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PICK'S THEOREM WITH OPERATOR-VALUED HOLOMORPHIC FUNCTIONS

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§ 1. Introduction.

Let D be a domain (or a complex manifold) in C^n and let $H(D)$ be the class of holomorphic functions (or forms) in D. Let $\mathscr{H}(D)$ be a Hilbert space of elements in $H(D)$ with a reproducing kernel $K(z, \bar{\zeta})$. More specifically, we shall assume that $\mathscr{H}(D)$ is the space of all $f \in H(D)$ with the norm

$$\|f\|^2=\int_{D_0} |f(z)|^2 d\mu(z)<\infty .$$

Here μ is a positive measure acting on D_0, where D_0 is either D or any part of the boundary ∂D which determines the holomorphic functions in D as, for example, the Šilov boundary of D. In the case that D_0 is not D, f in the last integral stands for the nontangential boundary values of the holomorphic function $f(z)$, $z \in D$. In this way we may regard $\mathscr{H}(D) \in H_2(D : \mu)$ as a closed subspace of $L_2(D_0 : \mu)$ in a natural manner. Examples of such spaces are the familiar Bergman and Hardy-Szegö spaces to name only a few (see [2, 6, 9, 13] for more details).

Let $T(z)$ be a holomorphic function of $z \in D$ and whose values are bounded linear operators of a Hilbert space U into a Hilbert space W with norm $\|T(z)\| \leqq 1$. Thus, $T(z)$ is a holomorphic contraction of U into W. Let I_U and I_W be the identity operators of U and W, respectively. For any $z \in D$,

$$\|T(z)u\|_W^2 \leqq \|u\|_U^2 ; \quad \|T(z)^* w\|_U^2 \leqq \|w\|_W^2$$

for every $u \in U$ and $w \in W$, where $T(z)^*$ is adjoint operator of $T(z)$. These generate the kernel

$$K_T(z, \bar{\zeta})=K(z, \bar{\zeta})[I_U-T(\zeta)^* T(z)] , \tag{1.1}$$

which is holomorphic in $(z, \bar{\zeta})$ for $(z, \zeta) \in D \times D$ and is a bounded linear operator from H into U, and the "dual" kernel

$$\mathcal{K}_{T^*}(\bar{\zeta}, z)=K(z, \bar{\zeta})[I_W-T(z)T(\zeta)^*] \tag{1.2}$$

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which is holomorphic in $(\bar{\zeta}, z)$ for $(\zeta, z) \in D\times D$ and is a bounded linear operator from W into W.

One of the main purposes of this paper is to prove the following statement:

THEOREM A. *The kernel* $\mathcal{K}_T(z, \bar{\zeta})$ *is positive definite, that is*

$$\sum_{m,n} (\mathcal{K}_T(z_m, \bar{z}_n)u_m, u_n)_U \geqq 0$$

for every finite system $\{z_m\}_{m=1}^N$ *of points in* D, *and every corresponding system* $\{u_m\}_{m=1}^N$ *of vectors in* U.

It is our purpose to also prove the corresponding dual theorem regarding the kernel $\mathcal{K}_{T*}(\bar{\zeta}, z)$.

When D is the unit disk $\Delta=\{z\in \boldsymbol{C}: |z|<1\}$, $U=W=\boldsymbol{C}$, $I_U=1$ and $K(z, \bar{\zeta}) =(1-z\bar{\zeta})^{-1}$ is the Szegö kernel of Δ, then this theorem is classical and is due to Pick (cf. Ahlfors [1, pp. 3-4]). A more general case of Pick's theorem, namely when in (1.1) $K(z, \bar{\zeta})$ is again the Szegö kernel of the disk was first proved by Rovnyak [8] and is discussed in Sz.-Nagy and Foias [13, pp. 231-233].

In our previous works [4, 5] we extended the theorem of Pick by replacing the Szegö kernel of the disk Δ with some reproducing kernel of $H_2(D:\mu)$. The method of proof of this general assertion is simpler then those found in [1, pp. 3-4] and [13, pp. 231-233] for the less general case when $D=\Delta$ and $K(z, \bar{\zeta})$ is its Szegö kernel. In fact, our method of proof conceals in it the proof of the much more general assertion embodied in the present Theorem A.

In the course of proving the above theorem we also prove some other relevant assertions which extend those found in [13, pp. 231-233]. The proof of this theorem appears in Theorem 2 of this paper. Moreover, when $D\notin O_{AB}$ is a plane domain we provide yet another generalization of Theorem A by letting $K(z, \bar{\zeta})$ be the "*generalized Szegö kernel*" of D. This positive definite kernel is induced from the analytic capacity of D and was first studied by Suita [11]. The present Theorem 3 is dedicated to this generalization.

§ 2. Preliminaries.

As usual, the space $L_2(D_0:\mu)$ stands for the Hilbert space of μ-measurable functions (or rather μ-equivalence classes of functions) f on D_0 for which

$$\|f\|^2=\int_{D_0} |f(z)|^2 d\mu(z)<\infty .$$

The inner product is, of course, given by

$$(f, g)=\int_{D_0} f(z)\overline{g(z)}d\mu(z)<\infty .$$

The space $H_2(D:\mu)$, mentioned in the introduction, may be regarded as a closed subspace of $L_2(D_0 : \mu)$ in a natural way. We shall write $k_\zeta(z)=K(z, \bar{\zeta})$; $z, \zeta\in D$, for the reproducing kernel of $H_2(D : \mu)$ and, therefore, $k_\zeta\in H_2(D : \mu)$ with

$$f(\zeta)=(f, k_\zeta), \qquad f\in H_2(D : \mu)$$

and

$$K(z, \bar{\zeta})=k_\zeta(z)=(k_\zeta, k_z)=\overline{K(\zeta, \bar{z})}, \qquad K(z, \bar{z})=(k_z, k_z)\geqq 0 .$$

Clearly, convergence in the norm in $H_2(D : \mu)$ implies uniform convergence on compacta of D.

Let U be any Hilbert space with the inner product $(\ ,\ )_U$. The space $L_2(U : \mu)$ consists of those functions $f(\cdot)\in U^{D_0}$ which are strongly μ-measurable and such that

$$\|f\|^2_{L_2(U\cdot\mu)}=\int_{D_0}\|f(z)\|^2_U d\mu(z)<\infty .$$

With this definition $L_2(U : \mu)$ becomes a Hilbert space with the inner product

$$(f, g)_{L_2(U\cdot\mu)}=\int_{D_0}(f(z), g(z))_U d\mu(z) ,$$

where, as usual, two functions in $L_2(U : \mu)$ are regarded as identical if they coincide μ-a. e. on D_0. Moreover, from every converging sequence of $L_2(U : \mu)$ one can extract a convergent subsequence which converges in the U-norm μ-a. e. on D_0.

The class of U-holomorphic functions in D will be denoted by $H(D : U)$. Thus, $H(D : U)$ is the class of all $f(\cdot)\in U^D$ for which $(f(z), u)_U$ is holomorphic in $z\in D$ for each $u\in U$. We also denote by $H_2(U : \mu)$ the subspace of $L_2(U : \mu)$ consisting of those functions $f\in L_2(U : \mu)$ for which $(f(\cdot), u)_U\in H_2(D : \mu)$ for each $u\in U$. For any $u\in U$ we have

$$\|(f(\cdot), u)_U\|^2=\int_{D_0}|(f(z), u)_U|^2 d\mu(z)\leqq\|u\|^2_U\|f\|^2_{L_2(U\cdot\mu)},$$

which shows that $H_2(U : \mu)$ is a closed subspace of $L_2(U : \mu)$. Also, one may regard $H_2(U : \mu)$ as a subclass of $H(D : U)$ in a natural way.

Some modifications of the arguments will be needed when D_0 is not D. This, however, does not constitute a major problem whenever $H_2(D : \mu)$ is well-defined. For sake of completeness we shall describe in some detail such an instance when D is a plane domain. This, along with other possible extensions, will be given in § 4.

We have the following obvious proposition:

PROPOSITION 1. *The following hold*:

(i) *For every* $(z, u)\in D\times U$ *and each* $f\in H_2(U : \mu)$

$$(f, k_z(\cdot)u)_{L_2(U\cdot\mu)}=(f(z), u)_U .$$

(ii) *For every* (z, u), $(\zeta, v)\in D\times U$

$$(k_\zeta(\cdot)v,\ k_z(\cdot)u)_{L_2(U\cdot\mu)}=(k_\zeta(z)v,\ u)_U=K(z,\ \bar{\zeta})(v,\ u)_U\,.$$

(iii) *For every* $(z,\ u)\in D\times U$

$$\|k_z(\cdot)u\|^2_{L_2(U\cdot\mu)}=K(z,\ \bar{z})\|u\|^2_U\,.$$

A kernel $\mathcal{K}(z,\ \zeta)$ defined on $D\times D$, whose values are bounded operators on U is said to be positive definite, in short $\mathcal{K}\gg 0$, if

$$\sum_{m,n}(\mathcal{K}(z_m,\ z_n)u_m,\ u_n)_U\geqq 0$$

for every finite system $\{z_m\}_{m=1}^N$ of points of D, and every corresponding vectors $\{u_m\}_{m=1}^N$ of U. For two kernels, the notation $\mathcal{K}_1\gg\mathcal{K}_2$ or $\mathcal{K}_2\ll\mathcal{K}_1$ will indicate that $\mathcal{K}_1-\mathcal{K}_2\gg 0$.

§ 3. Holomorphic Kernels.

Let U and W be two Hilbert spaces. The Banach algebra of all bounded linear operators from U into W is denoted by $\mathcal{B}(U:W)$. Let $T(z)\in\mathcal{B}(U:W)$ which is holomorphic in $z\in D$. Thus, for any $(u,\ w)\in U\times W$, $(T(z)u,\ w)_W$ is holomorphic in $z\in D$ i.e., $(T(\cdot)u,\ w)_W\in H(D)$. Since $T(z)$ is in $\mathcal{B}(U:W)$ for each $z\in D$, it is clear that its adjoint $T(z)^*$ is in $\mathcal{B}(W:U)$ and, moreover, it is antiholomorphic in $z\in D$. We shall assume that $T(z)$ is also a contraction, i.e., $\|T(z)\|=\|T(z)^*\|\leqq 1$ for all $z\in D$. This, of course, means

$$\|T(z)u\|^2_W\leqq\|u\|^2_U\,;\qquad \|T(z)^*w\|^2_U\leqq\|w\|^2_W$$

for every $z\in D$ and $(u,\ w)\in U\times W$. We therefore, have $T(z)^*T(z)\leqq I_U$ and $T(z)T(z)^*\leqq I_W$ for all $z\in D$ and so one can form the positive operators

$$\Omega_U(z)=[I_U-T(z)^*T(z)]^{1/2}$$

and

$$\Omega_W(z)=[I_W-T(z)T(z)^*]^{1/2},$$

which are self-adjoint on U and W respectively. Here, $0\leqq\|\Omega_U(z)\|,\ \|\Omega_W(z)\|\leqq 1$ for every $z\in D$.

We now define the following holomorphic kernels belonging to $\mathcal{B}(U:U)$. For $(z,\ \zeta)\in D\times D$ we let

$$\mathcal{I}_U(z,\ \bar{\zeta})=K(z,\ \bar{\zeta})I_U\,,\qquad \mathcal{R}_T(z,\ \bar{\zeta})=T(\zeta)^*T(z)\,,$$

$$\mathcal{B}_T(z,\ \bar{\zeta})=K(z,\ \bar{\zeta})\mathcal{R}_T(z,\ \bar{\zeta})$$

and thus, in view of (1.1),

$$\mathcal{K}_T(z,\ \bar{\zeta})=\mathcal{I}_U(z,\ \bar{\zeta})-\mathcal{B}_T(z,\ \bar{\zeta})\,.$$

These kernels are holomorphic in $(z,\ \bar{\zeta})$ for $(z,\ \zeta)\in D\times D$ and they are Hermitian,

i. e.,

$$\mathcal{G}_U(z, \bar{\zeta})^* = \mathcal{G}_U(\zeta, \bar{z})$$

and so on for the remaining kernels. A holomorphic and Hermitian kernel $L(z, \bar{\zeta})$ belonging to $\mathcal{B}(U:U)$ is said to be of class $H_2(U:\mu)$, if for any fixed $\zeta \in D$, $L(z, \bar{\zeta})u \in H_2(U:\mu)$, as a function of $z \in D$, for every $u \in U$.

We start with the following theorem, where some care is needed when D_0 is not D (see § 4).

THEOREM 1. *$\mathcal{G}_U(z, \bar{\zeta})$, $\mathcal{B}_T(z, \bar{\zeta})$ and $\mathcal{K}_T(z, \bar{\zeta})$ are of class $H_2(U:\mu)$. Moreover, $\mathcal{R}_T(z, \bar{\zeta})$ is of class $H_2(U:\mu)$ provided $\mu(D_0) < \infty$.*

Proof.

$$\|\mathcal{G}_U(\cdot, \bar{\zeta})u\|^2_{L_2(U\cdot\mu)} = \int_{D_0} \|K(z, \bar{\zeta})u\|^2_U d\mu(z) = \|u\|^2_U \int_{D_0} |K(z, \bar{\zeta})|^2 d\mu(z)$$
$$= K(\zeta, \bar{\zeta})\|u\|^2_U < \infty .$$

Next,

$$\|\mathcal{B}_T(\cdot, \bar{\zeta})u\|^2_{L_2(U\cdot\mu)} = \int_{D_0} |K(z, \bar{\zeta})|^2 \|T(\zeta)^* T(z) u\|^2_U d\mu(z)$$
$$\leqq \|u\|^2_U K(\zeta, \bar{\zeta}) < \infty .$$

It therefore, also follows that $\mathcal{K}_T(z, \bar{\zeta})$ is of class $H_2(U;\mu)$. Moreover, when $\mu(D_0) < \infty$ we have

$$\|\mathcal{R}_T(\cdot, \bar{\zeta})u\|^2_{L_2(U\cdot\mu)} = \int_{D_0} \|T(\zeta)^* T(z) u\|^2_U d\mu(z)$$
$$\leqq \|u\|^2_U \int_{D_0} d\mu(z) = \|u\|^2_U \mu(D_0) < \infty ,$$

which concludes the proof.

In analogy to the previous holomorphic kernels we also define the following kernels belonging $\mathcal{B}(W:W)$. For $(\zeta, z) \in D \times D$ we let

$$\mathcal{G}_W(\bar{\zeta}, z) = K(z, \bar{\zeta}) I_W , \qquad \mathcal{R}_{T*}(\bar{\zeta}, z) = T(z) T(\zeta)^*$$
$$\mathcal{B}_{T*}(\bar{\zeta}, z) = K(z, \bar{\zeta}) \mathcal{R}_{T*}(\bar{\zeta}, z)$$

and thus, in view of (1.2),

$$\mathcal{K}_{T*}(\bar{\zeta}, z) = \mathcal{G}_W(\bar{\zeta}, z) - \mathcal{B}_{T*}(\bar{\zeta}, z) .$$

Clearly, these kernels are holomorphic in $(\bar{\zeta}, z)$ for $(\zeta, z) \in D \times D$ and they are Hermitian (for example, $\mathcal{B}_{T*}(\bar{\zeta}, z)^* = \mathcal{B}_{T*}(\bar{z}, \zeta)$). A holomorphic and Hermitian kernel $L(\bar{\zeta}, z)$ belonging to $\mathcal{B}(W:W)$, is said to be of class $H_2(W:\mu)$, if for any fixed $\zeta \in D$, $L(\zeta, z)w \in H_2(W:\mu)$, as a function of $z \in D$, for every $w \in W$.

Similarly to Theorem 1 we have:

THEOREM 1.* *$\mathcal{J}_W(\bar{\zeta}, z)$, $\mathcal{B}_{T^*}(\bar{\zeta}, z)$ and $\mathcal{K}_{T^*}(\bar{\zeta}, z)$ are of class $H_2(W:\mu)$. Moreover, $\mathcal{R}_{T^*}(\bar{\zeta}, z)$ is of class $H_2(W:\mu)$ provided $\mu(D_0)<\infty$.*

The following lemma is crucial and, again, one must make the appropriate interpretation when D_0 is not D.

LEMMA 1. *Let $(z, \zeta)\in D\times D$ and $(u, v)\in U\times U$. Then $\mathcal{K}_T(z, \bar{\zeta})\in H_2(U:\mu)$ and*

$$(\mathcal{K}_T(z, \bar{\zeta})u, v)_U=(\bar{k}_z[T(\cdot)-T(z)]u, \bar{k}_\zeta[T(\cdot)-T(\zeta)]v)_{L_2(W\cdot\mu)}$$
$$+(\bar{k}_z\Omega_U(\cdot)u, \bar{k}_\zeta\Omega_U(\cdot)v)_{L_2(U\cdot\mu)}\,.$$

Proof. We use the reproducing property of $k_\zeta(z)=K(z, \bar{\zeta})$ as given in Proposition 1. Thus,

$$\begin{aligned}(\mathcal{K}_T(z, \bar{\zeta})u, v)_U&=(\mathcal{K}_T(\cdot, \bar{\zeta})u, k_z(\cdot)v)_{L_2(U\cdot\mu)}\\&=(k_\zeta(\cdot)[I_U-T(\zeta)^*T(\cdot)]u, k_z(\cdot)v)_{L_2(U\cdot\mu)}\\&=(k_\zeta(\cdot)[I_U-T(\cdot)^*T(\cdot)]u, k_z(\cdot)v)_{L_2(U\cdot\mu)}\\&\quad+(k_\zeta(\cdot)[T(\cdot)^*-T(\zeta)^*]T(\cdot)u, k_z(\cdot)v)_{L_2(U\cdot\mu)}\\&=(\bar{k}_z\Omega_U(\cdot)u, \bar{k}_\zeta\Omega_U(\cdot)v)_{L_2(U\cdot\mu)}\\&\quad+(k_\zeta(\cdot)T(\cdot)u, k_z(\cdot)[T(\cdot)-T(\zeta)]v)_{L_2(W\cdot\mu)}\,.\end{aligned}$$

However,

$$\begin{aligned}&(k_\zeta(\cdot)T(\cdot)u, k_z(\cdot)[T(\cdot)-T(\zeta)]v)_{L_2(W\cdot\mu)}\\&=(\bar{k}_z[T(\cdot)-T(z)]u, \bar{k}_\zeta[T(\cdot)-T(\zeta)]v)_{L_2(W\cdot\mu)}\\&\quad+(k_\zeta(\cdot)T(z)u, k_z(\cdot)[T(\cdot)-T(\zeta)]v)_{L_2(W\cdot\mu)}\,.\end{aligned}$$

We will therefore, in view of Theorem 1, conclude the proof by showing that the last term is zero. In fact, in view of Proposition 1,

$$(k_\zeta(\cdot)T(z)u, k_z(\cdot)T(\zeta)v)_{L_2(W\cdot\mu)}=K(z, \bar{\zeta})(T(z)u, T(\zeta)v)_W$$

and

$$\begin{aligned}(k_\zeta(\cdot)T(z)u, k_z(\cdot)T(\cdot)v)_{L_2(W\cdot\mu)}&=\overline{(k_z(\cdot)T(\cdot)v, k_\zeta(\cdot)T(z)u)}_{L_2(W\cdot\mu)}\\&=\overline{(k_z(\zeta)T(\zeta)v, T(z)u)_W}=\overline{k_z(\zeta)(T(\zeta)v, T(z)u)_W}\\&=K(z, \bar{\zeta})(T(z)u, T(\zeta)v)_W\end{aligned}$$

which concludes the proof.

Similarly to this lemma we also obtain:

LEMMA 1.* *Let* $(\zeta, z)\in D\times D$ *and* $(\omega, w)\in W\times W$. *Then* $\mathcal{K}_{T*}(\bar{\zeta}, z)\in H_2(W:\mu)$ *and*

$$(\mathcal{K}_{T*}(\bar{\zeta}, z)\omega, w)_W=(k_\zeta[T(\cdot)^*-T(\zeta)^*]\omega, k_z[T(\cdot)^*-T(z)^*]w)_{L_2(U:\mu)}$$
$$+(k_\zeta\Omega_W(\cdot)\omega, k_z\Omega_W(\cdot)w)_{L_2(W:\mu)}.$$

We are now is a position to prove our main results.

THEOREM 2. *The kernels* $\mathcal{G}_U(z, \bar{\zeta})$, $\mathcal{R}_T(z, \bar{\zeta})$, $\mathcal{B}_T(z, \bar{\zeta})$ *and* $\mathcal{K}_T(z, \bar{\zeta})$ *are positive definite. In fact,*

$$\mathcal{R}_T\gg 0,\quad \mathcal{G}_U\gg\mathcal{B}_T\gg 0;\quad \mathcal{K}_T\gg 0.$$

Proof. For $\{z_m\}_{m=1}^N$ in D and corresponding vectors $\{u_m\}_{m=1}^N$ of U we have:

$$\sum_{m,n}(\mathcal{R}_T(z_m, \bar{z}_n)u_m, u_n)_U=\sum_{m,n}(T(z_n)^*T(z_m)u_m, u_n)_U$$
$$=\sum_{m,n}(T(z_m)u_m, T(z_n)u_n)_W=\|\sum_m T(z_m)u_m\|_W^2\geqq 0,$$

showing that $\mathcal{R}_T\gg 0$. Further, by Proposition 1,

$$\sum_{m,n}(\mathcal{B}_T(z_m, \bar{z}_n)u_m, u_n)_U=\sum_{m,n}K(z_m, \bar{z}_n)(T(z_n)^*T(z_m)u_m, u_n)_U$$
$$=\sum_{m,n}K(z_m, \bar{z}_n)(T(z_m)u_m, T(z_n)u_n)_W$$
$$=\sum_{m,n}(\bar{k}_{z_m}T(z_m)u_m, \bar{k}_{z_n}T(z_n)u_n)_{L_2(W:\mu)}$$
$$=\|\sum_m \bar{k}_{z_m}T(z_m)u_m\|_{L_2(W:\mu)}^2\geqq 0$$

and therefore $\mathcal{B}_T\gg 0$. Finally, by virtue of Lemma 1, we have

$$\sum_{m,n}(\mathcal{K}_T(z_m, \bar{z}_n)u_m, u_n)_U=\sum_{m,n}(\bar{k}_{z_m}[T(\cdot)-T(z_m)]u_m, \bar{k}_{z_n}[T(\cdot)-T(z_n)]u_n)_{L_2(W:\mu)}$$
$$+\sum_{m,n}(\bar{k}_{z_m}\Omega_U(\cdot)u_m, \bar{k}_{z_n}\Omega_U(\cdot)u_n)_{L_2(U:\mu)}$$
$$=\|\sum_m \bar{k}_{z_m}[T(\cdot)-T(z_m)]u_m\|_{L_2(W:\mu)}^2$$
$$+\|\sum_m \bar{k}_{z_m}\Omega_U(\cdot)u_m\|_{L_2(U:\mu)}^2\geqq 0,$$

which shows that $\mathcal{K}_T\gg 0$ or $\mathcal{G}_U\gg\mathcal{B}_T$. This concludes the proof.
Similarly, by appealing to Lemma 1*, we also obtain:

THEOREM 2.* *The kernels* $\mathcal{G}_W(\bar{\zeta}, z)$, $\mathcal{R}_{T*}(\bar{\zeta}, z)$, $\mathcal{B}_{T*}(\bar{\zeta}, z)$ *and* $\mathcal{K}_{T*}(\bar{\zeta}, z)$ *are positive definite. In fact,*

$$\mathcal{R}_{T*}\gg 0,\quad \mathcal{G}_W\gg\mathcal{B}_{T*}\gg 0;\quad \mathcal{K}_{T*}\gg 0.$$

The most important results of the previous two theorems are, of course, that $\mathcal{K}_T \gg 0$ or $\mathcal{K}_{T^*} \gg 0$. Since these generalize the classical result of Pick (see Ahlfors [1, pp. 3-4]) one might expect that the positive definiteness of these kernels would lead to some distortion theorems of the Schwarz-Pick type. That this is indeed the case is shown in the following corollaries (see also [4, 5]).

COROLLARY 1. *Let u be any vector in U and let $\{z_m\}_{m=1}^N$ be points in D. Then*

$$\det[(\mathcal{K}_T(z_m, \bar{z}_n)u,\ u)_U]_{m,n}^N \geqq 0\,.$$

Proof. Let $\{\alpha_m\}_{m=1}^N$ be scalars in $\boldsymbol{C}$ and define $u_m = \alpha_m u$, $1 \leqq m \leqq N$. Since $\mathcal{K}_T \gg 0$, it follows that

$$\sum_{m,n} (\mathcal{K}_T(z_m, \bar{z}_n)u_m,\ u_n)_U = \sum_{m,n} (\mathcal{K}_T(z_m, \bar{z}_n)u,\ u)\alpha_m \bar{\alpha}_n \geqq 0$$

and the result follows.

We say that D is of class $\mathcal{M}$ if for any $\zeta \in D$ there exists an $f \in H_2(D:\mu)$ so that $f(\zeta) \neq 0$. Clearly, $D \in \mathcal{M}$ if and only if $K(z, \bar{z}) > 0$ for every $z \in D$.

COROLLARY 2. *Let $D \in \mathcal{M}$ and let $u \in U$ be a unit vector. Suppose further that $\|T(z)u\|_W$ is not a constant throughout D. Then*

$$\frac{|K(z, \bar{\zeta})|^2}{K(z, \bar{z})K(\zeta, \bar{\zeta})} \leqq \frac{[1 - \|T(z)u\|_W^2]\cdot[1 - \|T(\zeta)u\|_W^2]}{|1 - (T(z)u,\ T(\zeta)u)_W|^2}$$

for any $z, \zeta \in D$.

Proof. This follows from Corollary 1 by taking $N=2$ and $z_1 = z$, $z_2 = \zeta$. Note also that the denominator of the right hand side of the inequality does not vanish for every $z, \zeta \in D$. Indeed, $\|T(z)u\|_W \leqq 1$ and by the maximum principle (see [7, p. 100]), in view of the non-constancy of $\|T(z)u\|_W$, we actually have $\|T(z)u\|_W < 1$ for each $z \in D$. Consequently, $|(T(z)u,\ T(\zeta)u)_W| < 1$ for every $z, \zeta \in D$. The left hand side of the inequality is, of course, well defined because $D \in \mathcal{M}$. This concludes the proof.

If f is a C^1-function near $z = (z_1, \cdots, z_n) \in \boldsymbol{C}^n$, and, $v = (v_1, \cdots, v_n) \in \boldsymbol{C}^n$, we write

$$\partial_v f(z) = \sum_{j=1}^n v_j \partial_{z_j} f(z)\,, \qquad \bar{\partial}_v f(z) = \sum_{j=1}^n \bar{v}_j \partial_{\bar{z}_j} f(z)\,.$$

Let $D \in \mathcal{M}$, $\zeta \in D$ and $v \in \boldsymbol{C}^n$. We write

$$b_{\mu(D)}^2(\zeta : v) = \partial_v \bar{\partial}_v \log K\,, \qquad K = K(\zeta, \bar{\zeta})\,,$$

and note that when $d\mu$ is the usual (Lebesgue) volume element then this expression is, precisely, the classical Bergman metric for D. We also have

$$b_{\mu(D)}(\zeta : v) - \sqrt{K(\zeta, \bar{\zeta})} \max\{|\partial_v f(z)| : f \in \mathcal{B}_\zeta(D : \mu)\}\,,$$

where

$$\mathcal{B}_{\zeta}(D:\mu)=\{f\in H_2(D:\mu):\ \|f\|\leqq 1,\ f(\zeta)=0\},$$

(see [2, p. 26] and [3]).

COROLLARY 3. *Let the assumptions of Corollary 2 prevail. Assume further that* $v\in \boldsymbol{C}^n$. *Then*

$$b^2_{\mu(D)}(z:v)\geqq \frac{[1-\|T(z)u\|_W^2]\cdot\|\partial_v T(z)u\|_W^2+|(\partial_v T(z)u,\ T(z)u)_W|^2}{[1-\|T(z)u\|_W^2]^2}$$

and thus

$$b_{\mu(D)}(z\,;\,v)\geqq \|\partial_v T(z)u\|_W$$

for any $z\in D$.

Proof. For a fixed $\zeta\in D$, consider the function

$$g(z,\bar{z})=\log\frac{[1-\|T(z)u\|_W^2]\cdot[1-\|T(\zeta)u\|_W^2]}{[1-(T(z)u,\ T(\zeta)u)_W]^2}-\log\frac{|K(z,\bar{\zeta})|^2}{K(z,\bar{z})K(\zeta,\bar{\zeta})}.$$

According to Corollary 2, $g\geqq 0$ and, moreover, g assumes a local minimum at $z=\zeta$. Therefore, for each direction $v\in \boldsymbol{C}^n$ the Hessian of g

$$\Delta_v g=4\partial_v\bar{\partial}_v g$$

is non-negative at $z=\zeta$. However, by a direct computation,

$$\Delta_v g=-4\left\{\frac{\|\partial_v T(z)u\|_W^2}{1-\|T(z)u\|_W^2}+\frac{|(\partial_v T(z)u,\ T(z)u)_W|^2}{[1-\|T(z)u\|_W^2]^2}-\partial_v\bar{\partial}_v\log K(z,\bar{z})\right\}$$

and the corollary follows.

The last corollary also shows that the so-called "*Carathéodory-Reiffen metric*" of D is always dominated by $b_{\mu(D)}(z:v)$ (see [3, 4] for more details).

Remark. The previous results were proven under the assumptions that $T(z)$ is contractive and thus, $T(z)^*T(z)\leqq I_U$ and $T(z)T(z)^*\leqq I_W$. These results can be further generalized by assuming instead that $T(z)^*T(z)\leqq A$ and $T(z)T(z)^*\leqq B$ where $A\in\mathcal{B}(U:U)$ and $B\in\mathcal{B}(W:W)$ are constant operators. Indeed, we can, for example, write $T(z)=S(z)A^{1/2}$ with $S(z)\in\mathcal{B}(U:W)$ and $\|S(z)\|\leqq 1$. Then $A-T(z)^*T(z)=A^{1/2}[I_U-S(z)^*S(z)]A^{1/2}$ and one proceeds as before. We omit the details.

§ 4. The Szegö Kernel.

The results exhibited earlier are valid for holomorphic reproducing kernels of the Hilbert spaces $H_2(D:\mu)$. However, some care is needed when $d\mu$ does not act on D but rather, say, on the Šilov boundary of D. This care is mostly needed in providing a precise meaning for $H_2(U:\mu)$ or $H_2(W:\mu)$ where the inner products are defined via the boundary values of certain holomorphic vector functions. As an example we shall treat the case when D is a plane domain and, in so doing, we even obtain an extension of the previous results.

Let D be a plane domain whose boundary consists of a finite number of rectifiable curves. We shall assume that ∂D is in fact smooth or more generally of Smirnov type. Let $L_2(\partial D)$ designate the customary Hilbert space of functions on ∂D with the inner product

$$(f, g)=\int_{\partial D} f(z)\overline{g(z)}\,|dz|\ ;\ \|f\|=\sqrt{(f, f)},$$

that is, here we are taking $D_0=\partial D$ and $d\mu(z)=|dz|$. Let $L_2^+(\partial D)$ be the closed linear subspace of $L_2(\partial D)$ consisting of those functions $\nu\in L_2(\partial D)$ for which

$$\frac{1}{2\pi i}\int_{\partial D}\frac{\nu(z)}{z-\zeta}\,dz=0\,, \qquad \zeta\in E_D\equiv\hat{C}-\bar{D}\,.$$

Then

$$f(\zeta)=\frac{1}{2\pi i}\int_{\partial D}\frac{\nu(z)}{z-\zeta}\,dz\,, \qquad \zeta\in D\,, \tag{4.1}$$

is holomorphic in D and ν coincides almost everywhere on ∂D with the nontangential limit of f.

We now introduce the classical Hardy-Szegö space $H_2(D)$. A function $f\in H(D)$ is said to belong to $H_2(D)$ if there exists an exhaustion $\{D_m\}$ of D such that each ∂D_m is smooth, the lengths of the ∂D_m are bounded, and

$$\|f\|_H^2\equiv\overline{\lim_{m\to\infty}}\int_{\partial D_m}|f(z)|^2|dz|<\infty\,. \tag{4.2}$$

It is well known that every $f\in H_2(D)$ has a nontangential limit ν almost everywhere on ∂D with $\|f\|_H=\|\nu\|$ and that f can be recovered from its boundary function ν via a Cauchy integral over ∂D. Since ∂D is of Smirnov type, the set of boundary functions of $H_2(D)$ is precisely the class $L_2^+(\partial D)$. Therefore, each $\nu\in L_2^+(\partial D)$ determines exactly one holomorphic function $f\in H_2(D)$ by (4.1) with $\|f\|_H=\|\nu\|$, and, because of this uniqueness we shall not destinguish between $\nu(\zeta)$ and $f(\zeta)$, $\zeta\in\partial D$. We can therefore, identify $H_2(D)$ with $L_2^+(\partial D)$, thus providing $H_2(D)$ with the Hilbert structure of $L_2^+(\partial D)$ and embedding it in $L_2(\partial D)$ as a closed subspace.

In view of (4.1) point evaluations are bounded linear functionals on $H_2(D)$ and therefore, $H_2(D)$ admits a reproducing kernel $k_\zeta(z)=K(z, \bar{\zeta})$ which is the classical Szegö kernel for D. Let U be any Hilbert space with the inner product $(\ ,\ )_U$. As before, we consider the space $L_2(U)$ of functions $f(\cdot)\in U^{\partial D}$ which are (strongly) measurable on ∂D and for which

$$\|f\|_{L_2(U)}^2=\int_{\partial D}\|f(z)\|_U^2\,|dz|<\infty\,.$$

This is, of course, a Hilbert space with the inner product

$$(f, g)_{L_2(U)}=\int_{\partial D}(f(z), g(z))_U |dz|.$$

We let $L_2^+(U)$ be the closed subspace of $L_2(U)$ consisting of those functions $\nu\in L_2(U)$ for which $(\nu(z), u)_U\in L_2^+(\partial D)$, as a function of $z\in\partial D$, for every $u\in U$. Here, $(\nu(z), u)_U$ is the boundary value of the holomorphic function $(f(z), u)_U$ where

$$f(\zeta)=\frac{1}{2\pi i}\int_{\partial D}\frac{\nu(z)}{z-\zeta}dz, \qquad \zeta\in D, \tag{4.3}$$

the integral being in the sense of Bochner [7, p. 95]. Analogously, we can define the class $H_2(U)$, thus $f\in H(D:U)$ belongs to $H_2(U)$ if

$$\overline{\lim_{m\to\infty}}\int_{\partial D_m}\|f(z)\|_U^2|dz|<\infty,$$

where $\{D_m\}$ is an exhaustion of D as that in (4.2). Therefore, every $f\in H_2(U)$ has a nontangential limit (in the U-norm convergence) $\nu\in L_2^+(U)$ almost everywhere on ∂D and can be recovered from ν by (4.3). Consequently, we can identify $H_2(U)$ and $L_2^+(U)$ in the previous natural way. An account for the main part of the above exposition can be found, for example, in [6, pp. 167-185] and [13, pp. 183-190] where further literature is quoted.

Once the spaces $L_2(U)$ and $H_2(U)$ have been introduced it obviously follows that Proposition 1, Lemma 1 (or 1*), Theorems 1, 2 (or 1*, 2*) and their corollaries hold true in this setting.

We now pass to the more general case where D is merely assumed to be $D\notin O_{AB}$, i.e., D admits a non-constant bounded holomorphic function. In this case one can define the "*analytic capacity*" of D at $\zeta\in D$ by

$$C_D(\zeta)=\max\{|f'(\zeta)| : f\in H_\zeta(D:\Delta)\},$$

where, $H_\zeta(D:\Delta)$ is the family of all holomorphic functions f from D into the unit disk Δ with $f(\zeta)=0$. Then, there exists a unique $F\in H_\zeta(D:\Delta)$, called the "*Ahlfors function*" $F(z)=F(z:\zeta)$, with $F'(\zeta:\zeta)=C_D(\zeta)>0$. Moreover, if D is a plane domain with a smooth boundary then $C_D(\zeta)=2\pi K(\zeta, \bar\zeta)$, where $K(z, \bar\zeta)$ is the Szegö kernel for D (see [2, p. 118]).

Let $\{D_m\}$ be a canonical exhaustion of $D\notin O_{AB}$ such that each D_m consists of a finite number of analytic curves. Here, D_m eventually contains each compact subset of D. In every D_m we have the Szegö kernel $K_m(z, \bar\zeta)$, the analytic capacity $C_m(z)$ and the Ahlfors function $F_m(z)=F_m(z:\zeta)$. Obviously, $F'_m(\zeta)=C_m(\zeta)=2\pi K_m(\zeta, \bar\zeta)$ and $F_m(\zeta)=0$. Under these circumstances, $\{F_m(z)\}$ and $\{C_m(z)\}$ converge uniformly on compacta of D to $F(z)$ and $C_D(z)$, respectively. The same is also true for the sequence $\{K_m(z, \bar\zeta)\}$ as was pointed out by Suita [11]. Indeed, $\{K_m(z, \bar\zeta)\}$ converges uniformly on compacta of D to a function $K(z, \bar\zeta)$ which is holomorphic in $(z, \bar\zeta)$ with $(z, \zeta)\in D\times D$.

The function $K(z, \bar{\zeta})$ will be called the "*generalized Szegö kernel*" for $D \notin O_{AB}$. It is clearly an Hermitian positive definite kernel with $2\pi K(z, \bar{z}) = C_D(z)$ for every $z \in D$. Let $T(z)$ be a holomorphic contraction in D from U into W. Then the kernels $\mathcal{G}_U^{(m)}(z, \bar{\zeta})$, $\mathcal{R}_T^{(m)}(z, \bar{\zeta})$ and so on are, of course, well defined. In particular, $\{\mathcal{K}_T^{(m)}(z, \bar{\zeta})\}$, given by

$$\mathcal{K}_T^{(m)}(z, \bar{\zeta}) = K_m(z, \bar{\zeta})[I_U - T(\zeta)^* T(z)],$$

converges uniformly on compacta of D to $\mathcal{K}_T(z, \bar{\zeta}) = K(z, \bar{\zeta})[I_U - T(\zeta)^* T(z)]$. This leads to an extension of Theorems 2 and 2* as follows:

THEOREM 3. *Let D be a plane domain, $D \notin O_{AB}$, and let $K(z, \bar{\zeta})$ be its generalized Szegö kernel. Let $\mathcal{K}_T(z, \bar{\zeta}) = K(z, \bar{\zeta})[I_U - T(\zeta)^* T(z)]$ have the same meaning as before. Then $\mathcal{K}_T \gg 0$ and the same holds for the corresponding kernel $\mathcal{K}_{T^*}(\bar{\zeta}, z)$.*

It is clear that Corollaries 1-3 are also valid for this generalized Szegö kernel. This leads to many interesting distortion theorems. In fact, already even in the scalar case where $U = W = \boldsymbol{C}$ we have, in view of Corollary 2,

$$\frac{|K(z, \bar{\zeta}|^2}{K(z, \bar{z})K(\zeta, \bar{\zeta})} \leqq \frac{[1-|f(z)|^2][1-|f(\zeta)|^2]}{[1-f(z)\overline{f(\zeta)}]^2}; \qquad z, \zeta \in D, \quad (D \notin O_{AB}), \tag{4.4}$$

where $f \in H(D : \varDelta)$. Here, $H(D : \varDelta)$ is the family of holomorphic functions from D into $\varDelta$. This inequality constitutes an improvement of the well known inequality $|K(z, \bar{\zeta})|^2 \leqq K(z, \bar{z})K(\zeta, \bar{\zeta})$. From (4.4) one deduces, as in Corollary 3, that the curvature of the analytic capacity is $\leqq -4$ (see [3] and [12] for details). It also shows that

$$\left|\frac{f(z)-f(\zeta)}{1-f(z)\overline{f(\zeta)}}\right| \leqq \left[1 - \frac{|K(z, \bar{\zeta})|^2}{K(z, \bar{z})K(\zeta, \bar{\zeta})}\right]^{1/2}; \qquad f \in H(D : \varDelta), \tag{4.5}$$

which means that the so-called "*Ahlfors distance*" is dominated by the right hand side of (4.5). Evidently, these results may be also formulated in terms of weighted Szegö kernels and the so-called "*Rudin kernels*" (see Saitoh [9, 10] for further information on these kernels). We omit the details.

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