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L^p p-HARMONIC 1-FORMS ON LOCALLY CONFORMALLY FLAT RIEMANNIAN MANIFOLDS

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Abstract

In this paper, we obtain some vanishing and finiteness theorems for L^p p-harmonic 1-forms on a locally conformally flat Riemmannian manifolds which satisfies an integral pinching condition on the traceless Ricci tensor, and for which the scalar curvature satisfies pinching curvature conditions or the first eigenvalue of the Laplace-Beltrami operator of M is bounded by a suitable constant.

1. Introduction

Let us recall that an m-dimensional Riemannian manifold (M^m, g) is said to be locally conformally flat if it admits a coordinate covering $\{U_\alpha, \varphi_\alpha\}$ such that the map $\varphi_\alpha; (U_\alpha, \varphi_\alpha) \to (S^m, g_0)$ is a conformal map, where g_0 is the standard metric on S^m. It is well known that a conformally flat manifold is a higher dimensional generalization of a Riemannian surface. But not every higher dimensional manifold admits a locally conformally flat structure, and it is difficult to give a good classification of locally conformally flat Riemannaian manifolds. However, by adding various geometric conditions, many authors have given some partial classification for locally conformally flat Riemannian manifolds (for examples, [2, 4, 5, 7, 11, 12, 13], etc.).

For a compact Riemannian manifold M^m, according to Hodge theory, the space of harmonic 1-forms on M^m is isomorphic to its first de Rham cohomology group. And it is well known that there are no harmonic p-forms, $0 < p < m$, on a compact conformally flat manifold M^m with positive Ricci curvature. When M is non-compact, the Hodge theory does not work anymore, hence it is natural to consider L^2-harmonic forms, as is showed that L^2-Hodge theory remains valid in complete non-compact manifolds as classical Hodge theory works well in the compact case. In [10], Li-Tam showed that the theory of L^2-harmonic 1-forms can be used to study the topology at infinity of a complete Riemannian manifold. Recently, H. Z. Lin [11], investigated the L^2 harmonic 1-form on locally con-

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formally flat Riemannian manifolds and obtain some vanishing and finiteness theorems for L^2 harmonic 1-forms. For p-harmonic 1-forms, Zhang [14] obtained vanishing results for p-harmonic 1-form. Chang [3] obtained the compactness for any bounded set of p-harmonic 1-forms. The first author in [8] investigated L^p p-harmonic 1-forms on complete noncompact submanifolds in a Hadamard manifold, and obtained some vanishing and finiteness theorems for these forms.

Let (M^m, g) be a Riemannian manifold, and let u be a real C^∞ function on M^m. Fix $p \in R$, $p \geq 2$ and consider a compact domain $\Omega \subset M^m$. The p-energy of u on Ω, is defined to be

$$E_p(\Omega, u) = \frac{1}{p}\int_\Omega |\nabla u|^p.$$

The function u is said to be p-harmonic on M^m if u is a critical point of $E_p(\Omega, *)$ for every compact domain $\Omega \subset M^m$. Equivalently, u satisfies the Euler-Lagrange equation.

$$div(|\nabla u|^{p-2}\nabla u) = 0.$$

Thus, the concept of p-harmonic function is a natural generalization of that of harmonic function, that is, of a critical point of the 2-energy functional.

Definition 1.1. A p-harmonic 1-form is a differentiable 1-form on M^m satisfying the following properties:

$$\begin{cases} d\omega = 0, \\ \delta(|\omega|^{p-2}\omega) = 0, \end{cases}$$

where δ is the codifferential operator. It is easy to see that the differential of a p-harmonic function is a p-harmonic 1-from.

In this paper, we investigate the properties for p-harmonic 1-form on locally conformally flat Riemannian manifolds. We assume that M^m is a complete noncompact manifold and define the space of the L^p p-harmonic 1-froms on M by

$$H^{1,p}(M) = \left\{\omega \,\middle|\, \int_M |\omega|^p < \infty, d\omega = 0 \text{ and } \delta(|\omega|^{p-2}\omega) = 0\right\}$$

where $p \geq 2$. We obtain the following results:

Theorem 1.2 (cf. Theorem 3.1). *Let (M^m, g), $m \geq 3$, be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold. Assume that*

$$\int_M |R|^{m/2} < \infty \quad \text{and} \quad \int_M |T|^{m/2} < \infty.$$

Then we have $\dim H^{1,p}(M) < \infty$ *for* $p \geq 2$, *where Ric*, R *and* $T = Ric - \frac{R}{m}g$ *are the Ricci curvature tensor, the scalar curvature and the traceless Ricci tensor respectively of* (M^m, g).

THEOREM 1.3 (cf. Theorem 3.2). *Let* (M^m, g), $m \geq 3$, *be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold. Then there exists a positive constant* $\Lambda < \frac{4[(m-1)(p-1)+1]}{Sp^2(m-1)}$ *such that if*

$$\left(\int_M |T|^{m/2}\right)^{2/m} + \frac{1}{\sqrt{m}}\left(\int_M |R|^{m/2}\right)^{2/m} \leq \Lambda,$$

then we have $H^{1,p}(M) = \{0\}$ *for* $p \geq 2$, *where* S *is a positive constant in the inequality* (6).

THEOREM 1.4 (cf. Theorem 3.3). *Let* (M^m, g), $m \geq 3$, *be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold. Assume that* $\int_M |T|^{m/2}\, dv < \infty$ *and* R *is bounded on* M *and* $\sup_M |R| > 0$. *If the first eigenvalue of the Laplace-Beltrami operator of* M *satisfies*

$$\lambda_1(M) > \frac{(m-1)p^2 \sup_M |R|}{4\sqrt{m}[(m-1)(p-1)+1]},$$

then we have $\dim H^{1,p}(M) < \infty$ *for* $p \geq 2$.

THEOREM 1.5 (cf. Theorem 3.4). *Let* (M^m, g), $m \geq 3$, *be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold. Assume that* R *is bounded on* M *and* $\sup_M |R| > 0$. *Assume the first eigenvalue of the Laplace-Beltrami operator of* M *satisfies*

$$\lambda_1(M) > \frac{(m-1)p^2 \sup_M |R|}{4\sqrt{m}[(m-1)(p-1)+1]}.$$

then there exists a positive constant Λ *such that if* $(\int_M |T|^{m/2})^{2/m} < \Lambda$, *then we have* $H^{1,p}(M) = \{0\}$ *for* $p \geq 2$.

THEOREM 1.6 (cf. Theorem 3.5). *Let* (M^m, g), $m > p^4$, *be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold with* $R \leq 0$. *Assume that* $\int_M |T|^{m/2} < \infty$. *Then we have* $\dim H^{1,p}(M) < \infty$ *for* $p \geq 2$.

From the proof of Theorem 1.6, we can obtain the following result.

THEOREM 1.7. *Let* (M^m, g), $m > p^4$, *be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold with* $R \leq 0$. *Then there* *exists a positive constant* $\Lambda < \left[\frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(m-1)}{\sqrt{m}(m-2)}\right] Q(S^m)$ *such that if*

$$\left(\int_M |T|^{m/2}\right)^{2/m} \leq \Lambda,$$

then we have $H^{1,p}(M) = \{0\}$ *for* $p \geq 2$, *where* $Q(S^m) = \frac{m(m-2)\omega_m^{2/m}}{4}$ *is the Yamabe constant of* S^m *and* ω_m *is the volume of the unit sphere in* R^m.

2. Preliminaries

In order to prove our main result, we need the following results:

LEMMA 2.1 ([9]). *Let E be a finite dimensional subspace of the space L^2 q-forms on a compact Riemannian manifold $\tilde{M}^m$. Then there exists $\omega \in E$ such that*

$$\frac{\dim E}{Vol(\tilde{M})}\int_{\tilde{M}} |\omega|^2 \, dv \leq \min\{\tbinom{m}{q}, \dim E\} \sup_{\tilde{M}} |\omega|^2.$$

From Lemma 2.1, the first author in [8] obtained the following result.

LEMMA 2.2 ([8]). *Let E be a finite dimensional subspace of the space L^p q-forms on a compact Riemannian manifold $\tilde{M}^m$. Then there exists $\omega \in E$ such that*

$$\frac{\dim E}{Vol(\tilde{M})}\int_{\tilde{M}} |\omega|^p \, dv \leq \min\{C_p\tbinom{m}{q}, \dim E\} \sup_{\tilde{M}} |\omega|^p,$$

where C_p is a positive constant depending only p and $p \geq 2$.

We also need a Kato type inequality for p-harmonic 1-form.

LEMMA 2.3 ([8]). *Let ω be a p-harmonic 1-form on Riemannian manifold M^m. Then we have the following inequality*

$$|\nabla(|\omega|^{p-2}\omega)|^2 \geq \left(1 + \frac{1}{(m-1)(p-1)^2}\right)|\nabla|\omega|^{p-1}|^2, \tag{1}$$

where $p \geq 2$.

Using Bochner's formula [1], we have the following results.

LEMMA 2.4. *Let ω be a p-harmonic 1-form on Riemannian manifold M^m. Then we have*

$$(2)\qquad \frac{1}{2}\triangle|\omega|^{2(p-1)} = |\nabla(|\omega|^{p-2})\omega|^2 - \langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle + |\omega|^{2(p-2)}\, Ric^M(\omega,\omega).$$

From (1) and (2), we have

$$(3)\qquad |\omega|^{p-1}\triangle|\omega|^{p-1} \geq \frac{1}{(m-1)(p-1)^2}|\nabla|\omega|^{p-1}|^2 - \langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle, + |\omega|^{2(p-2)}\, Ric^M(\omega,\omega),$$

where ω is a p-harmonic 1-form on Riemannian manifold M^m.
In [11], H. Z. Lin obtained following result.

Lemma 2.5 ([11]). *Let (M^m, g) be an m-dimensional complete Riemannian manifold. Then*

$$(4)\qquad Ric \geq -|T|g - \frac{|R|}{\sqrt{m}}g$$

in the sense of quadratic forms, where Ric, R and $T = Ric - \frac{R}{m}g$ are the Ricci curvature tensor, the scalar curvature and the traceless Ricci tensor respectively of (M^m, g).

It is known that a simply connected, locally conformally flat manifold M^m, $m \geq 3$, has a conformal immersion into S^m and according to [6], the Yamabe constant of M^m satisfies $Q(M^m) = Q(S^m) = \frac{m(m-2)\omega_m^{2/m}}{4}$, which ω_m is the volume of the unit sphere in R^m. Therefore the following inequality

$$(5)\qquad Q(S^m)\left(\int_M f^{2m/(m-2)}\right)^{(m-2)/m} \leq \int_M |\nabla f|^2 + \frac{m-2}{4(m-1)}\int_M Rf^2$$

holds for all $f \in C_0^\infty(M)$. From (5), H. Z. Lin in [11] proved the following result.

Lemma 2.6 ([11]). *Let (M^m, g), $m \geq 3$, be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold with $R \leq 0$ or $\int_M |R|^{m/2}\, dv < \infty$. Then the following Sobolev inequality*

$$(6)\qquad \left(\int_M f^{2m/(m-2)}\right)^{(m-2)/m} \leq S\int_M |\nabla f|^2, \quad \forall f \in C_0^\infty(M)$$

holds for some constant $S > 0$, *which is equal to* $Q(S^m)^{-1}$ *in the case where* $R \leq 0$. *In particular*, M *has infinite volume*.

In [8], the first author proved the following result.

LEMMA 2.7 ([8]). *Let* $f : M^m \to R$ *be a smooth function on Riemannian manifold* M, *and* ω *be a closed* 1*-form on* M. *Then we have* $|d(f\omega)| \leq |df|\,|\omega|$.

3. Proof of the main results

THEOREM 3.1. *Let* (M^m, g), $m \geq 3$, *be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold. Assume that*

$$\int_M |R|^{m/2} < \infty \quad \text{and} \quad \int_M |T|^{m/2} < \infty.$$

Then we have $\dim H^{1,p}(M) < \infty$ *for* $p \geq 2$.

Proof. Assume that ω is a p-harmonic 1-form on M^m, i.e. $\omega \in H^{1,p}(M)$. From (3) and (4), we have

$$\begin{aligned}|\omega|^{p-1}\triangle|\omega|^{p-1} &\geq -\langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle + \frac{1}{(m-1)(p-1)^2}|\nabla|\omega|^{p-1}|^2 \\ &\quad - |T|\,|\omega|^{2(p-1)} - \frac{|R|}{\sqrt{m}}|\omega|^{2(p-1)} \\ &\geq -\langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle + \frac{1}{m-1}\frac{4}{p^2}|\omega|^{p-2}|\nabla|\omega|^{p/2}|^2 \\ &\quad - |T|\,|\omega|^{2(p-1)} - \frac{|R|}{\sqrt{m}}|\omega|^{2(p-1)}.\end{aligned}$$

So we have

$$|\omega|\triangle|\omega|^{p-1} \geq -\langle \delta d(|\omega|^{p-2}\omega), \omega\rangle + \frac{1}{m-1}\frac{4}{p^2}|\nabla|\omega|^{p/2}|^2 - |T|\,|\omega|^p - \frac{|R|}{\sqrt{m}}|\omega|^p. \tag{7}$$

Fix a point $x_0 \in M$. Let $\mu \in C_0^\infty(M)$ be a smooth function with compact support on M. Multiplying (7) by μ^2 and integrating over on M, we have

$$\begin{aligned}&-\int_M \mu^2\langle\nabla|\omega|, \nabla|\omega|^{p-1}\rangle - 2\int_M \mu|\omega|\langle\nabla\mu, \nabla|\omega|^{p-1}\rangle + \int_M \mu^2\langle\delta d(|\omega|^{p-2}\omega), \omega\rangle \\ &\quad \geq \frac{1}{m-1}\frac{4}{p^2}\int_M \mu^2|\nabla|\omega|^{p/2}|^2 - \int_M |T|\mu^2|\omega|^p - \frac{1}{\sqrt{m}}\int_M |R|\mu^2|\omega|^p\end{aligned} \tag{8}$$

From Lemma 2.7, we have

$$
\begin{aligned}
(9)\qquad \left|\int_M \mu^2\langle \delta d(|\omega|^{p-2}\omega),\omega\rangle\right| &= \left|\int_M \langle d(|\omega|^{p-2}\omega), d(\mu^2\omega)\rangle\right| \\
&\le \int_M |d(|\omega|^{p-2}\omega)|\,|d(\mu^2\omega)| \\
&\le 2\int_M \mu|d\mu|\,|\omega|^2|d|\omega|^{p-2}| \\
&= \frac{4(p-2)}{p}\int_M \mu|\nabla\mu|\,|\omega|^{p/2}|\nabla|\omega|^{p/2}|.
\end{aligned}
$$

By direct computation, we get

$$
\begin{aligned}
(10)\qquad & -\int_M \mu^2\nabla|\omega|\nabla|\omega|^{p-1} - 2\int_M \mu|\omega|\langle\nabla\mu,\nabla|\omega|^{p-1}\rangle \\
&= -\frac{4(p-1)}{p^2}\int_M \mu^2|\nabla|\omega|^{p/2}|^2 - \frac{4(p-1)}{p}\int_M \mu\langle\nabla\mu,\nabla|\omega|^{p/2}\rangle|\omega|^{p/2} \\
&\le -\frac{4(p-1)}{p^2}\int_M \mu^2|\nabla|\omega|^{p/2}|^2 + \frac{4(p-1)}{p}\int_M \mu|\nabla\mu|\,|\omega|^{p/2}|\nabla|\omega|^{p/2}|.
\end{aligned}
$$

From (8), (9) and (10), we have

$$
\begin{aligned}
(11)\qquad 0 \le & -\frac{4(p-1)}{p^2}\int_M \mu^2|\nabla|\omega|^{p/2}|^2 + \frac{4(2p-3)}{p}\int_M \mu|\nabla\mu|\,|\omega|^{p/2}|\nabla|\omega|^{p/2}| \\
& - \frac{4}{p^2(m-1)}\int_M \mu^2|\nabla|\omega|^{p/2}|^2 + \int_M |T|\mu^2|\omega|^p + \frac{1}{\sqrt{m}}\int_M |R|\mu^2|\omega|^p.
\end{aligned}
$$

For $\varepsilon_1 > 0$, we apply the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
(12)\qquad & \left[\frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p}\varepsilon_1\right]\int_M \mu^2|\nabla|\omega|^{p/2}|^2 \\
&\le \frac{(2p-3)}{p}\frac{1}{\varepsilon_1}\int_M |\omega|^p|\nabla\mu|^2 + \int_M |T|\mu^2|\omega|^p + \frac{1}{\sqrt{m}}\int_M |R|\mu^2|\omega|^p.
\end{aligned}
$$

From the assumption and Lemma 2.6, we know the Sobolev inequality (6) holds on M. Now since $m \ge 3$, we use Hölder, Sobolev inequality (6), and Cauchy-Schwartz inequalities to obtain

$$
\begin{aligned}
(13)\qquad \int_M |T|\mu^2|\omega|^p &\le \left(\int_{supp(\mu)} |T|^{m/2}\right)^{2/m}\left(\int_M (\mu|\omega|^{p/2})^{2m/(m-2)}\right)^{(m-2)/m} \\
&\le S\left(\int_{supp(\mu)} |T|^{m/2}\right)^{2/m}\int_M |\nabla(\mu|\omega|^{p/2})|^2 \\
&\le \phi(\mu)\left[(1+\varepsilon_2)\int_M \mu^2|\nabla|\omega|^{p/2}|^2 + \left(1+\frac{1}{\varepsilon_2}\right)\int_M |\omega|^p|\nabla\mu|^2\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(14) \qquad \int_M |R|\mu^2|\omega|^p &\le \left(\int_{supp(\mu)} |R|^{m/2}\right)^{2/m} \left(\int_M (\mu|\omega|^{p/2})^{2m/(m-2)}\right)^{(m-2)/m} \\
&\le S\left(\int_{supp(\mu)} |R|^{m/2}\right)^{2/m} \int_M |\nabla(\mu|\omega|^{p/2})|^2 \\
&\le \varphi(\mu)\left[(1+\varepsilon_3)\int_M \mu^2|\nabla|\omega|^{p/2}|^2 + \left(1+\frac{1}{\varepsilon_3}\right)\int_M |\omega|^p|\nabla\mu|^2\right]
\end{aligned}
$$

for $\varepsilon_2 > 0,\ \varepsilon_3 > 0$ where

$$
\phi(\mu) = S\left(\int_{supp(\mu)} |T|^{m/2}\right)^{2/m} \quad \text{and} \quad \varphi(\mu) = S\left(\int_{supp(\mu)} |R|^{m/2}\right)^{2/m}.
$$

From (12), (13) and (14), we have

$$
(15) \qquad A\int_M \mu^2|\nabla|\omega|^{p/2}|^2 \le B\int_M |\omega|^p|\nabla\mu|^2,
$$

where

$$
\begin{aligned}
A &= \frac{4}{p^2}\,\frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p}\varepsilon_1 - \phi(\mu)(1+\varepsilon_2) - \frac{1}{\sqrt{m}}\varphi(\mu)(1+\varepsilon_3) \\
B &= \frac{(2p-3)}{p}\,\frac{1}{\varepsilon_1} + \phi(\mu)\left(1+\frac{1}{\varepsilon_2}\right) + \frac{1}{\sqrt{m}}\varphi(\mu)\left(1+\frac{1}{\varepsilon_3}\right).
\end{aligned}
$$

Since $\int_M |T|^{m/2} < \infty$ and $\int_M |R|^{m/2} < \infty$, we can choose r_0 large enough such that

$$
(16) \qquad \int_{M\setminus B_{x_0}(r_0)} |T|^{m/2} \le \left(\frac{(m-1)(p-1)+1}{p^2(m-1)S(1+\varepsilon_2)}\right)^{2/m}
$$

and

$$
(17) \qquad \int_{M\setminus B_{x_0}(r_0)} |R|^{m/2} \le \left(\frac{(m-1)(p-1)+1}{p^2(m-1)S(1+\varepsilon_3)}\sqrt{m}\right)^{2/m},
$$

where $B_p(r_0)$ is the geodesic ball centered at p of radius r_0. Let $\mu = \mu_{r_0} \in C_0^\infty(M)$ be any smooth function with compact support which satisfies $\mathrm{Supp}(\mu) \subset M\setminus B_{x_0}(r_0)$. From (16) and (17), we have

$$
(18) \qquad \phi(\mu) \le \frac{(m-1)(p-1)+1}{p^2(m-1)(1+\varepsilon_2)} \quad \text{and} \quad \varphi(\mu) \le \frac{(m-1)(p-1)+1}{p^2(m-1)(1+\varepsilon_3)}\sqrt{m}.
$$

From (18), we have

$$\frac{4[(m-1)(p-1)+1]}{p^2(m-1)} - \phi(\mu)(1+\varepsilon_2) - \frac{1}{\sqrt{m}}\varphi(\mu)(1+\varepsilon_3)$$
$$\geq \frac{2[(m-1)(p-1)+1]}{p^2(m-1)} > 0$$

Hence we can choose $\varepsilon_1 > 0$ small enough such that

$$A = \frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p}\varepsilon_1 - \phi(\mu)(1+\varepsilon_2) - \frac{1}{\sqrt{m}}\varphi(\mu)(1+\varepsilon_3) > 0.$$

Therefore, (15) reduces to

$$\int_{M\backslash B_{x_0}(r_0)} \mu^2|\nabla|\omega|^{p/2}|^2 \leq C\int_{M\backslash B_{x_0}(r_0)} |\omega|^p|\nabla\mu|^2, \tag{19}$$

where $C = C(m,p) > 0$ depends only on m, p. On the other hand, applying the Sobolev inequality (6) to $\mu|\omega|^{p/2}$, we have

$$\begin{aligned}\left(\int_{M\backslash B_{x_0}(r_0)} (\mu|\omega|^{p/2})^{2m/(m-2)}\right)^{(m-2)/m} &\leq S\int_{M\backslash B_{x_0}(r_0)} |\nabla(\mu|\omega|^{p/2})|^2 \\ &\leq 2S\left[\int_{M\backslash B_{x_0}(r_0)} (\mu^2|\nabla|\omega|^{p/2}|^2 + |\omega|^p|\nabla\mu|^2)\right]\end{aligned} \tag{20}$$

From (19) and (20), we have

$$\left(\int_{M\backslash B_{x_0}(r_0)} (\mu|\omega|^{p/2})^{2m/(m-2)}\right)^{(m-2)/m} \leq C_1\left[\int_{M\backslash B_{x_0}(r_0)} |\omega|^p|\nabla\mu|^2)\right] \tag{21}$$

where $C_1 = C_1(m,p) > 0$ depends only on m, p. Let $\rho(x)$ be the geodesic distance on M from x_0 to x. Let us choose $\mu \in C_0^\infty(M)$ satisfying

$$\mu(x) = \begin{cases} 0 & \text{on } B_{x_0}(r_0) \cup (M\backslash B_{x_0}(2r)), \\ \rho(x) - r_0 & \text{on } B_{x_0}(r_0+1)\backslash B_{x_0}(r_0), \\ 1 & \text{on } B_{x_0}(r)\backslash B_{x_0}(r_0+1), \\ \dfrac{2r-\rho(x)}{r} & \text{on } B_{x_0}(2r)\backslash B_{x_0}(r), \end{cases}$$

where $r > r_0 + 1$. From (21) and the definition of μ, we have

$$\left(\int_{B_{x_0}(r)\backslash B_{x_0}(r_0)} |\omega|^{pm/(m-2)}\right)^{(m-2)/m} \leq C_1\int_{B_{x_0}(r_0+1)\backslash B_{x_0}(r_0)} |\omega|^p + \frac{C_1}{r^2}\int_{B_{x_0}(2r)\backslash B_{x_0}(r)} |\omega|^p$$

since $|\omega| \in L^p(M)$, taking $r \to \infty$, we have

$$\left(\int_{M \backslash B_{x_0}(r_0)} |\omega|^{pm/(m-2)} \right)^{(m-2)/m} \leq C_1 \int_{B_{x_0}(r_0+1) \backslash B_p(r_0)} |\omega|^p. \tag{22}$$

It follows from the Hölder inequality that

$$\begin{aligned} &\int_{B_{x_0}(r_0+2) \backslash B_{x_0}(r_0+1)} |\omega|^p \\ &\quad \leq [\mathrm{Vol}(B_{x_0}(r_0+2))]^{2/m} \left(\int_{B_{x_0}(r_0+2) \backslash B_{x_0}(r_0+1)} |\omega|^{pm/(m-2)} \right)^{(m-2)/m}. \end{aligned} \tag{23}$$

From (22) and (23), we have

$$\int_{B_{x_0}(r_0+2)} |\omega|^p \leq C_2 \int_{B_{x_0}(r_0+1)} |\omega|^p, \tag{24}$$

where C_2 depends on $\mathrm{Vol}(B_{x_0}(r_0+2))$, m and p.

From (7), we have

$$|\omega| \triangle |\omega|^{p-1} \geq -\langle \delta d(|\omega|^{p-2}\omega), \omega \rangle + \frac{1}{m-1} \frac{4}{p^2} |\nabla |\omega|^{p/2}|^2 - F|\omega|^p. \tag{25}$$

where $F : M \to [0, +\infty)$ is the function given by $F = |T| + \dfrac{|R|}{\sqrt{m}}$.

Fix $x \in M$ and take $\eta \in C_0^\infty(B_x(1))$. Multiply both sides of (25) by $\eta^2 |\omega|^{pq/2-p}$, with $q \geq 2$, and integrating by parts we obtain

$$\begin{aligned} &-\frac{4(p-1)}{p} \int_{B_x(1)} \eta |\omega|^{pq/2-p/2} \langle \nabla \eta, \nabla |\omega|^{p/2} \rangle \\ &\quad \geq \left[\frac{2(p-1)(pq-2p+2)}{p^2} + \frac{4}{p^2(m-1)} \right] \int_{B_x(1)} |\omega|^{pq/2-p} |\nabla |\omega|^{p/2}|^2 \eta^2 \\ &\qquad - F \int_{B_x(1)} \eta^2 |\omega|^{pq/2} - \int_{B_x(1)} \langle d(|\omega|^{p-2}\omega), d(\eta^2 |\omega|^{pq/2-p}\omega) \rangle. \end{aligned} \tag{26}$$

From Lemma 2.7 and Cauchy-Schwatz inequality, we have

$$\begin{aligned} &\int_{B_x(1)} |\langle d(|\omega|^{p-2}\omega), d(\eta^2 |\omega|^{pq/2-p}\omega) \rangle| \\ &\quad \leq \int_{B_x(1)} |d(|\omega|^{p-2}\omega)| \, |d(\eta^2 |\omega|^{pq/2-p}\omega)| \\ &\quad \leq \int_{B_x(1)} |\nabla |\omega|^{p-2}| \, |\omega|^2 |[d(\eta^2)|\omega|^{pq/2-p} + \eta^2 d|\omega|^{pq/2-p}]| \end{aligned} \tag{27}$$

$$\begin{aligned}
&\le \frac{4(p-2)}{p}\int_{B_x(1)} \eta|\omega|^{pq/2-p/2}|\nabla\eta|\,|\nabla|\omega|^{p/2}| \\
&\quad + \frac{2(p-2)(q-2)}{p}\int_{B_x(1)} \eta^2|\omega|^{pq/2-p}|\nabla|\omega|^{p/2}|^2 \\
&\le \frac{2}{p^2(m-1)}\int_{B_x(1)} |\omega|^{pq/2-p}|\nabla|\omega|^{p/2}|^2\eta^2 \\
&\quad + 2(p-2)^2(m-1)\int_{B_x(1)} |\nabla\eta|^2|\omega|^{pq/2} \\
&\quad + \frac{2(p-2)(q-2)}{p}\int_{B_x(1)} \eta^2|\omega|^{pq/2-p}|\nabla|\omega|^{p/2}|^2
\end{aligned}$$

and

$$\begin{aligned}
-\frac{4(p-1)}{p}&\int_{B_x(1)} \eta|\omega|^{pq/2-p/2}\langle\nabla\eta,\nabla|\omega|^{p/2}\rangle \\
&\le \frac{2}{p^2(m-1)}\int_{B_x(1)} |\omega|^{pq/2-p}|\nabla|\omega|^{p/2}|^2\eta^2 \\
&\quad + 2(p-1)^2(m-1)\int_{B_x(1)} |\nabla\eta|^2|\omega|^{pq/2}.
\end{aligned} \tag{28}$$

From (26), (27) and (28), we have

$$\begin{aligned}
&\left[\frac{2(p-1)(pq-2p+2)}{p^2} - \frac{2(p-2)(q-2)}{p}\right]\int_{B_x(1)} |\omega|^{pq/2-p}|\nabla|\omega|^{p/2}|^2\eta^2 \\
&\quad \le F\int_{B_x(1)} \eta^2|\omega|^{pq/2} + [2(p-1)^2 + 2(p-2)^2](m-1)\int_{B_x(1)} |\nabla\eta|^2|\omega|^{pq/2}.
\end{aligned} \tag{29}$$

By using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\int_{B_x(1)} |\nabla(\eta|\omega|^{pq/4})|^2 \\
&\quad \le (1+q)\left[\int_{B_x(1)} |\omega|^{pq/2}|\nabla\eta|^2 + \frac{q}{4}\int_{B_x(1)} \eta^2|\omega|^{pq/2-p}|\nabla|\omega|^{p/2}|^2\right].
\end{aligned} \tag{30}$$

From (29) and (30), we have

$$\int_{B_x(1)} |\nabla(\eta|\omega|^{pq/4})|^2 \le C_3\int_{B_x(1)} |\omega|^{pq/2}|\nabla\eta|^2 + C_4\int_{B_x(1)} F\eta^2|\omega|^{pq/2}, \tag{31}$$

where

$$C_3 = 1 + q + \frac{q}{4}(q+1)[2(p-1)^2 + 2(p-2)^2](m-1)$$
$$\left[\frac{2(p-1)(pq-2p+2)}{p^2} - \frac{2(p-2)(q-2)}{p}\right]^{-1} \le C(p)mq,$$
$$C_4 = \frac{q}{4}(q+1)\left[\frac{2(p-1)(pq-2p+2)}{p} - \frac{2(p-2)(q-2)}{p}\right]^{-1} \le C(p)q,$$

where $C(p)$ is a positive constant depending only on p. Applying (5) to $\eta|\omega|^{pq/4}$ and using (31), we have

$$\begin{aligned} Q(S^m)&\left(\int_{B_x(1)} (\eta|\omega|^{pq/4})^{2m/(m-2)}\right)^{(m-2)/m} \\ &\le \int_{B_x(1)} |\nabla(\eta|\omega|^{pq/4})|^2 + \frac{m-2}{4(m-1)}\int_{B_x(1)} R\eta^2|\omega|^{pq/2} \\ &\le \int_{B_x(1)} \left[C_4F + \frac{m-2}{4(m-1)}R\right]\eta^2|\omega|^{pq/2} + C_3\int_{B_x(1)} |\omega|^{pq/2}|\nabla\eta|^2. \end{aligned}$$

so we have

$$\left(\int_{B_x(1)} (\eta|\omega|^{pq/4})^{2m/(m-2)}\right)^{(m-2)/m} \le qC_5\int_{B_x(1)} [\eta^2 + |\nabla\eta|^2]|\omega|^{pq/2}, \tag{32}$$

for a constant $C_5 > 0$ depending m, p, $\mathrm{Vol}(B_x(1))$, $\sup_{B_x(1)} F$ and $\sup_{B_x(1)}|R|$.

Given an integer $k \ge 0$, we set $q_k = \dfrac{2m^k}{(m-2)^k}$ and $\rho_k = \dfrac{1}{2} + \dfrac{1}{2^{k+1}}$. Take a function $\xi_k \in C_0^\infty(B_x(\rho_k))$ satisfying $\eta_k \ge 0$, $\eta_k = 1$ on $B_x(\rho_{k+1})$ and $|\nabla\eta_k| \le 2^{k+3}$. Replacing q and η in (32) by q_k and η_k respectively, we have

$$\left(\int_{B_x(\rho_{k+1})} |\omega|^{(pq_{k+1})/2}\right)^{1/q_{k+1}} \le (q_kC_54^{k+4})^{1/q_k}\left(\int_{B_x(\rho_k)} |\omega|^{pq_k/2}\right)^{1/q_k}. \tag{33}$$

Applying the Moser iteration to (33), we conclude that

$$|\omega|^p(x) \le \|\omega\|^p_{L^\infty(B_x(1/2))} \le C_6\int_{B_x(1)} |\omega|^p \tag{34}$$

for a constant $C_6 > 0$ depending only on m, p, $\mathrm{Vol}(B_x(1))$, $\sup_{B_x(1)} F$ and $\sup_{B_x(1)}|R|$. Take $x \in B_{x_0}(r_0+1)$ such that

$$|\omega|^p(x) = \sup_{B_{x_0}(r_0+1)} |\omega|^p. \tag{35}$$

From (34) and (35), we have

$$\sup_{B_{x_0}(r_0+1)} |\omega|^p \le C_6 \int_{B_{x_0}(r_0+2)} |\omega|^p. \tag{36}$$

From (24) and (36), we have

$$\sup_{B_{x_0}(r_0+1)} |\omega|^p \le C_7 \int_{B_{x_0}(r_0+1)} |\omega|^p, \tag{37}$$

where $C_7 > 0$ is a constant depending on m, p, $\mathrm{Vol}(B_x(r_0+2))$, $\sup_{B_x(r_0+2)} F$ and $\sup_{B_x(r_0+2)}|R|$.

Finally, let V be any finite-dimensional subspace of $H^{1,p}(M)$. From Lemma 2.2, there exists $\omega \in V$ such that

$$\frac{\dim V}{Vol(B_{x_0}(r_0+1))} \int_{B_{x_0}(r_0+1)} |\omega|^p \le \min\{C_p(\tbinom{m}{q}), \dim V\} \sup_{B_{x_0}(r_0+1)} |\omega|^p. \tag{38}$$

From (37) and (38), we have $\dim V \le C_8$, where $C_8 > 0$ depends only on m, p, $\mathrm{Vol}(B_x(r_0+2))$, $\sup_{B_x(r_0+2)} F$ and $\sup_{B_x(r_0+2)}|R|$. This implies that $H^{1,p}(M)$ has finite dimension. □

From the proof of Theorem 3.1, we can obtain the following result:

THEOREM 3.2. *Let (M^m, g), $m \ge 3$, be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold. Then there exists a positive constant $\Lambda < \dfrac{4[(m-1)(p-1)+1]}{Sp^2(m-1)}$ such that if*

$$\left(\int_M |T|^{m/2}\right)^{2/m} + \frac{1}{\sqrt{m}}\left(\int_M |R|^{m/2}\right)^{2/m} \le \Lambda,$$

then we have $H^{1,p}(M) = \{0\}$ for $p \ge 2$.

Proof. For a point p and take a cut-off function μ satisfying $0 \le \mu \le 1$, $\mu = 1$ on $B_{x_0}(r)$, $\mu = 0$ on $M \backslash B_{x_0}(2r)$ and $|d\mu| \le \dfrac{c}{r}$, where c is a positive constant. From (12), (13) and (14), we have

$$A \int_M \mu^2 |\nabla |\omega|^{p/2}|^2 \le B \int_M |\omega|^p |\nabla \mu|^2, \tag{39}$$

where

$$A = \frac{4}{p^2}\,\frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p}\varepsilon_1 - \phi(\mu)(1+\varepsilon_2) - \frac{1}{\sqrt{m}}\varphi(\mu)(1+\varepsilon_3)$$

$$B = \frac{(2p-3)}{p}\,\frac{1}{\varepsilon_1} + \phi(\mu)\left(1+\frac{1}{\varepsilon_2}\right) + \frac{1}{\sqrt{m}}\varphi(\mu)\left(1+\frac{1}{\varepsilon_3}\right).$$

Now we choose ε small enough such that

$$0 < \varepsilon < \frac{(m-1)(p-1) + 1 - \frac{1}{4} S\Lambda p^2(m-1)}{(2p-3)p(m-1) + (p-1)(m-1) + 1},$$

and $\varepsilon_i < \varepsilon$, $i = 1, 2, 3$, so we have

$$\begin{aligned}(40) \quad A &= \frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p}\varepsilon_1 - \phi(\mu)(1+\varepsilon_2) - \frac{1}{\sqrt{m}}\varphi(\mu)(1+\varepsilon_3) \\ &> \left[\frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - S\Lambda\right] - \left[\frac{4(2p-3)}{p} + S\Lambda\right]\varepsilon \\ &> \left[\frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - S\Lambda\right] \\ &\quad - \left[\frac{4(2p-3)}{p} + \frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1}\right]\varepsilon > 0.\end{aligned}$$

From (39) and (40), we have

$$(41) \qquad \int_M \mu^2 |\nabla|\omega|^{p/2}|^2 \le \frac{B}{A}\int_M |\omega|^p |\nabla\mu|^2.$$

From (41) and the definition of μ, we have

$$\int_{B_{x_0}(r)} |\nabla|\omega|^{p/2}|^2 \le \frac{Bc^2}{A}\frac{1}{r^2}\int_M |\omega|^p.$$

Taking $r \to \infty$, we have $|\nabla|\omega|^{p/2}| = 0$. Then $|\omega|$ is constant. Since M has infinite volume and $\int_M |\omega|^p < \infty$. So we obtain that $\omega = 0$. Hence $H^{1,p}(M) = \{0\}$. □

THEOREM 3.3. *Let (M^m, g), $m \ge 3$, be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold. Assume that $\int_M |T|^{m/2}\, dv < \infty$ and R is bounded on M and $\sup_M|R| > 0$. If the first eigenvalue of the Laplace-Beltrami operator of M satisfies*

$$\lambda_1(M) > \frac{(m-1)p^2 \sup_M|R|}{4\sqrt{m}[(m-1)(p-1)+1]},$$

then we have $\dim H^{1,p}(M) < \infty$ *for* $p \ge 2$.

Proof. By using the similar method in the proof in Theorem 3.1, we will prove this theorem. From the formulas (12) and (13), we have

$$(42) \qquad D\int_M \mu^2|\nabla|\omega|^{p/2}|^2 \le E\int_M |\omega|^p|\nabla\mu|^2 + \frac{1}{\sqrt{m}}\int_M |R|\mu^2|\omega|^p,$$

where

$$D = \frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p}\varepsilon_1 - \phi(\mu)(1+\varepsilon_2)$$

$$E = \frac{(2p-3)}{p}\frac{1}{\varepsilon_1} + \phi(\mu)\left(1+\frac{1}{\varepsilon_2}\right).$$

Since $\int_M |T|^{m/2} < \infty$, we can choose r_0 large enough such that

$$\int_{M\setminus B_{x_0}(r_0)} |T|^{m/2} \le \varepsilon_3, \tag{43}$$

where $B_p(r_0)$ is the geodesic ball centered at p of radius r_0 and ε_3 is small enough positive constant. So we have

$$\begin{aligned} D &= \frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p}\varepsilon_1 - \phi(\mu)(1+\varepsilon_2) \\ &> \frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p}\varepsilon_1 - \varepsilon_3(1+\varepsilon_2) > 0 \end{aligned} \tag{44}$$

where ε_1, ε_2, ε_3 are small enough. From (42) and (44), we have

$$\int_{M\setminus B_{x_0}(r_0)} \mu^2|\nabla|\omega|^{p/2}|^2 \le \frac{E}{D}\int_{M\setminus B_{x_0}(r_0)} |\omega|^p|\nabla\mu|^2 + \frac{1}{D\sqrt{m}}\int_{M\setminus B_{x_0}(r_0)} |R|\mu^2|\omega|^p. \tag{45}$$

Now we recall that the first eigenvalue $\lambda_1(M)$ of the Laplacian of M satisfies

$$\lambda_1(M)\int_M \varphi^2 \le \int_M |\nabla\varphi|^2 \tag{46}$$

for any $\varphi \in C_0^\infty(M)$. Applying (46) with $\varphi = \mu|\omega|^{p/2}$, we have

$$\begin{aligned} \lambda_1(M)\int_M \mu^2|\omega|^p &\le \int_M |\nabla[\mu|\omega|^{p/2}]|^2 \\ &= \int_M [\mu^2|\nabla|\omega|^{p/2}| + 2\mu|\omega|^{p/2}\langle\nabla\mu, \nabla|\omega|^{p/2}\rangle + |\omega|^p|\nabla\mu|^2] \end{aligned}$$

By using the Cauchy-Schwarz inequality, we have for $\varepsilon_4 > 0$

$$\lambda_1(M)\int_M \mu^2|\omega|^p \le \int_M [(1+\varepsilon_4)\mu^2|\nabla|\omega|^{p/2}| + \left(1+\frac{1}{\varepsilon_4}\right)|\omega|^p|\nabla\mu|^2]. \tag{47}$$

From (45) and (47), we have

$$\left(\lambda_1(M) - (1+\varepsilon_4)\frac{1}{D\sqrt{m}}\sup|R|\right)\int_{M\setminus B_{x_0}(r_0)}\mu^2|\omega|^p \tag{48}$$
$$\leq \left[(1+\varepsilon_4)\frac{E}{D} + \left(1+\frac{1}{\varepsilon_4}\right)\right]\int_{M\setminus B_{x_0}(r_0)}|\omega|^p|\nabla\mu|^2$$

Thus, if $\lambda_1(M) > \dfrac{(m-1)p^2\sup_M|R|}{4\sqrt{m}[(m-1)(p-1)+1]}$, then we can choose ε_i, $i = 1,\dots,4$ small enough and depending on m, p, $\lambda_1(M)$ and $\sup_M|R|$, so that $(\lambda_1(M) - (1+\varepsilon_4)\dfrac{1}{D\sqrt{m}}\sup|R|) > 0$. Then we have

$$\int_{M\setminus B_{x_0}(r_0)}\mu^2|\omega|^p \leq \tilde{D}\int_{M\setminus B_{x_0}(r_0)}|\nabla\mu|^2|\omega|^p \tag{49}$$

where $\tilde{D} = \tilde{D}(m, p, \lambda_1(M), \sup_M|R|)$ is a positive constant. From (49) and the proof of Theorem 3.1, we can complete this proof. □

From the proof of Theorem 3.3, we can obtain the following result.

THEOREM 3.4. *Let (M^m, g), $m \geq 3$, be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold. Assume that R is bounded on M and $\sup_M|R| > 0$. Assume the first eigenvalue of the Laplace-Beltrami operator of M satisfies*

$$\lambda_1(M) > \frac{(m-1)p^2\sup_M|R|}{4\sqrt{m}[(m-1)(p-1)+1]}.$$

then there exists a positive constant Λ such that if $(\int_M |T|^{m/2})^{2/m} < \Lambda$, then we have $H^{1,p}(M) = \{0\}$ for $p \geq 2$.

Proof. We choose ε

$$\left(0 < \varepsilon < \frac{1}{2}\left\{-\left[1+\frac{4(2p-3)}{p}\right] + \sqrt{\left[1+\frac{4(2p-3)}{p}\right]^2 + \frac{16[(m-1)(p-1)+1]}{p^2(m-1)}}\right\}\right),$$

$0 < \varepsilon_1, \varepsilon_2 < \varepsilon$ and a positive constant $\Lambda = \Lambda(\varepsilon) > 0$ satisfying:

$$D = \frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p}\varepsilon_1 - \phi(\mu)(1+\varepsilon_2)$$
$$> \frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p}\varepsilon - \varepsilon(1+\varepsilon) > 0$$

and

$$S\Lambda < \varepsilon.$$

Assume $\|T\|_{L^{m/2}} < \Lambda$. From (42), we have

$$\int_M \mu^2 |\nabla|\omega|^{p/2}|^2 \le \frac{E}{D}\int_M |\omega|^p |\nabla\mu|^2 + \frac{1}{D\sqrt{m}}\int_M |R|\mu^2|\omega|^p. \tag{50}$$

From (47) and (50), we have

$$\begin{aligned}\left(\lambda_1(M) - (1+\varepsilon_4)\frac{1}{D\sqrt{m}}\sup|R|\right)\int_M \mu^2|\omega|^p \\ \le \left[(1+\varepsilon_4)\frac{E}{D} + \left(1+\frac{1}{\varepsilon_4}\right)\right]\int_M |\omega|^p|\nabla\mu|^2\end{aligned} \tag{51}$$

Thus, if $\lambda_1(M) > \dfrac{(m-1)p^2 \sup_M|R|}{4\sqrt{m}[(m-1)(p-1)+1]}$, then we can choose ε_i, $i = 1, \ldots, 4$ small enough and depending on m, p, $\lambda_1(M)$ and $\sup_M|R|$, so that $\left(\lambda_1(M) - (1+\varepsilon_4)\dfrac{1}{D\sqrt{m}}\sup|R|\right) > 0$. Then we have

$$\int_M \mu^2|\omega|^p \le \tilde{D}\int_M |\nabla\mu|^2|\omega|^p \tag{52}$$

where $\tilde{D} = \tilde{D}(m, p, \lambda_1(M), \sup_M|R|)$ is a positive constant. For a point p and take a cut-off function μ satisfying $0 \le \mu \le 1$, $\mu = 1$ on $B_{x_0}(r)$, $\mu = 0$ on $M \backslash B_{x_0}(2r)$ and $|d\mu| \le \dfrac{c}{r}$, where c is a positive constant.

From (52) and the definition of μ, we have

$$\int_{B_{x_0}(r)} |\omega|^p \le \tilde{D}\frac{c^2}{r^2}\int_M |\omega|^p \tag{53}$$

Let $r \to \infty$ and since $\|\omega\|_{L^p(M)} < \infty$, we have $|\omega|^p = 0$ i.e. $\omega = 0$. Hence $H^{1,p}(M) = \{0\}$. □

THEOREM 3.5. *Let (M^m, g), $m > p^4$, be an m-dimensional complete, simply connected, locally conformally flat Riemannian manifold with $R \le 0$. Assume that $\int_M |T|^{m/2} < \infty$. Then we have* $\dim H^{1,p}(M) < \infty$ *for* $p \ge 2$.

Proof. Since M^m is a locally conformally flat Riemannian manifold with $R \le 0$, and the inequality (5), we can obtain

$$\frac{m-2}{4(m-1)}\int_M |R| f^2 \le \int_M |\nabla f|^2 \tag{54}$$

for any $f \in C_0^\infty(M)$. Applying the inequality (54) to $\mu|\omega|^{p/2}$, we have

$$\frac{m-2}{4(m-1)}\int_M |R|\mu^2|\omega|^p \le \int_M |\nabla(\mu|\omega|^{p/2})|^2. \tag{55}$$

From (55) and Cauchy-Schwarz inequality, we have

$$\int_M |R|\mu^2|\omega|^p \le \frac{4(m-1)}{(m-2)}\left[(1+\varepsilon_5)\int_M \mu^2|\nabla|\omega|^{p/2}|^2 + \left(1+\frac{1}{\varepsilon_5}\right)\int_M |\omega|^p|\nabla\mu|^2\right] \tag{56}$$

where $\varepsilon_5 > 0$ is a positive constant. From (12), (13) and (56), we have

$$\tilde{A}\int_M \mu^2|\nabla|\omega|^{p/2}|^2 \le \tilde{B}\int_M |\omega|^p|\nabla\mu|^2, \tag{57}$$

where

$$\tilde{A} = \frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p}\varepsilon_1 - \phi(\mu)(1+\varepsilon_2) - \frac{4(m-1)}{\sqrt{m}(m-2)}(1+\varepsilon_5)$$

$$\tilde{B} = \frac{(2p-3)}{p}\frac{1}{\varepsilon_1} + \phi(\mu)\left(1+\frac{1}{\varepsilon_2}\right) + \frac{4(m-1)}{\sqrt{m}(m-2)}\left(1+\frac{1}{\varepsilon_5}\right).$$

Since $m > p^4$, we have

$$\frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(m-1)}{\sqrt{m}(m-2)} > 0$$

Since $\int_M |T|^{m/2} < \infty$, we can choose r_0 large enough such that

$$\left(\int_{M\setminus B_{x_0}(r_0)} |T|^{m/2}\right)^{2/m} < \frac{1}{S}\left[\frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(m-1)}{\sqrt{m}(m-2)}\right]$$

Hence we can choose ε_1, ε_2, ε_5 small enough such that

$$\tilde{A} = \frac{4}{p^2}\frac{(m-1)(p-1)+1}{m-1} - \frac{4(2p-3)}{p}\varepsilon_1 - \phi(\mu)(1+\varepsilon_2) - \frac{4(m-1)}{\sqrt{m}(m-2)}(1+\varepsilon_5) > 0$$

Therefore, (57) can be written as

$$\int_{M\setminus B_{x_0}(r_0)} \mu^2|\nabla|\omega|^{p/2}|^2 \le \tilde{C}\int_{M\setminus B_{x_0}(r_0)} |\omega|^p|\nabla\mu|^2, \tag{58}$$

where $\tilde{C} > 0$ depends only on m, p. From (58) and the proof of Theorem 3.1, we can complete this proof. □

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