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ON FACTORIZATION OF ENTIRE FUNCTIONS

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1. Introduction. A meromorphic function $F(z)=f(g(z))$ is said to have f and g as left and right factors respectively, provided that f is meromorphic and g is entire (g may be meromorphic when f is rational). $F(z)$ is said to be prime (pseudo-prime, left-prime, right-prime) if every factorization of the above form into factors implies either f is linear or g is linear (either f is rational or g is a polynomial, f is linear whenever g is transcendental, g is linear whenever f is transcendental). When factors are restricted to entire functions, it is called to be a factorization in entire sense.

Gross [4] posed the following problem:

(A) Given any entire function f, does there exist a polynomial Q such that $f+Q$ is prime?

Further, Gross-Yang-Osgood [6] posed the following problem:

(B) Given any entire function f, does there exist an entire function g such that fg is prime?

In this paper we shall give affirmative answers to the above two problems (Theorem 2 and Theorem 3). Further we shall show a similar result for periodic entire functions (Theorem 4). In each case it can be shown that almost all functions are prime.

According to [9], [10], we shall make use of the simultaneous equations

$$\begin{cases} F(z)=c, \\ F'(z)=0. \end{cases}$$

Theorem 1 and Theorem 5 are extensions of theorem 1 and theorem 2 in [10].

2. In this section we shall state the following two theorems which are used in the proof of Theorem 2 and Theorem 3.

THEOREM A (a modified version of theorem 2 in [9]). *Let $F(z)$ be a transcendental entire function satisfying $N(r, 0, F')>km(r, F')$ on a set of r of infinite*

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measure for some $k>0$. *Assume that the simultaneous equations*

$$\begin{cases} F(z)=c\,, \\ F'(z)=0 \end{cases}$$

have only finitely many common roots for any constant c. *Then* $F(z)$ *is left-prime in entire sense.*

The proof is essentially the same as that of theorem 2 in [9], hence omitted. The following theorem is an extension of theorem 2 in [10].

THEOREM 1. *Let* $F(z)$ *be a transcendental entire function with at least one simple zero satisfying*

$$N(r, 0, F')-(N(r, 0, F)-\bar{N}(r, 0, F))>kT(r, F'/F) \tag{2.1}$$

on a set of r *of infinite measure for some* $k>0$. *Assume that the simultaneous equations*

$$\begin{cases} F(z)=c\,, \\ F'(z)=0 \end{cases}$$

have only finitely many common roots for any non-zero constant c. *Then* $F(z)$ *is left-prime in entire sense.*

Proof. Let $F(z)=f(g(z))$.

a) f and g are transcendental entire. We consider the following two cases.

(1) There exists a complex number w_0 such that $f'(w_0)=0$ and $f(w_0)\neq 0$.

(2) If p is a zero of $f'(w)$, then $f(p)=0$.

Firstly we consider the case (1). By the assumption $g(z)$ must be of the form

$$g(z)=w_0+P(z)e^{G(z)}\,,$$

where $P(z)$ is a polynomial and $G(z)$ a non-constant entire function. Further if x is a zero of $f'(w)$ other than w_0, then $f(x)=0$. Thus

$$\begin{aligned} N(r, 0, F')&=N(r, 0, f'\circ g)+N(r, 0, g') \\ &\leqq (N(r, 0, F)-\bar{N}(r, 0, F))+N(r, 0, g')+O(\log r)\,. \end{aligned}$$

Therefore

$$N(r, 0, F')-(N(r, 0, F)-\bar{N}(r, 0, F))\leqq m(r, G')+O(\log r)\leqq O(m(r, G)) \tag{2.2}$$

outside a set of r of finite measure. Let p be a zero of $f(w)$. Then $p\neq w_0$. By the second fundamental theorem

$$(1-t)m(r, g)<\bar{N}(r, p, g)\leqq \bar{N}(r, 0, F) \tag{2.3}$$

outside a set of r of finite measure, where t is an arbitrarily fixed number in $(0, 1)$. By (2.1), (2.2) and (2.3)

$$m(r, g)<O(m(r, G))$$

on a set of r of infinite measure. By Clunie's theorem [1] we have a contradiction.

Secondly we consider the case (2). In this case

$$N(r, 0, F')=N(r, 0, f'\circ g)+N(r, 0, g')$$
$$\leqq N(r, 0, F)-\bar{N}(r, 0, F)+N(r, 0, g').$$

Thus

$$N(r, 0, F')-(N(r, 0, F)-\bar{N}(r, 0, F))\leqq O(m(r, g)) \tag{2.4}$$

outside a set of r of finite measure. There are the following two subcases.

(2, a) $f(w)$ has infinitely many zeros $\{w_n\}_{n=1}^{\infty}$.

(2, b) $f(w)$ has at most finitely many zeros.

In the case (2, a), by the second fundamental theorem,

$$(1-t)M\cdot m(r, g)<\sum_{n=1}^{2M}\bar{N}(r, w_n, g)\leqq\bar{N}(r, 0, F) \tag{2.5}$$

outside a set of r of finite measure, where t is an arbitrarily fixed number in $(0, 1)$ and M an arbitrarily fixed positive integer. By (2.1) and (2.5)

$$(1-t)kM\cdot m(r, g)<N(r, 0, F')-(N(r, 0, F)-\bar{N}(r, 0, F)) \tag{2.6}$$

on a set of r of infinite measure. Since M can be taken arbitrarily large, from (2.4) and (2.6) we have a contradiction.

In the case (2, b) $f(w)$ is of the form

$$f(w)=P(w)e^{H(w)}, \tag{2.7}$$

where $P(w)$ is a non-constant polynomial and $H(w)$ a non-constant entire function. Suppose that $H(w)$ is transcendental entire. Since

$$F'(z)/F(z)=g'(z)(P'(g(z))+P(g(z))H'(g(z)))/P(g(z)),$$

$$T(r, F'/F)\sim m(r, H'\circ g) \tag{2.8}$$

holds outside a set of r of finite measure. By (2.1), (2.4), (2.8) and Clunie's theorem [1], we have a contradiction. Thus $H(w)$ must be a polynomial.

Since

$$f'(w)=(P'(w)+P(w)H'(w))e^{H(w)},$$

by (2) and (2.7) we see that any root x of

$$P'(w)+P(w)H'(w)=0$$

satisfies

$$P(x)=P'(x)=0 . \tag{2.9}$$

By (2.9) $P(w)$ has at least one multiple zero. Let $\{a_i\}_i$ be the set of multiple zeros of $P(w)$ and n_i the multiplicity at a_i. Put

$$Q(w)=(P'(w)+P(w)H'(w))/\prod_i (w-a_i)^{n_i-1} .$$

Then $Q(w)$ is a polynomial satisfying $Q(a_i)\neq 0$ for every i. If x is a zero of $Q(w)$, then

$$P'(x)+P(x)H'(x)=0 .$$

Thus by (2.9) $x=a_i$ for some i. This is a contradiction. Thus $Q(w)$ is equal to a constant. Hence

$$\deg(P'+PH')=\sum_i (n_i-1) .$$

On the other hand the left side is not less than deg(P). And $\deg(P)\geqq\sum_i n_i$. Thus we have a contradiction. Therefore $F(z)$ is pseudo-prime in entire sense.

b) f is a polynomial of degree d ($\geqq 2$) and g is transcendental entire. We consider the same conditions (1) and (2) as in the case a). If the case (2) occurs, then it is easily seen that $f(w)$ must be of the form

$$f(w)=A(w-B)^d ,$$

where A and B are constants. This is a contradiction, since $F(z)$ has at least one simple zero. If the case (1) occurs, then using the same argument as in the case a) we have again a contradiction.

Theorem 1 is thus proved.

3. Problem (A).

THEOREM 2. *Let $f(z)$ be a transcendental entire function. Then the set*

$$\{a\in \boldsymbol{C}\,;\ f(z)+az \text{ is not prime}\}$$

is at most a countable set.

We shall first prove

LEMMA 1. *Let $f(z)$ be a transcendental entire function. Then there is a countable set E of complex numbers such that the simultaneous equations*

$$\begin{cases} f(z)-az=c , \\ f'(z)-a=0 \end{cases}$$

have at most one common root for any constant c $(\in \boldsymbol{C})$ *provided that* a *is in* $\boldsymbol{C}\backslash E$.

Proof. Let us write

$$A=\boldsymbol{C}-\{p\in\boldsymbol{C};\ f''(p)=0\}.$$

We choose open sets $\{c_i\}_{i=1}^{\infty}$ of A satisfying the following conditions.

(1) $\bigcup_{i=1}^{\infty} c_i=A$.

(2) $f'(z)$ is univalent in c_i $(i=1, 2, \cdots)$.

(3) $\{f'(z);\ z\in c_i\}$ is a disk $(i=1, 2, \cdots)$.

Put

$$D_i=\{f'(z);\ z\in c_i\} \qquad (i=1, 2, \cdots),$$

$$F(z)=f(z)-z\cdot f'(z), \tag{3.1}$$

$$u_i(w)=(f'|c_i)^{-1}(w) \qquad (w\in D_i,\ i=1, 2, \cdots), \tag{3.2}$$

$$v_i(w)=F(u_i(w)) \qquad (w\in D_i,\ i=1, 2, \cdots), \tag{3.3}$$

$$I=\{(i, j)\in\boldsymbol{N}\times\boldsymbol{N};\ D_i\cap D_j\neq\emptyset,\ v_i(w)\not\equiv v_j(w)\ (w\in D_i\cap D_j)\},$$

$$S_{i,j}=\{w\in D_i\cap D_j;\ v_i(w)=v_j(w)\} \qquad ((i, j)\in I), \tag{3.4}$$

$$E_0=\Big(\bigcup_{i=1}^{\infty} D_i\Big)-\Big(\{f'(p);\ f''(p)=0,\ p\in\boldsymbol{C}\}\cup\Big(\bigcup_{(i,j)\in I} S_{i,j}\Big)\Big). \tag{3.5}$$

Then $E=\boldsymbol{C}\backslash E_0$ is a countable set.

Let $a\in E_0$. If

$$v_i(a)=v_j(a) \tag{3.6}$$

for some i, j, then by (3.4) and (3.5)

$$v_i(w)\equiv v_j(w) \qquad (w\in D_i\cap D_j).$$

Thus

$$v_i'(a)=v_j'(a).$$

By (3.1), (3.2) and (3.3) we have

$$v_k'(a)=-u_k(a) \qquad (k=i, j).$$

Hence

$$u_i(a)=u_j(a). \tag{3.7}$$

From (3.1), (3.2) and (3.3) we have

$$v_k(a)=f(u_k(a))-a\cdot u_k(a) \qquad (k=i, j).$$

Thus from (3.6) and (3.7) we see that if

$$f(u_i(a))-a\cdot u_i(a)=f(u_j(a))-a\cdot u_j(a),$$

then

$$u_i(a)=u_j(a).$$

On the other hand, by (3.2) and (3.5), the set

$$\{u_k(a);\ a\in D_k,\ k=1, 2, \cdots\}$$

coincides with the set of distinct a-points $\{z_n\}_n$ of $f'(z)$. Therefore if $z_n\neq z_m$, then $f(z_n)-az_n\neq f(z_m)-az_m$. Thus the simultaneous equations

$$\begin{cases} f(z)-az=c, \\ f'(z)-a=0 \end{cases}$$

have at most one common root for any constant c. Lemma 1 is thus proved.

Proof of Theorem 2. Let $t\in(0, 1/2)$. Then by Lemma 1 and the second fundamental theorem there is a countable set E_1 of complex numbers such that the conclusion of Lemma 1 holds with E replaced by E_1 and that

(3.8) $$N(r, a, f')>t\cdot m(r, f')$$

holds on a set of r of infinite measure for every a in $\boldsymbol{C}\backslash E_1$. Hence by Theorem A $f(z)-az$ is left-prime in entire sense for every a in $\boldsymbol{C}\backslash E_1$.

We next show the right-primeness of $f(z)-az$ in entire sense ($a\in\boldsymbol{C}\backslash E_1$). Let $f(z)-az=g(P(z))$, where g is transcendental entire and P is a polynomial of degree d ($\geqq 2$). Then $f'(z)-a=g'(P(z))P'(z)$. From (3.8) g' has infinitely many zeros $\{w_n\}_n$. For sufficiently large n the equation $w_n=P(z)$ has d distinct roots, which are also common roots of the simultaneous equations

$$\begin{cases} f(z)-az=g(w_n), \\ f'(z)-a=0. \end{cases}$$

This is a contradiction. Thus $f(z)-az$ is prime in entire sense for every a in $\boldsymbol{C}\backslash E_1$.

If for some constants a, b ($a\neq b$) the functions $f(z)-az$ and $f(z)-bz$ are periodic with periods x and y respectively, then $f'(z)$ has periods x and y. Hence x/y must be a real number. Thus $f(z)-az$ and $f(z)-bz$ are both bounded on the straight line $\{tx;\ t\in(-\infty, +\infty)\}$. This is impossible. Thus $f(z)-az$ is not periodic for every a ($\in\boldsymbol{C}$) with at most one exception.

Therefore by Gross' theorem [3] we conclude that $f(z)-az$ is prime for every a in $\boldsymbol{C}\backslash E_1$ with at most one exception. Theorem 2 is thus proved.

4. Problem (B).

THEOREM 3. *Let $f(z)$ be a transcendental entire function. Then the set*

$$\{a\in \boldsymbol{C}\,;\ f(z)\cdot(z-a) \text{ is not prime}\}$$

is at most a countable set.

We need the following lemmas.

LEMMA 2. *Let $f(z)$ be a transcendental entire function. Then there is a countable set E' of complex numbers such that the simultaneous equations*

$$\begin{cases} f(z)\cdot(z-a)=c\,, \\ \dfrac{d}{dz}(f(z)\cdot(z-a))=0 \end{cases}$$

have at most one common root for any non-zero constant c ($\in \boldsymbol{C}$) provided that a is in $\boldsymbol{C}\backslash E'$.

Proof. Put

$$h(z)=z+(f(z)/f'(z))\,,$$

$$A'=\boldsymbol{C}-\{p\in \boldsymbol{C}\,;\ p \text{ is a zero or a pole of } h'(z)\}\,.$$

We choose open sets $\{c_i'\}_{i=1}^{\infty}$ of A' satisfying the following conditions.

(1) $\bigcup\limits_{i=1}^{\infty} c_i'=A'$.

(2) $h(z)$ is univalent in c_i' ($i=1, 2, \cdots$).

(3) $\{h(z)\,;\ z\in c_i'\}$ is a disk ($i=1, 2, \cdots$).

Put

$$D_i'=\{h(z)\,;\ z\in c_i'\} \qquad (i=1, 2, \cdots)\,,$$

(4.1) $$H(z)=(z-h(z))\cdot f(z)\,,$$

(4.2) $$x_i(w)=(h\,|\,c_i')^{-1}(w) \qquad (w\in D_i',\ i=1, 2, \cdots)\,,$$

(4.3) $$y_i(w)=H(x_i(w)) \qquad (w\in D_i',\ i=1, 2, \cdots)\,,$$

$$I'=\{(i,\ j)\in \boldsymbol{N}\times \boldsymbol{N};\ D_i'\cap D_j'\neq\varnothing,\ y_i(w)\not\equiv y_j(w)\ (w\in D_i'\cap D_j')\}\,,$$

(4.4) $$S_{i,j}'=\{w\in D_i'\cap D_j'\,;\ y_i(w)=y_j(w)\} \qquad ((i,\ j)\in I')\,,$$

(4.5) $$E_0'=\Big(\bigcup_{i=1}^{\infty} D_i'\Big)-\Big(\{h(p)\,;\ h'(p)=0,\ p\in \boldsymbol{C}\}\cup\Big(\bigcup_{(i,j)\in I'} S_{i,j}'\Big)$$
$$\cup\Big(\bigcup_{i=1}^{\infty}\{p\in D_i'\,;\ f\circ x_i(p)=0\}\Big)\Big)\,.$$

As in the case of Lemma 1 we can show the following four facts.

1) $E'=\boldsymbol{C}\backslash E_0'$ is a countable set.

2) $y_k(w)=(x_k(w)-w)\cdot f(x_k(w)) \quad (w\in D_k')$.

3) If $y_i(a)=y_j(a)$ for some a in E_0', then $x_i(a)=x_j(a)$.

4) If a is in E_0', then the set $\{x_k(a)\,;\ a\in D_k',\ k=1, 2, \cdots\}$ contains the set $\left\{p\in\boldsymbol{C}\,;\ \frac{d}{dz}(f(z)(z-a))|_{z=p}=0,\ f(p)(p-a)\neq 0\right\}$.

1) and 2) are immediate consequences of (4.1)-(4.5).

Next we shall show 3). From (4.4) and (4.5) we deduce that $y_i(w)\equiv y_j(w)$ $(w\in D_i'\cap D_j')$. Thus $y_i'(a)=y_j'(a)$. Since $H'(z)=-f(z)h'(z)$, from (4.2) and (4.3) we have

$$y_k'(a)=-f(x_k(a)) \qquad (k=i,\ j)\,. \tag{4.6}$$

From (4.5) we have $f(x_k(a))\neq 0$ $(k=i,\ j)$. Thus by 2) and (4.6) we obtain $x_i(a)=x_j(a)$. 3) is thus proved.

Finally, we shall show 4). If $\frac{d}{dz}(f(z)(z-a))|_{z=p}=f'(p)(p-a)+f(p)=0$ and $f(p)(p-a)\neq 0$ for some p in $\boldsymbol{C}$, then $f'(p)\neq 0$. Thus $a=p+(f(p)/f'(p))=h(p)$. Therefore by (4.5) we have $h'(p)\neq 0$. Thus $p\in c_k'$ for some k in $\boldsymbol{N}$. Hence we have $p=x_k(a)$ and $a\in D_k'$. 4) is thus proved.

From 1), 2), 3) and 4) we have the desired result.

LEMMA A [6]. *Let $F(z)$ be a transcendental entire function. Then except for a countable set of $a\in\boldsymbol{C}$, the function $(z-a)\cdot F(z)$ has no factorization of form $(z-a)\cdot F(z)=g(P(z))$, where g is transcendental entire and P is a polynomial of degree at least two.*

Proof of Theorem 3. Let us write

$$h(z)=z+(f(z)/f'(z))\,,$$

$$F_a(z)=(z-a)\cdot f(z)\,,$$

$$E_1'=\{p\,;\ p \text{ is a zero of } f(z)\}\cup\{h(p)\,;\ p \text{ is a zero of } h'(z)\}\,.$$

Let $a\in\boldsymbol{C}\backslash E_1'$. Then $F_a(z)$ has at least one simple zero and

$$N(r,\ a,\ h)=\bar{N}(r,\ a,\ h)\leqq N(r,\ 0,\ F_a')-(N(r,\ 0,\ F_a)-\bar{N}(r,\ 0,\ F_a))\,.$$

Let $t\in(0,\ 1/3)$. Then by the second fundamental theorem

$$N(r,\ a,\ h)>tT(r,\ h)$$

holds on a set of r of infinite measure for every complex number a with at most two exceptions. Further we see that for some k (>0)

$$T(r,\ h)\sim kT(r,\ F_a'/F_a)\,.$$

By Theorem 1, Lemma 2, Lemma A and the above consideration we deduce that there is a countable set E_2' of complex numbers such that $F_a(z)$ is prime in entire sense for every a in $C\backslash E_2'$.

It is easily seen that there is a countable set E_3' of complex numbers such that $F_a(z)$ is not periodic for every a in $C\backslash E_3'$. Therefore by Gross' theorem [3] $F_a(z)$ is prime for every a in $C\backslash(E_2'\cup E_3')$. Theorem 3 is thus proved.

5. In this section we shall prove.

THEOREM 4. *Let $h(w)$ be a one-valued regular function in $0<|w|<\infty$, having essential singularities at $w=0$ and $w=\infty$. Let n be a non-zero integer. Then the set*

$$\{a\in \boldsymbol{C}\,;\ h(e^z)+ae^{nz} \text{ is not prime}\}$$

is at most a countable set.

By the same method as in the proof of Lemma 1 we can show

LEMMA 3. *Let $h(w)$ and n satisfy the assumption of Theorem 4. Then there is a countable set E'' of complex numbers such that any two common roots s, t of the simultaneous equations*

$$\begin{cases} h(w)+aw^n=c\,, \\ h'(w)+anw^{n-1}=0 \end{cases}$$

satisfy $s^n=t^n$ for any constant c $(\in \boldsymbol{C})$ provided that a is in $\boldsymbol{C}\backslash E''$.

Proof. Put

$$k(w)=-h'(w)/nw^{n-1}\,,\quad A''=\boldsymbol{C}-(\{0\}\cup\{p\in \boldsymbol{C}-\{0\}\,;\ k'(p)=0\})\,.$$

We choose open sets $\{c_i''\}_{i=1}^{\infty}$ of A'' satisfying the following conditions.

(1) $\bigcup_{i=1}^{\infty} c_i''=A''$.

(2) $k(w)$ is univalent in c_i'' $(i=1, 2, \cdots)$.

(3) $\{k(w)\,;\ w\in c_i''\}$ is a disk $(i=1, 2, \cdots)$.

Put

$$K(w)=h(w)+w^nk(w)\,,\quad D_i''=\{k(w)\,;\ w\in c_i''\}\quad (i=1, 2, \cdots)\,,$$

$$q_i(x)=(k|_{c_i''})^{-1}(x)\qquad (x\in D_i'',\ i=1, 2, \cdots)\,,$$

$$r_i(x)=K(q_i(x))\qquad (x\in D_i'',\ i=1, 2, \cdots)\,,$$

$$I''=\{(i,\ j)\in \boldsymbol{N}\times \boldsymbol{N};\ D_i''\cap D_j''\neq\emptyset,\ r_i(x)\not\equiv r_j(x)\ (x\in D_i''\cap D_j'')\}\,,$$

$$S_{i,j}''=\{x\in D_i''\cap D_j''\,;\ r_i(x)=r_j(x)\}\qquad ((i,\ j)\in I'')\,,$$

$$E_0''=\left(\bigcup_{i=1}^{\infty} D_i''\right)-\left(\{k(p)\,;\ k'(p)=0,\ p\in \boldsymbol{C}-\{0\}\}\cup\left(\bigcup_{(i,j)\in I''} S_{i,j}''\right)\right).$$

As in the case of Lemma 1 we can show the following four facts.

1) $E''=\boldsymbol{C}\backslash E_0''$ is a countable set.

2) $r_k(x)=h(q_k(x))+q_k(x)^n x \quad (x\in D_k'')$.

3) If $r_i(a)=r_j(a)$ for some a in E_0'', then $q_i(a)^n=q_j(a)^n$.

4) If a is in E_0'', then the set $\{q_k(a)\,;\ a\in D_k'',\ k=1,2,\cdots\}$ coincides with the set of roots of $h'(w)+anw^{n-1}=0$.

From 1), 2), 3) and 4) we have the desired result.

Proof of Theorem 4. Put

$$H_a(z)=h(e^z)+ae^{nz}.$$

Let $t\in(0,1/2)$. Then Lemma 3 and the second fundamental theorem imply that there is a countable set E_0'' of complex numbers such that the conclusion of Lemma 3 holds with E'' replaced by E_0'' and that the inequalities

$$N(r,0,H_a')\geqq tm(r,h'(e^z)), \tag{5.1}$$

$$N(r,c,H_a)\geqq tm(r,h(e^z)) \tag{5.2}$$

hold on a set of r of infinite measure for any complex number c, provided that a is in $\boldsymbol{C}\backslash E_0''$.

In what follows we shall assume that a is in $\boldsymbol{C}\backslash E_0''$ and prove that $H_a(z)$ is prime.

Let $H_a(z)=f(g(z))$.

a) f and g are transcendental entire. We shall make use of Kobayashi's theorem [7]. This idea is due to theorem 3 in [11]. Since $H_a'(z)=f'(g(z))g'(z)$, by (5.1) $f'(w)$ has infinitely many zeros $\{w_n\}_{n=1}^{\infty}$. Then any root of $g(z)=w_n$ is also a common root of the simultaneous equations

$$\begin{cases} H_a(z)=f(w_n)\,, \\ H_a'(z)=0\,. \end{cases}$$

Therefore, since $a\notin E_0''$, all the roots of $g(z)=w_n$ lie on a straight line of the complex plane $(n=1,2,\cdots)$. Thus by Kobayashi's theorem [7]

$$g(z)=P(e^{Az})$$

with a quadratic polynomial $P(z)$ and a non-zero constant A. It is easily seen that $A=n/N$ with an integer N. Thus

$$H_a(z)=f(P(e^{nz/N}))\,.$$

Put $w=e^{z/N}$. Then

$$h(w^N)+aw^{nN}=f(P(w^n)).$$

The right side is regular at $w=0$ but the left side is not. This is a contradiction.

b) f is transcendental meromorphic (not entire) and g is transcendental entire. This case can be treated by the same method as in the case a).

c) f is transcendental entire and g is a polynomial of degree at least two. By Rényi's theorem [13] g is a quadratic polynomial. Put $g(z)=s(z-u)^2+v$ with constants s, u, v. Let $\{w_m\}_m$ be the zeros of $f'(w)$ and let p_m and q_m be the two roots of $g(z)=w_m$. Then p_m and q_m are also common roots of the simultaneous equations

$$\begin{cases} H_a(z)=f(w_m), \\ H_a'(z)=0. \end{cases}$$

Therefore, since $a\notin E_0''$, $e^{np_m}=e^{nq_m}$. Thus $\operatorname{Re} p_m=\operatorname{Re} q_m=\operatorname{Re} u$. Hence

$$\begin{aligned} N(r, 0, H_a')&=N(r, 0, f'\circ g)+N(r, 0, g') \\ &=O(r)+O(\log r)=o(m(r, h'(e^z))). \end{aligned}$$

This contradicts (5.1).

d) f is a polynomial of degree d ($\geqq 2$) and g is transcendental entire. By Rényi's theorem [13] g is periodic. Put $g(z)=k(e^{Az})$, where $k(w)$ is a regular function in $0<|w|<\infty$ and A a non-zero constant. Since 0 and ∞ are essential singularities of H_a, they are also essential singularities of k.

Let x be a zero of f'. Then by $a\notin E_0''$ $k(w)=x$ has at most finitely many roots. Thus f' has exactly one zero, say x. Therefore $f'(w)=b(w-x)^{d-1}$, $f(w)=bd^{-1}(w-x)^d+c$ with constants b ($\neq 0$), c. Thus $H_a(z)=bd^{-1}(g(z)-x)^d+c$. Hence

$$N(r, c, H_a)=dN(r, x, g). \tag{5.3}$$

Since $k(w)=x$ has at most finitely many roots,

$$N(r, x, g)=O(r)=o(m(r, h(e^z))).$$

This contradicts (5.2) and (5.3).

e) f is rational (not a polynomial) and g is transcendental entire. Then

$$f(w)=\frac{P(w)}{(w-w_0)^q} \qquad (P(w_0)\neq 0), \tag{5.4}$$

$$g(z)=w_0+e^{G(z)}, \tag{5.5}$$

where P is a polynomial, G a non-constant entire function and q a positive integer [8, proposition 2].

By the theorem in [5, p. 59], g is periodic. Put $g(z)=k(e^{Az})$, where $k(w)$ is a regular function in $0<|w|<\infty$ and A a non-zero constant. Since 0 and ∞ are essential singularities of H_a, they are also essential singularities of k. Thus

$$\lim_{r\to\infty} m(r, g)/r=\infty . \tag{5.6}$$

If x is a zero of f', then $x\neq w_0$. Further, by $a\notin E_0''$, $k(w)=x$ has at most finitely many roots. Thus

$$N(r, x, g)=O(r) . \tag{5.7}$$

From (5.5), (5.6), (5.7) and the second fundamental theorem, we have a contradiction. Thus f' has no zero.

From (5.4)

$$f'(w)=(P'(w)(w-w_0)-qP(w))/(w-w_0)^{q+1}=b/(w-w_0)^{q+1} ,$$

where b is a non-zero constant. Hence $f(w)=d(w-w_0)^{-q}+c$ with constants c, d ($\neq 0$). Thus from (5.5) $H_a(z)=de^{-qG(z)}+c$. This contradicts (5.2).

f) f is rational (not a polynomial) and g is transcendental meromorphic (not entire). This case can be reduced to the case d) or the case e).

Theorem 4 is thus proved.

A remark should be mentioned here. Theorem 4 indicates that there are prime periodic entire functions of arbitrarily rapid growth.

6. In this section we shall give an extension of theorem 1 in [10].

THEOREM 5. *Let $F(z)$ be a transcendental entire function of finite order and R an arbitrarily fixed positive number. Assume that the simultaneous equations*

$$\begin{cases} F(z)=c , \\ F'(z)=0 \end{cases}$$

have only finitely many common roots for any constant c satisfying $|c|>R$. Then $F(z)$ is pseudo-prime.

Examples. The functions $\cos z$ and $P(Q(z)e^{S(z)})$, where P and S are non-constant polynomials and Q is a non-zero polynomial, satisfy the assumption of Theorem 5.

Proof of Theorem 5. Let $F(z)=f(g(z))$.

a) f and g are transcendental entire. By Pólya's theorem [12] $f(z)$ is of order zero. Let $\{z_n\}_{n=1}^{\infty}$ be the zeros of $f'(z)$. Then by the assumption $|f(z_n)| \leqq R$ for every z_n with at most one exception. Hence there is a positive number A satisfying

$$|f(z_n)|<A \qquad (n=1, 2, \cdots). \tag{6.1}$$

By Wiman's theorem and (6.1) we can see that $\{z\in \boldsymbol{C}\,;\ |f(z)|\leqq A\}$ consists of infinitely many bounded components $\{D_i\}_{i=1}^{\infty}$ and that ∂D_i consists of one closed Jordan curve $(i=1, 2, \cdots)$. Let E_r $(r>0)$ be that component of $\{z\in \boldsymbol{C}\,;\ |f(z)|\leqq M(r, f)\}$ which contains the circle $|z|=r$. Then, as in the case of D_i, ∂E_r consists of one closed Jordan curve for every r satisfying $M(r, f)>A$. Let $I(r)=\{i\,;\ D_i\subset E_r\}$ $(M(r, f)>A)$.

For a subset X of the complex plane and an entire function h, we denote by $n(X, h)$ the number of zeros (counting multiplicity at multiple zeros) of h in X. If $M(r, f)>A$, then

$$n(E_r, f)=\sum_{i\in I(r)} n(D_i, f). \tag{6.2}$$

On the other hand, if $M(r, f)>A$, then by the argument principle

$$n(E_r, f')=n(E_r, f)-1, \tag{6.3}$$

$$n(D_i, f')=n(D_i, f)-1 \qquad (i=1, 2, \cdots). \tag{6.4}$$

By (6.1) we have

$$n(E_r, f')=\sum_{i\in I(r)} n(D_i, f'). \tag{6.5}$$

Since the number of the elements of $I(r)$ tends to infinity as $r\rightarrow\infty$, from (6.2)-(6.5) we have a contradiction.

b) f is transcendental meromorphic (not entire) and g is transcendental entire. Then by proposition 2 in [8]

$$f(w)=\frac{f^*(w)}{(w-w_0)^m}, \qquad f^*(w_0)\neq 0,$$

where f^* is transcendental entire and m a positive integer. By Edrei-Fuchs' theorem [2] f is of order zero. Then by the same argument as in the case a), we have a contradiction. The detail is omitted.

The following corollary is an extension of theorem 1 in [10].

COROLLARY 1. *Let $F(z)$ be a transcental entire function of finite order with at least one but at most finitely many simple zeros. Assume that the simultaneous equations*

$$\begin{cases} F(z)=c\,, \\ F'(z)=0 \end{cases}$$

have only finitely many common roots for any non-zero constant c. Then F(z) is left-prime in entire sense.

Proof. By Theorem 5 $F(z)$ is pseudo-prime. Let $F(z)=P(g(z))$, where g is transcendental entire and P is a polynomial of degree d ($\geqq 2$). We consider the following two cases.

(1) There exists a complex number w_0 such that $P'(w_0)=0$ and $P(w_0)\neq 0$.
(2) If x is a zero of $P'(w)$, then $P(x)=0$.

Firstly we consider the case (1). By the assumption $g(z)$ must be of the form

$$g(z)=w_0+Q(z)e^{R(z)}\,, \tag{6.6}$$

where $Q(z)$ and $R(z)$ are polynomials. By the assumption $P(w)$ has a simple zero b. Then $b\neq w_0$. Thus from (6.6) and the second fundamental theorem we have

$$\Theta(b,\, g)=1-\limsup_{r\to\infty}(\bar{N}(r,\, b,\, g)/m(r,\, g))=0\,.$$

Thus $g(z)$ has infinitely may simple b-points. Hence $F(z)=P(g(z))$ has infinitely many simple zeros. This is a contradiction.

Secondly we consider the case (2). In this case $P(w)$ must be of the form $P(w)=a(w-b)^d$ with constants a, b. This is a contradiction, since $F(z)=P(g(z))$ has a simple zero.

Corollary 1 is thus proved.

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