P}^N$ defined over the field $\mathbf{Q}(t)$.) Indeed, by definition of n_Y (see Theorem 2.4) and specialization $t \mapsto 0$ one finds that n_{X_d} divides $p = n_X$, and once $n_{X_d} = 1$ we get (by the same reasoning) that $n_X = 1$ as well, a contradiction.

3. Proof of Theorem 1.2

We keep on with notation of Section 2.

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1) Existence of similar hypersurfaces X of general type, as suggested by [26, Example 6.4], is not clear in the current setting because the condition $n_X = p$ need not be satisfied (compare with Proposition 2.6 below).

3.1. Let X be as in Proposition 2.6. Consider a cone $\hat{X} \subset \mathbf{P}^N$ over X of sufficiently large dimension and a family $\mathscr{X}$ of degree p hypersurfaces ($\subset \mathbf{P}^N$) over $\mathbf{Q}$ that smooths out the singularities of $\hat{X}$. We may regard (the general fiber of) $\mathscr{X}$ as a smooth $\mathbf{k}_0$-hypersurface of degree p in $\mathbf{P}^N$ for some purely transcendental field $\mathbf{k}_0 \supset \mathbf{Q}$.

Next, using the (dominant) projections $Z \to \mathbf{P}, \mathrm{Gr}$ one can see by the same argument as in **2.1** that the set of all $\mathbf{k}$-points $(X_p, \Lambda) \in Z$ is dense on Z in the complex analytic topology, and so $\mathscr{X}$ can be approximated by $\mathbf{k}$-hypersurfaces X_p (for $\mathscr{X}$, X_p treated as points on $\mathbf{P}$, with $\mathbf{k} \subset \mathbf{C}$). The hypersurfaces $\mathscr{X}$ and X_p can actually be put on an affine line $\mathbf{A}^1_{\mathbf{k}} \subset \mathbf{P}$ in such a way that the preimage $\widetilde{\mathbf{A}^1_{\mathbf{k}}} \subset Z$ of $\mathbf{A}^1_{\mathbf{k}}$ has all fibers = some projective spaces and generic fiber isomorphic to Λ (apply the reasoning from **2.1** to this family over $\mathbf{A}^1_{\mathbf{k}}$ considered as a degree p hypersurface over $\mathbf{k}_0(t)$).

Lemma 3.2. *In the previous setting, the hypersurface $\mathscr{X}$ contains a projective subspace $\subset \mathbf{P}^N$, isomorphic to Λ and defined over* $\mathbf{k}$.

Proof. It suffices to show that there exists a $\mathbf{k}$-point on Z whose projection to $\mathbf{P}$ coincides with $\mathscr{X}$. Indeed, the preceding natural projection $\widetilde{\mathbf{A}^1_{\mathbf{k}}} \to \mathbf{A}^1_{\mathbf{k}}$ is $\mathrm{Gal}(\bar{\mathbf{k}}/\mathbf{k})$-equivariant, hence its fiber over $\mathscr{X}$ is a projective $\mathbf{k}$-space. □

Without loss of generality we will assume that $\mathbf{k}_0 = \mathbf{Q}(t)$ in what follows.

Lemma 3.3. *There exists a* $\mathbf{k}$-*endomorphism* $f : \mathscr{X} \dashrightarrow \mathscr{X}$ *such that* $\deg f > 1$.

Proof. It follows from Lemma 3.2 and Theorem 2.2 (applied to $X_d := \mathscr{X}$) that $\mathscr{X}$ is $\mathbf{k}$-unirational. This yields two rational dominant $\mathbf{k}$-maps $\phi : \mathbf{P}^{N-1} \dashrightarrow \mathscr{X}$ and $\psi : \mathscr{X} \dashrightarrow \mathbf{P}^{N-1}$. We may assume one of ϕ, ψ to have degree > 1. Then it remains to take $f := \phi \circ \psi$. □

3.4. Let f be as in Lemma 3.3. Note that f is induced by a rational self-map of $\mathbf{P}^N \supset \mathscr{X}, \hat{X}$. Indeed, since $p \geq 5$ and thus $\mathrm{Pic}(\mathscr{X}) = \mathbf{Z}$ (Lefschetz), the map f is given by such a linear system on $\mathscr{X}$ that is obtained via restriction of a linear system from $\mathbf{P}^N$.

Put f_0 to be the specialization of f at the fiber $\hat{X}$ of the family $\mathscr{X}$ (recall that $\mathbf{k}_0 = \mathbf{Q}(t)$). More precisely, if $\mathscr{L}_t$ is a (movable) linear system defining f, then its specialization $\mathscr{L}_0$ to $\hat{X}$ may acquire some divisorial components in the base locus.[2)] Subtracting all these we arrive at a linear system which we set to define f_0.

Lemma 3.5. *f_0 is not induced by a self-map of* $\mathrm{Sing}(\hat{X})$ (*for an appropriate f*). *In other words, if $x_0 = \cdots = x_p = 0$ are the equations of the singular*

[2)] The latter stems from the fact that f may not be defined (in codimension 2) on $\hat{X}$.

locus $\mathrm{Sing}(\hat{X}) \subset \hat{X}$, *then* $f_0 \neq \mathrm{id}, 0$ *on the subspace* $\mathbf{P}^p \subset \mathbf{P}^N$ *complementary to* $\mathrm{Sing}(\hat{X})$.

Proof. Let the notation be as in the proof of Lemma 3.3. One can choose a $\mathbf{k}$-point $o \in \mathscr{X}$ such that ψ (resp. ϕ) is unramified at o (resp. $\psi(o)$).

Take $f := \phi \circ \sigma \circ \psi$ for some $\sigma \in \mathrm{PGL}(N, \mathbf{k})$ (to be specified further) in such a way that $f(o) = o$. Namely, since ψ, ϕ induce isomorphisms of the tangent spaces $T_o = T_{\phi(\psi(o))}, T_{\psi(o)}$, and (adjoint of) σ can act transitively on the N-tuples of $\mathbf{k}$-vectors in $T_{\psi(o)}$, we arrive at such f whose Jacobian $\mathrm{Jac}_o(f)$ is a $\mathbf{k}$-matrix with pairwise distinct eigen values and all defined over $\mathbf{Q}$. Then, considering f_0 as a rational self-map of $\mathbf{P}^N$ (cf. the discussion at the beginning of **3.4**), we obtain that all non-zero eigen values of the matrix $\mathrm{Jac}_{o'}(f_0)$ are pairwise distinct as well. Here $o' \in \hat{X}$ is the specialization of o.[3)]

Now suppose that f_0 *is* induced by some rational endomorphism of $\mathrm{Sing}(\hat{X})$. Again we regard f_0 as a self-map of $\mathbf{P}^N$. Then f_0 is regular at some point $o' \in \mathbf{P}^N$, $f_0(o') = o'$ and the non-zero eigen values of $\mathrm{Jac}_{o'}(f_0)$ are all different. On the other hand, $f_0 = \mathrm{id}$ on the subspace $\mathbf{P}^p \subset \mathbf{P}^N$ complementary to $\mathrm{Sing}(\hat{X})$, a contradiction.

Same argument shows that f_0 has non-trivial components on $\mathbf{P}^p$, i.e. $f_0 \neq 0$ there, which concludes the proof. □

Note that generic subspace $\mathbf{P}^p \subset \mathbf{P}^N$ cuts out a subvariety on $\hat{X}$ isomorphic X (cf. the beginning of **3.1**). Identify X with such a section $\mathbf{P}^p \cap \hat{X}$ so that the restriction $f_0|_X$ is a well-defined rational map. Let $f_X : X \dashrightarrow X$ be the composition of $f_0|_X$ and the linear projection $\hat{X} \dashrightarrow X$ from $\mathrm{Sing}(\hat{X})$.

Lemma 3.6. $f_X \neq \mathrm{id}$ *and is dominant.*

Proof. Indeed, f_0 induces a non-identical map on X by Lemma 3.5, and it remains to apply Corollary 2.5 to the $\mathbf{Q}(\sqrt{-1})$-map f_X. □

Remark 3.7. For the last part of Lemma 3.6, observe that the fact $\deg f_X \neq 0$ is not immediate from the proof of Lemma 3.5, and hence one needs an additional argument (results of the second half of Section 2 for instance) in order to proceed.

Theorem 1.2 (for the given X) now follows from

Proposition 3.8. $\deg f_X > 1$.

3) Note that o' is not contained in the indeterminacy locus of f_0 for the indicated o and σ. Thus f_0 is defined at o' and satisfies $f_0(o') = o'$.

Proof. Regard $\mathscr{X}$ as a (flat) family of hypersurfaces $\mathscr{X}_t \subset \mathbf{P}^N \times t \subset \mathbf{P}^N \times \mathbf{P}^1$ (with $\mathscr{X}_0 := \hat{X}$). Let $\pi : \mathscr{X} \to \mathbf{P}^1$ be the natural projection (so that $\pi^{-1}(t) = \mathscr{X}_t$). Let also $\mathscr{L}_t$ be as in the second paragraph of **3.4**.

LEMMA 3.9. *The linear system $\mathscr{L}_0$ is non-trivial on $\hat{X}$ (unless $\deg f_X > 1$).*

Proof. Suppose the contrary (i.e. $\mathscr{L}_0 = \{0\}$). Then the π-map $f : \mathscr{X} \dashrightarrow \mathscr{X}$ has indeterminacies along $\hat{X}$. Resolve these by some π-blow-up $\sigma : \mathscr{Y} \to \mathscr{X}$. Let $\mathscr{Y} \xrightarrow{a} \mathscr{Z} \xrightarrow{b} \mathscr{X}$ be the Stein factorization of the resolved f. Here a, b are some π-morphisms, with a inducing a birational isomorphism between $\mathscr{Y}$ and $\mathscr{Z}$. Moreover, since f is not defined on $\hat{X}$, the proper transform $\sigma_*^{-1}\hat{X}$ of $\hat{X}$ on $\mathscr{Y}$ belongs to the exceptional locus of a. Composing further with b yields a rational self-map $\mu : \hat{X} \dashrightarrow \hat{X}$ of degree 0.

Notice that μ does not coincide with the projection $\hat{X} \dashrightarrow X$ from $\mathrm{Sing}(\hat{X})$ because $\deg b > 1$ (cf. Lemma 3.3) and $\mathscr{X}_0 = \hat{X}$ is a non-multiple fiber of π (so that $\hat{X}$ is not a branching divisor of b). Thus one may assume $X = \mathbf{P}^p \cap \hat{X}$ intersects generic fiber of μ at >1 points. Composing $\mu|_X$ with $\hat{X} \dashrightarrow X$ either gives a self-map $X \dashrightarrow X$ of degree > 1 (which we take for f_X), or that X is not incompressible, in contradiction with Corollary 2.5. □

Let $\mathscr{Y}, \sigma, a, \ldots$ be as above. It follows from Lemma 3.9 that $\hat{X} = \mathscr{X}_0$ and the scheme $\mathscr{Z}_0 := b^{-1}(0)$ are birationally isomorphic via $a \circ \sigma^{-1}$ (recall that b is *finite*). In particular, $\mathscr{Z}_0$ is not a ramification divisor of b, which implies that $\deg b|_{\mathscr{Z}_t} = \deg b|_{\mathscr{Z}_0}$ for all $t \in \mathbf{C}$ close to 0.

Now, since $\deg b|_{\mathscr{Z}_t} > 1$ by construction, we deduce that $\deg f_0 = \deg b|_{\mathscr{Z}_0} > 1$ as well. The latter also gives $\deg f_X > 1$. Indeed, otherwise restricting $\mathscr{L}_0$ to generic $X = \mathbf{P}^p \cap \hat{X}$, we get $f_X \in \mathbf{Aut}(X)$ according to Lemma 3.6 and [10]. But then, since f_X is composed of $f_0|_X$ and projection $\hat{X} \dashrightarrow X$ to the base of the cone, we obtain that f_0 must be induced by some projective transformation of $\mathbf{P}^N \supset \hat{X}$. The latter obviously contradicts $\deg f_0 > 1$ and the proof of Proposition 3.8 is finished. □

To complete the proof of Theorem 1.2 we simply apply Remark 2.7 and the fact that all the preceding arguments go verbatim with $\mathbf{Q}$ replaced by $\mathbf{Q}(t)$. It remains to set $t := t_0 \in \mathbf{Q}$—a given general parameter value—to obtain hypersurfaces as in the statement of Theorem 1.2. Furthermore, in the generic setting one can replace the argument with $f_X \in \mathbf{Aut}(X)$ at the end of the proof of Proposition 3.8 by that with $f_X = \mathrm{id}$ (cf. [21, Theorem 5]), thus getting a contradiction with Lemma 3.6.[4)]

[4)] This argument may be considered as another way to prove Theorem 1.2.

4. Proof of Corollary 1.6

4.1. We proceed with applying the constructions of Section 3 to study the arithmetics of hypersurfaces $X_p \subset \mathbf{P}^p$ (the notation is as earlier).[5)]

Let again X be as in Proposition 2.6.

LEMMA 4.2. *f_X is non-periodic.*

Proof. Indeed, otherwise we have $f_X^k = \mathrm{id}$ for some k, so that both f_X, f_X^{k-1} are invertible. But this contradicts Proposition 3.8. $\square$

Fix f_X as in Lemma 4.2. Then after possibly replacing f_X by f_X^k, $k \gg 1$, we obtain a point $o \in X(K)$ such that $f_X(o) = o$ and f_X is defined at o (see [11]). We also have $\det \mathrm{Jac}_o(f_X) \neq 0$ because f_X is dominant.

LEMMA 4.3. *There exists a* $\mathbf{k}$*-cube* $\square_o \subset X$ *(i.e.* $\square_o$ *is given by linear inequalities with coefficients in* $\mathbf{k}$*) centered at* o *and invariant under* f_X.

Proof. Let $\square \subset X$ be some cube containing o and defined over $\mathbf{Q}$. Then the set $\bigcup_{k>0} X \backslash f_X^k(\square)$ is not everywhere dense in X. Indeed, otherwise there would exist a subsequence $f_X^{k_i}(\square) \to o$ for $i \to \infty$, which implies that $o \in X(\mathbf{Q})$ and contradicts $n_X = p$. It remains to take f_X-invariant $\square_o \subseteq \bigcap_{k>0} f_X^k(\square)$. $\square$

It follows from Lemma 4.3 (and the implicit function theorem) that the eigen values λ_i of the matrix $\mathrm{Jac}_o(f_X)$ are algebraic numbers (from K), all having norms $|\lambda_i| = 1$, $1 \le i \le p-1$. Furthermore, making if necessary a coordinate change on $\mathbf{P}^p \supset X$ of the form $x_j \mapsto \alpha_j x_j$, $0 \le j \le p$ (i.e. rescaling the metric on X), for some $\alpha_j \in \mathbf{Q}^*$, we may assume λ_i to be algebraic integers (from O_K).

LEMMA 4.4. *The matrix* $\mathrm{Jac}_o(f_X)$ *is semi-simple. Moreover, there is* $j < p-2$ *such that* $\lambda_1 = \cdots = \lambda_j = \pm 1$*, while* $\lambda_{j+1}, \ldots, \lambda_{p-1}$ *are multiplicatively independent.*

Proof. The first claim follows from the fact that f_X linearizes on $\square_o$. Now, if $\lambda_i = \pm 1$ (or equivalently $\lambda_i \in \mathbf{R}$) for all i, then $f_X^2 = \mathrm{id}$, a contradiction.

Further, Dirichlet's unit theorem yields $\lambda_{j+1} \cdots \lambda_{p-1} = |\lambda_{j+1} \cdots \lambda_{p-1}| = 1$, $\lambda_j = \lambda_k^{-1}$ as the only relations between $\lambda_{j+1}, \ldots, \lambda_{p-1} \in O_K^*$. In particular, since $j < p-1$, $\lambda_{j+1}, \ldots, \lambda_{p-1}$ generate a free subgroup $\subseteq \mathbf{Z}^{p-j-1} \subset O_K^*$.

Suppose that $j = p-2$. Then $\lambda_{j+1}, \ldots, \lambda_{p-1}$ are all equal to $\lambda_{j+1}^{\pm 1}$ say. On the other hand, characteristic polynomial of $\mathrm{Jac}_o(f_X)$ is defined over $\mathbf{k}$

[5)] Note that one can not specialize the (abundance of) $\mathbf{k}$-points on $\mathscr{X}$ to "many" $\mathbf{Q}$-points on the cone $\hat{X}$ (as we did with (some of) endomorphisms) because $n_X = p$ and so all the points on $\hat{X}$ we obtain this way are concentrated on $\mathrm{Sing}(\hat{X})$.

(by construction of $\square_o$ in Lemma 4.3), and hence the minimal polynomial of λ_{j+1} divides it. All together, this implies that $\lambda_i^2 = \pm 1$ for all $i \geq j+1$, a contradiction.

Thus we get $j < p - 2$ and the claim follows. $\square$

The arguments in [1, Section 2] and Lemma 4.4 imply that Zariski closure of the f_X-orbit of the locus $X(K)$ has dimension $\geq p - j - 1 > 1$.

Finally, to complete the proof of Corollary 1.6 one replaces $\mathbf{Q}$ by $\mathbf{Q}(t)$, as at the end of Section 3, and repeats the previous arguments.

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