

COHOMOLOGY FORMULA FOR OBSTRUCTIONS TO ASYMPTOTIC CHOW SEMISTABILITY

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Abstract

Odaka [16] and Wang [19] proved the intersection formula for the Donaldson-Futaki invariant. In this paper, we generalize this result for the higher Futaki invariants, which are obstructions to asymptotic Chow semistability.

1. Introduction

Let M be a complex manifold of dimension n , and L be an ample line bundle over M . The existence of constant scalar curvature Kähler metrics (cscK metrics for short) in $c_1(L)$ is an important problem in Kähler geometry. The conjecture of Yau, Tian and Donaldson asserts that the existence of cscK metrics in $c_1(L)$ is equivalent to a certain GIT stability of the polarized variety (M, L) . In fact, Chen, Donaldson and Sun [2], Tian [18] gave the solution of this problem when the polarization is the anti-canonical bundle of M . The relevant GIT condition in this case is called K-stability. On the other hand, K-stability is not the only stability notion related to the existence of cscK metrics, and asymptotic Chow stability is one of stability notions. Donaldson [4] proved that if the automorphism group $\text{Aut}(M, L)$ is discrete, then the existence of cscK metrics implies the asymptotic Chow semistability of (M, L) . Mabuchi [14] extended the result for the case $\text{Aut}(M, L)$ is not discrete. More precisely, he proved that there exists an obstruction to asymptotic Chow semistability and if the obstruction vanishes, the existence of cscK metrics in $c_1(L)$ implies the asymptotic Chow polystability of (M, L) . After that, the obstruction was reformulated by Futaki [7] as an integral invariants of (M, L) . They are called higher Futaki invariants.

Della Vedova and Zuddas [3] showed that the higher Futaki invariants are closely related to the Chow weight. Let $(\mathcal{X}, \mathcal{L})$ be a test configuration of (M, L) . Then the central fiber $(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$ is a polarized scheme endowed with a \mathbf{C}^* -action. For an integer k , let $\chi(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0})$ be the Euler-Poincaré characteristic of $(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0})$ and $w(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0})$ be the weight of the \mathbf{C}^* -action on the

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one-dimensional vector space $\bigotimes_{i=0}^n (\bigwedge^{\max} H^i(M, L)^{(-1)^i})$. Then the Chow weight $\text{Chow}(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0})$ is defined by using $\chi(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0})$ and $w(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0})$. We are interested in the asymptotic behavior of $\text{Chow}(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0})$ when k grows. Della Vedova and Zuddas showed that if $(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0})$ is smooth, the higher Futaki invariants are equal to the coefficients of polynomial expansion of $\text{Chow}(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0})$ with respect to k . The leading coefficient is called the Donaldson-Futaki invariant.

On the other hand, Odaka [16] and Wang [19] showed that there exists another formula of the Donaldson-Futaki invariant. A test configuration $(\mathcal{X}, \mathcal{L})$ has a natural compactification $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$. Then the Donaldson-Futaki invariant of $(\mathcal{X}, \mathcal{L})$ is expressed by intersection numbers of (M, L) and $(\mathcal{X}, \mathcal{L})$. Theorem 1.1, which is our main result, is a generalization of this result.

THEOREM 1.1. *Let (M, L) be a polarized variety of complex dimension n . Let $(\mathcal{X}, \mathcal{L})$ be a test configuration of (M, L) . If \mathcal{X} is smooth, then the ℓ -th Futaki invariant $F_\ell(\mathcal{X}, \mathcal{L})$ can be computed by the following formula for all ℓ :*

$$F_\ell(\mathcal{X}, \mathcal{L}) = \frac{1}{n!(n - \ell + 1)!(L^n)^2} \times [(n + 1)(L^n)\{c_1(\bar{\mathcal{L}})^{n+1-\ell} \text{Td}_\ell(\bar{\mathcal{X}}) - c_1(L)^{n+1-\ell} \text{Td}_{\ell-1}(M)\} - (n - \ell + 1)(\bar{\mathcal{L}}^{n+1})(c_1(L)^{n-\ell} \text{Td}_\ell(M))],$$

where $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ is the natural compactification of $(\mathcal{X}, \mathcal{L})$.

The notation $(\bar{\mathcal{L}}^{n+1})$ means the intersection number $\bar{\mathcal{L}} \cdots \bar{\mathcal{L}}$ in $\bar{\mathcal{X}}$ and (L^n) means $L \cdots L$ in M , and so on. This theorem allows us to compute $F_\ell(\mathcal{X}, \mathcal{L})$ in terms of characteristic classes of (M, L) and $(\mathcal{X}, \mathcal{L})$.

This paper is organized as follows. In section 2 we recall some definitions and theorems we mentioned above. The proof of Theorem 1.1 is given in section 3. Then we give the compactification of test configurations. In section 4, we give the localization formula of Theorem 1.1. This is given by Futaki [6]. The localization formula gives an alternative proof for Theorem 1.1 at least for the product configurations.

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2. Background

2.1. Asymptotic Chow stability. First, we recall the definition of asymptotic Chow stability. Let $X \subset \mathbf{P}(V)$ be an n -dimensional subvariety of degree d . The set of $n + 1$ hyperplanes that have a common intersection with X

$$\{(H_1, \dots, H_{n+1}) \in \mathbf{P}(V^*) \times \mathbf{P}(V^*) \times \cdots \times \mathbf{P}(V^*) \mid H_1 \cap \cdots \cap H_{n+1} \cap X \neq \emptyset\}$$

is an irreducible hypersurface in $(\mathbf{P}(V^*))^{n+1}$. The defining polynomial P_X is homogeneous of multi-degree $d = \deg(X)$. The polynomial $P_X \in \text{Sym}^d(V)^{\otimes n+1}$ is called the Chow form.

Let (M, L) be a polarized variety of dimension n . Let $M_k \subset \mathbf{P}(H^0(M, L^k)^*) = \mathbf{P}(V_k)$ be the image of the Kodaira embedding of M . For M_k we obtain the Chow form $P_{M_k} \in \text{Sym}^{d_k}(V_k)^{\otimes n+1}$. Consider the $\text{SL}(V_k)$ -action on $\text{Sym}^{d_k}(V_k)^{\otimes n+1}$.

- DEFINITION 2.1 (Chow stability). (1) The variety M is Chow polystable with respect to L^k if the $\text{SL}(V_k)$ -orbit of P_{M_k} in $\text{Sym}^{d_k}(V_k)^{\otimes n+1}$ is closed. (2) The variety M is Chow stable with respect to L^k if M is polystable and the stabilizer at P_{M_k} is finite. (3) The variety M is Chow semistable with respect to L^k if the closure of the $\text{SL}(V_k)$ -orbit of P_{M_k} in $\text{Sym}^{d_k}(V_k)^{\otimes n+1}$ does not contain the origin $\mathbf{o} \in \text{Sym}^{d_k}(V_k)^{\otimes n+1}$.

DEFINITION 2.2 (asymptotic Chow stability). The variety M is asymptotically Chow polystable (respectively stable or semistable) with respect to L if there is a $k_0 > 0$ such that for any $k > k_0$, M is Chow polystable (respectively stable or semistable) with respect to L^k .

2.2. The relationship to cscK metrics. There is a relation between GIT stabilities and canonical metrics, called the Kobayashi-Hitchin correspondence. The well-known conjecture of Yau, Tian and Donaldson asserts that the polarized manifold (M, L) is “K-polystable” if and only if constant scalar curvature Kähler metrics (cscK metrics for short) exist in $c_1(L)$. K-stability is a little different notion from Chow stability, but asymptotic Chow semistability implies K-semistability. Chow stability also has a relationship to cscK metric, as explained below.

We denote by $\text{Aut}(M)$ the group of automorphisms of M and by $\text{Aut}(L)$ the group of bundle automorphisms of L . Let $\text{Aut}(M, L)$ be the subgroup of automorphism group $\text{Aut}(L)$ of L consisting of all automorphisms of L commuting with the \mathbf{C}^* -action on the fiber. Such automorphisms of L descend to automorphisms of M . So we can consider $\text{Aut}(M, L)$ as a subgroup of $\text{Aut}(M)$. Donaldson proved the following theorem.

THEOREM 2.1 (Donaldson [4]). *Let (M, L) be a polarized manifold. Assume that $\text{Aut}(M, L)$ is discrete. If M admits cscK metrics in $c_1(L)$, then (M, L) is asymptotically Chow stable.*

The assumption for $\text{Aut}(M, L)$ means the finiteness of the stabilizer. This theorem gives us a differential geometric criterion of asymptotic Chow stability. Note that we can not omit the assumption. The following example is known.

THEOREM 2.2 (Ono-Sano-Yotsutani [17], Nill-Paffenholtz [15]). *There is a toric Fano 7-manifold which admits Kähler-Einstein metrics (so cscK metrics) but not asymptotically Chow semistable.*

We will explain the example of Ono-Sano-Yotsutani in Section 4.

In the case when $\text{Aut}(M, L)$ is not discrete, Mabuchi extended the theorem of Donaldson.

THEOREM 2.3 (Mabuchi [13], [14]). *If $\text{Aut}(M, L)$ is not discrete, there exists an obstruction to asymptotic Chow semistability. If the obstruction vanishes, then the existence of cscK metrics in $c_1(L)$ implies the asymptotic Chow polystability of (M, L) .*

This obstruction was reformulated by Futaki [7]. We will explain that in the next section.

2.3. Higher Futaki invariants. Here we recall the definition of the Futaki invariant. First let $\mathfrak{h}(M)$ be the complex Lie algebra of $\text{Aut}(M)$ which consists of holomorphic vector fields over M . When we consider $\text{Aut}(M, L)$ as the Lie subgroup of $\text{Aut}(M)$, its Lie algebra $\mathfrak{h}_0(M)$ is a Lie subalgebra of $\mathfrak{h}(M)$. Second we fix a Kähler form ω representing $c_1(L)$. Then for any $X \in \mathfrak{h}_0$ there exists a complex valued function u_X such that

$$(1) \quad i(X)\omega = -\bar{\partial}u_X,$$

$$(2) \quad \int_M u_X \omega^n = 0.$$

The function u_X is called the Hamiltonian function of X . The existence of such u_X is well known, see [11]. Let ∇ be the Chern connection of the Kähler metric associated to ω , and Θ be the curvature of ∇ . Put $L(X) = \nabla_X - \mathcal{L}_X = \nabla X$ where \mathcal{L}_X is the Lie derivative. The operator $L(X)$ defines a smooth section of $\text{End}(T^{1,0}M)$. Let ϕ be a $\text{GL}(n, \mathbf{C})$ -invariant polynomial of degree ℓ on $\mathfrak{gl}(n, \mathbf{C})$. We define $\mathcal{F}_\phi : \mathfrak{h}_0(M) \rightarrow \mathbf{C}$ by

$$\mathcal{F}_\phi(X) = (n - \ell + 1) \int_M \phi(\Theta) \wedge u_X \omega^{n-\ell} + \int_M \phi(L(X) + \Theta) \wedge \omega^{n-\ell+1}.$$

Then \mathcal{F}_ϕ does not depend on the choice of ω and depends only on the Kähler class $c_1(L)$ (see [7]). Let Td_ℓ be the ℓ -th Todd polynomial, which is $\text{GL}(n, \mathbf{C})$ -invariant polynomial of degree ℓ on $\mathfrak{gl}(n, \mathbf{C})$. Then $\mathcal{F}_{\text{Td}_\ell}$ is an obstruction to the asymptotically Chow semistability.

THEOREM 2.4 (Futaki [7]). *If (M, L) is asymptotically Chow semistable, then*

$$\mathcal{F}_{\text{Td}_\ell}(X) = 0$$

holds for any $1 \leq \ell \leq n$ and X in a maximal reductive subalgebra \mathfrak{h}_r of $\mathfrak{h}_0(M)$. The vanishing of all invariants $\mathcal{F}_{\text{Td}_\ell}$ is equivalent to the vanishing of Mabuchi's obstruction.

The invariant $\mathcal{F}_{\text{Td}_\ell}$ is called the ℓ -th Futaki invariant. In particular first Futaki invariant $\mathcal{F}_{\text{Td}_1}$ is the same as classical one up to a constant factor.

2.4. Chow weight. There is another interpretation of $\mathcal{F}_{\text{Td}_\ell}$ by Della Vedova and Zuddas [3]. Given a one-parameter subgroup $\rho : \mathbf{C}^* \rightarrow \text{Aut}(M, L)$ with a lifting action on L , let $X \in \mathfrak{h}_0(M) = \text{Lie}(\text{Aut}(M, L))$ be the generator of ρ . We denote by $w(M, L)$ the weight of the \mathbf{C}^* -action induced on $\bigotimes_{i=0}^n (\bigwedge^{\max} H^i(M, L)^{(-1)^i})$, and by $\chi(M, L)$ the Euler-Poincaré characteristic $\sum_{i=0}^n (-1)^i \dim H^i(M, L)$. For sufficiently large k , we may assume $H^i(M, L^k) = 0$ for $i > 0$ by the Kodaira vanishing theorem. We have polynomial expansions with respect to k :

$$(3) \quad \chi(M, L^k) = a_0(M, L)k^n + a_1(M, L)k^{n-1} + \cdots + a_n(M, L),$$

$$(4) \quad w(M, L^k) = b_0(M, L)k^{n+1} + b_1(M, L)k^n + \cdots + b_{n+1}(M, L).$$

DEFINITION 2.3. The Chow weight of this action is defined by

$$\text{Chow}(M, L^k) = \frac{w(M, L^k)}{k\chi(M, L^k)} - \frac{b_0(M, L)}{a_0(M, L)}.$$

We can show

$$\begin{aligned} \text{Chow}(M, L^k) &= \frac{b_{n+1}(M, L)}{\chi(M, L^k)} \\ &+ \frac{a_0(M, L)}{k\chi(M, L^k)} \sum_{\ell=1}^n \frac{a_0(M, L)b_\ell(M, L) - b_0(M, L)a_\ell(M, L)}{a_0(M, L)^2} k^{n-\ell+1}. \end{aligned}$$

The first term is known to vanish in the smooth case. Therefore we define $F_\ell(M, L)$ by

$$(5) \quad F_\ell(M, L) = \frac{a_0(M, L)b_\ell(M, L) - b_0(M, L)a_\ell(M, L)}{a_0(M, L)^2}.$$

THEOREM 2.5 (Della Vedova-Zuddas [3]).

$$(6) \quad F_\ell(M, L) = \frac{1}{\text{Vol}(M, L)} \mathcal{F}_{\text{Td}_\ell}(X).$$

Paul and Tian showed that the first Futaki invariant F_1 can be considered as the Mumford weight of the CM-line λ_{Chow} on the Hilbert scheme. Della Vedova and Zuddas showed that the ℓ -th Futaki invariant is also the weight of some line $\lambda_{\text{Chow}, \ell}$, see [3] and [8] for detail.

2.5. Intersection formula of Donaldson-Futaki invariant. So far we have considered product configurations. Now we consider general test configurations. First we recall the definition. Note that Tian originally assumed that a central fiber of a test configuration \mathcal{X}_0 is normal but Donaldson defined a test configuration as a general scheme. After that Li and Xu proved we can assume its normality by using the minimal model program (see [12]).

DEFINITION 2.4 (test configuration). Let (M, L) be a polarized variety. A test configuration of (M, L) consists of the following data:

- (1) a scheme \mathcal{X} with \mathbf{C}^* -action;
- (2) a \mathbf{C}^* -equivariant relatively ample line bundle $\mathcal{L} \rightarrow \mathcal{X}$;
- (3) a flat \mathbf{C}^* -equivariant morphism $\pi : (\mathcal{X}, \mathcal{L}) \rightarrow \mathbf{C}$ such that we have $(\mathcal{X}_1, \mathcal{L}|_{\mathcal{X}_1}) = (M, L)$ where $\mathcal{X}_1 = \pi^{-1}(1)$.

When (M, L) has a \mathbf{C}^* -action, we can make a test configuration $\mathcal{X} = M \times \mathbf{C}$ with a diagonal \mathbf{C}^* -action, called a product configuration. Clearly the central fiber $(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$ is a polarized scheme endowed with a \mathbf{C}^* -action.

DEFINITION 2.5 (Donaldson-Futaki invariant). Let $(\mathcal{X}, \mathcal{L})$ be a test configuration of (M, L) . Then the Donaldson-Futaki invariant $\text{DF}(\mathcal{X}, \mathcal{L})$ is defined by $F_1(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})$.

The intersection formula for the Donaldson-Futaki invariant $\text{DF}(\mathcal{X}, \mathcal{L})$ was proved by Odaka and Wang.

THEOREM 2.6 (Odaka [16], Wang [19]). *If $(\mathcal{X}, \mathcal{L})$ is normal and \mathbf{Q} -Gorenstein, it follows that*

$$\text{DF}(\mathcal{X}, \mathcal{L}) = \frac{1}{2(n+1)(L^n)^2} ((n+1)(K_{\bar{\mathcal{X}}/\mathbf{P}^1} \cdot \bar{\mathcal{L}}^n)(L^n) - n(\bar{\mathcal{L}}^{n+1})(K_M \cdot L^{n-1})),$$

where $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ is the natural compactification of $(\mathcal{X}, \mathcal{L})$, explained in the proof of our Theorem 1.1.

Theorem 1.1 is a generalization of this result.

3. Proof of main result

First we recall the compactification of $(\mathcal{X}, \mathcal{L})$ in [19]. Let $[z_0 : z_1]$ be the homogeneous coordinates of \mathbf{CP}^1 . Let $0 = [1 : 0]$, $\infty = [0 : 1]$, Δ_0, Δ_∞ be the coordinate neighborhood of $0, \infty$ respectively and $\mu = z_1/z_0$ be the local coordinate in Δ_0 . The transition function $g : \Delta_0 \setminus 0 \rightarrow \Delta_\infty \setminus 0$ is given by $g(\mu) = 1/\mu$.

We define a \mathbf{C}^* -action on $\mathbf{CP}^1 \times M$ by

$$(7) \quad t \cdot ([z_0 : z_1], p) = ([z_0 : tz_1], p)$$

for $t \in \mathbf{C}^*$. In $\Delta_0 \times M$, this action is given by $t.(\mu, p) = (t\mu, p)$. Let $x \in \mathcal{X} \setminus \mathcal{X}_0$ and $\pi(x) = \mu$. Then

$$f : \mathcal{X} \setminus \mathcal{X}_0 \rightarrow \Delta_0 \setminus \{0\} \times M$$

$$x \mapsto (\mu, \rho(\mu)^{-1}x)$$

is isomorphic where $\rho : \mathbf{C}^* \rightarrow \text{Aut}(\mathcal{X}, \mathcal{L})$ is the \mathbf{C}^* -action and we use $\rho(\mu)^{-1}x \in \mathcal{X}_1 = M$. Moreover this map is \mathbf{C}^* -equivariant, that is, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{X} \setminus \mathcal{X}_0 & & \Delta_0 \setminus \{0\} \times M \\ & \xrightarrow{f} & \\ x & \xrightarrow{f} & (\mu, \rho(\mu)^{-1}x) \\ \downarrow \times t & & \downarrow \times t \\ \rho(t)x & \xrightarrow{f} & (t\mu, \rho(\mu)^{-1}x) \end{array}$$

So we can define

$$\bar{\mathcal{X}} = \mathcal{X}_0 \cup (\mathcal{X} \setminus \mathcal{X}_0) \bigcup_{(g \times id_M) \circ f} \Delta_\infty \times M$$

with the \mathbf{C}^* -action. By $\mathcal{L}|_{\mathcal{X} \setminus \mathcal{X}_0} = \Delta_0 \setminus 0 \times L$, we also have the \mathbf{C}^* -equivariant line bundle $\bar{\mathcal{L}} \rightarrow \bar{\mathcal{X}}$. Finally we get the \mathbf{C}^* -equivariant morphism $\pi : (\bar{\mathcal{X}}, \bar{\mathcal{L}}) \rightarrow \mathbf{CP}^1$. Note that the \mathbf{C}^* -action on $\bar{\mathcal{X}}_\infty = \pi^{-1}(\{\infty\})$ and $\bar{\mathcal{L}}|_{\bar{\mathcal{X}}_\infty}$ is trivial.

We can take an integer N such that $\bar{\mathcal{L}} + N\pi^*(\mathcal{O}_{\mathbf{CP}^1}(1)) \rightarrow \bar{\mathcal{X}}$ is ample since $\bar{\mathcal{L}} \rightarrow \bar{\mathcal{X}}$ is relatively ample. Put

$$(8) \quad \mathcal{N} = \bar{\mathcal{L}} + N\pi^*\mathcal{O}_{\mathbf{CP}^1}(1).$$

Let σ_0 and σ_∞ be the sections of $\mathcal{O}_{\mathbf{CP}^1}(1)$ corresponding to the divisors $[0]$ and $[\infty]$, respectively. Since the action on \mathbf{CP}^1 lifts to $\mathcal{O}_{\mathbf{CP}^1}(-1)$, the weight of σ_0 and σ_∞ are $-1, 0$, respectively. The short exact sequence

$$0 \longrightarrow k\mathcal{N}(-[\mathcal{X}_0]) \xrightarrow{\times \pi^* \sigma_0} k\mathcal{N} \longrightarrow k\mathcal{N}|_{\mathcal{X}_0} \longrightarrow 0$$

induces the following exact sequence

$$0 \longrightarrow H^0(\bar{\mathcal{X}}, k\mathcal{N}(-[\mathcal{X}_0])) \xrightarrow{\times \pi^* \sigma_0} H^0(\bar{\mathcal{X}}, k\mathcal{N}) \longrightarrow H^0(\mathcal{X}_0, k\mathcal{N}|_{\mathcal{X}_0}) \longrightarrow 0$$

by the Kodaira vanishing theorem. Put $V_1 = H^0(\bar{\mathcal{X}}, k\mathcal{N}(-[\mathcal{X}_0]))$, $V_2 = H^0(\bar{\mathcal{X}}, k\mathcal{N})$ and $V_3 = H^0(\mathcal{X}_0, k\mathcal{N}|_{\mathcal{X}_0})$. We denote by d_i the dimension of V_i and by w_i the total weights of the \mathbf{C}^* -action on V_i . We have

$$(9) \quad d_3 = d_2 - d_1,$$

$$(10) \quad w_3 = w_2 - (w_1 - d_1),$$

since the weight of $\pi^*\sigma_0$ is -1 . By definition, the \mathbf{C}^* -action on $\pi^*\mathcal{O}_{\mathbf{CP}^1}(1)|_{\mathcal{X}_0}$ is trivial, and hence $w_3 = w(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0})$. For σ_∞ , we have

$$0 \longrightarrow H^0(\bar{\mathcal{X}}, k\mathcal{N}(-[\bar{\mathcal{X}}_\infty])) \xrightarrow{\times \pi^*\sigma_\infty} H^0(\bar{\mathcal{X}}, k\mathcal{N}) \longrightarrow H^0(\bar{\mathcal{X}}_\infty, k\mathcal{N}|_{\bar{\mathcal{X}}_\infty}) \longrightarrow 0.$$

Put $V_4 = H^0(\bar{\mathcal{X}}_\infty, k\mathcal{N}|_{\bar{\mathcal{X}}_\infty})$. Similarly, we have

$$(11) \quad d_4 = d_2 - d_1,$$

$$(12) \quad w_4 = w_2 - w_1.$$

Note that $k\mathcal{N}|_{\bar{\mathcal{X}}_\infty} = k\bar{\mathcal{L}}|_{\bar{\mathcal{X}}_\infty} + kN\pi^*\mathcal{O}_{\mathbf{P}^1}([\infty])$ by (8). Since the action on $\bar{\mathcal{L}}|_{\bar{\mathcal{X}}_\infty}$ is trivial, we have $w_4 = -kNd_4$. It follows that

$$\begin{aligned} (13) \quad w(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0}) &= w_3 = w_2 - (w_1 - d_1) \\ &= d_1 + (w_2 - w_1) \\ &= d_1 + w_4 \\ &= d_2 - (kN + 1)d_3 \\ &= \dim H^0(\bar{\mathcal{X}}, k\mathcal{N}) - (kN + 1) \dim H^0(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0}) \end{aligned}$$

from (9) to (12).

Now, we calculate the weight $w(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0})$ by the Riemann-Roch-Hirzebruch theorem. Note that $\dim H^0(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0}) = \dim H^0(\mathcal{X}_1, k\mathcal{L}|_{\mathcal{X}_1}) = \dim H^0(M, kL)$ for sufficiently large k by the flatness of $\bar{\mathcal{X}} \rightarrow \mathbf{CP}^1$. We have

$$\begin{aligned} (14) \quad \dim H^0(M, kL) &= \int_M \text{ch}(kL) \text{Td}(M) \\ &= \sum_{\ell=0}^n \frac{1}{(n-\ell)!} \int_M c_1(L)^{n-\ell} \text{Td}_\ell(M) k^{n-\ell}, \end{aligned}$$

$$\begin{aligned} (15) \quad \dim H^0(\bar{\mathcal{X}}, k\mathcal{N}) &= \int_{\bar{\mathcal{X}}} \text{ch}(k\mathcal{N}) \text{Td}(\bar{\mathcal{X}}) \\ &= \sum_{\ell=0}^{n+1} \frac{1}{(n-\ell+1)!} \int_{\bar{\mathcal{X}}} c_1(\mathcal{N})^{n-\ell+1} \text{Td}_\ell(\bar{\mathcal{X}}) k^{n-\ell+1} \\ &= \sum_{\ell=0}^{n+1} \frac{1}{(n-\ell+1)!} \int_{\bar{\mathcal{X}}} c_1(\bar{\mathcal{L}})^{n-\ell+1} \text{Td}_\ell(\bar{\mathcal{X}}) k^{n-\ell+1} \\ &\quad + N \sum_{\ell=0}^n \frac{1}{(n-\ell)!} \int_{\bar{\mathcal{X}}_\infty} c_1(\bar{\mathcal{L}})^{n-\ell} \text{Td}_\ell(\bar{\mathcal{X}}) k^{n-\ell+1}. \end{aligned}$$

Here we use $\mathcal{N} = \bar{\mathcal{L}} + N\pi^*\mathcal{O}_{\mathbf{CP}^1}([\infty])$. Substituting (14) and (15) into (13), we get

$$\begin{aligned}
 w(\mathcal{X}_0, k\mathcal{L}|_{\mathcal{X}_0}) &= w_3 \\
 &= \left[\frac{1}{(n+1)!} c_1(\overline{\mathcal{L}})^{n+1} + N \frac{1}{n!} \left\{ \int_{\overline{\mathcal{X}}_\infty} c_1(\overline{\mathcal{L}})^n - \int_M c_1(L)^n \right\} \right] k^{n+1} \\
 &\quad + \sum_{\ell=1}^n \left[\frac{1}{(n-\ell+1)!} \{ c_1(\overline{\mathcal{L}})^{n-\ell+1} \text{Td}_\ell(\overline{\mathcal{X}}) - c_1(L)^{n-\ell+1} \text{Td}_{\ell-1}(M) \} \right. \\
 &\quad \left. + N \frac{1}{(n-\ell)!} \left\{ \int_{\overline{\mathcal{X}}_\infty} c_1(\overline{\mathcal{L}})^{n-\ell} \text{Td}_\ell(\overline{\mathcal{X}}) - \int_M c_1(L)^{n-\ell} \text{Td}_\ell(M) \right\} \right] k^{n-\ell+1} \\
 &\quad + \left[\int_{\overline{\mathcal{X}}} \text{Td}_{n+1}(\overline{\mathcal{X}}) - \int_M \text{Td}_n(M) \right].
 \end{aligned}$$

Note that this polynomial does not depend on N . Finally, we obtain

$$(16) \quad b_\ell = \frac{1}{(n-\ell+1)!} [c_1(\overline{\mathcal{L}})^{n-\ell+1} \text{Td}_\ell(\overline{\mathcal{X}}) - c_1(L)^{n-\ell+1} \text{Td}_{\ell-1}(M)]$$

for $1 \leq \ell \leq n$. This implies Theorem 1.1.

4. Localization and example

In this section, we will see that Theorem 1.1 is localized to the formula in [6] by the original Bott residue formula. This gives the alternative proof of the result of Della Vedova and Zuddas [3]. Finally we give the example of [15] calculated in [17].

For the convenience of reader, recall the Bott residue formula [1]. Let M be a compact complex manifold and φ be a $\text{GL}(n, \mathbb{C})$ -invariant polynomial of degree n on $\mathfrak{gl}(n, \mathbb{C})$. Let X be a holomorphic vector field. Assume that the zero set of X consists of manifolds $\{Z_\lambda\}$. Then $L(X) = \nabla_X - \mathcal{L}_X$ induces an endomorphism $L^\vee(X)$ of the normal bundle $\nu(Z_\lambda)$. Suppose that $L^\vee(X)$ is non-degenerate. Then it holds

$$(17) \quad \varphi(M) = \varphi(\Theta) = \sum_{\lambda \in \Lambda} \int_{Z_\lambda} \frac{\varphi(L(X) + \Theta)|_{Z_\lambda}}{\det \left(\frac{\sqrt{-1}}{2\pi} (L^\vee(X) + K) \right)},$$

where Θ and K is the curvature of tangent bundle TM and normal bundle $\nu(Z_\lambda)$ respectively. Note that Bott proved this for arbitrary equivariant vector bundle, not only for tangent bundle TM .

We consider the localization of Theorem 1.1 by the Bott residue formula. Let M be a Fano manifold and X a holomorphic vector field on M . Assume that zero set of X consists of isolated points and X is non-degenerate. Take the canonical lift of X to $-K_M$. Let $(\mathcal{X}, \mathcal{L})$ be the product configuration of the \mathbb{C}^* -action generated by X and $(\overline{\mathcal{X}}, \overline{\mathcal{L}})$ be the compactification of $(\mathcal{X}, \mathcal{L})$.

The set of fixed points of the \mathbf{C}^* -action on $\bar{\mathcal{X}}$ is the union of the whole fiber $\bar{\mathcal{X}}_\infty$ and points on the central fiber \mathcal{X}_0 . Thus, we have

$$(18) \quad \int_{\bar{\mathcal{X}}} c_1(\bar{\mathcal{L}})^{n-\ell+1} \text{Td}_\ell(\bar{\mathcal{X}}) = \int_{\bar{\mathcal{X}}_\infty} \frac{c_1(\bar{\mathcal{L}}|_{\bar{\mathcal{X}}_\infty})^{n-\ell+1} \text{Td}_\ell(\bar{L}(X) + \Theta)}{\det \frac{\sqrt{-1}}{2\pi} (\bar{L}^\vee(X) + K)} + \sum_{\mathbf{q}: \text{fixed point}} \frac{(c_1^{n-\ell+1} \text{Td}_\ell)(\bar{L}(X)_\mathbf{q})}{\det \frac{\sqrt{-1}}{2\pi} (\bar{L}(X)_\mathbf{q})},$$

where $\bar{L}(X)$ is the endomorphism of tangent bundle $T\bar{\mathcal{X}}$, $\bar{L}^\vee(X)$ is the induced endomorphism of the normal bundle $\nu(\bar{\mathcal{X}}_\infty)$, Θ is curvature of $T\bar{\mathcal{X}}$ and K is the curvature of $\nu(\bar{\mathcal{X}}_\infty)$. Here we use the fact that $\bar{\mathcal{L}}|_{\bar{\mathcal{X}}_\infty}$ is the anticanonical bundle and the \mathbf{C}^* -action on $\bar{\mathcal{L}}|_{\bar{\mathcal{X}}_\infty}$ is trivial. We consider the first term of (18), we omit the determinant since the codimension of $\bar{\mathcal{X}}_\infty$ is one. From the construction of $\bar{\mathcal{X}}$, $\nu(\bar{\mathcal{X}}_\infty)$ is trivial. Thus, $K = 0$ and $T\bar{\mathcal{X}}|_{\bar{\mathcal{X}}_\infty}$ is decomposed to $T\mathbf{P}^1|_{\{\infty\}} \oplus T\bar{\mathcal{X}}_\infty$. Then we have

$$(19) \quad \bar{L}(X) + \Theta = \left(\begin{array}{c|ccc} -1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \Theta_M & \\ 0 & & & \end{array} \right),$$

where Θ_M is the curvature of M . Since the Todd polynomial is multiplicative, it follows that

$$(20) \quad \begin{aligned} \text{Td}_\ell(\bar{L}(X) + \Theta) &= \text{Td}_\ell(M) + \text{Td}_1(-1) \text{Td}_{\ell-1}(\Theta_M) \\ &= \text{Td}_\ell(M) + \frac{1}{2} c_1(-1) \text{Td}_{\ell-1}(\Theta_M) \\ &= \text{Td}_\ell(M) - \frac{\sqrt{-1}}{4\pi} \text{Td}_{\ell-1}(\Theta_M). \end{aligned}$$

Substituting (19) and (20) to (18), we get

$$(21) \quad \begin{aligned} \int_{\bar{\mathcal{X}}} c_1(\bar{\mathcal{L}})^{n-\ell+1} \text{Td}_\ell(\bar{\mathcal{X}}) &= \frac{1}{2} \int_{\bar{\mathcal{X}}_\infty} c_1(\bar{\mathcal{L}}|_{\bar{\mathcal{X}}_\infty})^{n-\ell+1} \text{Td}_{\ell-1}(\Theta_M) + \sum_{\mathbf{q}: \text{fixed point}} \frac{(c_1^{n-\ell+1} \text{Td}_\ell)(\bar{L}(X))}{\det \frac{\sqrt{-1}}{2\pi} (\bar{L}(X))} \\ &= \frac{1}{2} \int_M c_1(L)^{n-\ell+1} \text{Td}_{\ell-1}(M) + \sum_{\mathbf{q}: \text{fixed point}} \frac{(c_1^{n-\ell+1} \text{Td}_\ell)(\bar{L}(X))}{\det \frac{\sqrt{-1}}{2\pi} (\bar{L}(X))}. \end{aligned}$$

Similarly we can calculate the second term of (21). On the central fiber \mathcal{X}_0 , we have

$$(22) \quad \bar{L}(X) = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & L(X) & \\ 0 & & & \end{array} \right)$$

and

$$(23) \quad \mathrm{Td}_\ell(\bar{L}(X)) = \mathrm{Td}_\ell(L(X)) + \frac{\sqrt{-1}}{4\pi} \mathrm{Td}_{\ell-1}(L(X)).$$

Substituting (22) and (23) to (21), we obtain

$$(24) \quad \int_{\bar{\mathcal{X}}} c_1(\bar{\mathcal{L}})^{n-\ell+1} \mathrm{Td}_\ell(\bar{\mathcal{X}}) \\ = \int_{\bar{\mathcal{X}}_\infty} c_1(\bar{\mathcal{L}}|_{\bar{\mathcal{X}}_\infty})^{n-\ell+1} \mathrm{Td}_{\ell-1}(\Theta) + \sum_{q:\text{fixed point}} \frac{(c_1^{n-\ell+1} \mathrm{Td}_\ell)(L(X)_q)}{\det \frac{\sqrt{-1}}{2\pi}(L(X)_q)}.$$

From (16), it follows that

$$(25) \quad b_\ell = \frac{1}{(n-\ell+1)!} \sum_{q:\text{fixed point}} \frac{(c_1^{n-\ell+1} \mathrm{Td}_\ell)(L(X)_q)}{\det \frac{\sqrt{-1}}{2\pi}(L(X)_q)}.$$

This is the localization formula in [17].

Similarly we obtain the localization formula of b_0 and a_ℓ :

$$(26) \quad b_0 = \frac{(\bar{\mathcal{L}}^{n+1})}{(n+1)!} \\ = \frac{1}{(n+1)!} \sum_{q:\text{fixed point}} \frac{c_1(L(X)_q)^{n+1}}{\det \frac{\sqrt{-1}}{2\pi}(L(X)_q)}, \\ a_\ell = \frac{1}{(n-\ell)!} \sum_{q:\text{fixed point}} \frac{c_1(L(X)_q)^{n-\ell} \mathrm{Td}_\ell(L(X)_q)}{\det \frac{\sqrt{-1}}{2\pi}(L(X)_q)}.$$

Next, see the example in [15]. We consider the Fano polytope in \mathbf{R}^7 whose vertices are given by the following matrix:

$$(27) \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 2 & 1 & -1 \end{pmatrix}.$$

Let M be the 7-dimensional toric Fano manifold associated with the polytope. Then M is a \mathbf{P}^1 -fibration on $(\mathbf{P}^1)^3 \times \mathbf{P}^3$ and admits Kähler-Einstein metrics (see [15]).

Note that b_0 in (26) coincides with the original Futaki invariant when we consider the canonical lift of X to $-K_M$ (see [9]). Now it is zero since M is Kähler-Einstein. So we just have to calculate b_i and a_0 .

Next, define a \mathbf{C}^* -action on M . Here we consider the following one-parameter subgroup. Let v_i be the i -th vertex in (27). Let $\text{Spec}(\mathbf{C}[X_1, X_2, X_3, Y_1, Y_2, Y_3, Z])$ be the affine toric variety which corresponds to the 7-dimensional cone generated by $\{v_1, v_2, v_3, v_7, v_8, v_9, v_{11}\}$. Here X_1, X_2, X_3 are affine coordinates of $(\mathbf{P}^1)^3$, Y_1, Y_2, Y_3 are affine coordinates of \mathbf{P}^3 and Z is an affine coordinate of the fiber. Then the one-parameter subgroup σ_t is defined by

$$\begin{aligned} \sigma_t \cdot (X_1, X_2, X_3, Y_1, Y_2, Y_3, Z) \\ = (e^{\alpha_1 t} X_1, e^{\alpha_2 t} X_2, e^{\alpha_3 t} X_3, e^{\beta_1 t} Y_1, e^{\beta_2 t} Y_2, e^{\beta_3 t} Y_3, e^{\gamma t} Z). \end{aligned}$$

This one-parameter subgroup is defined over the whole M . For a generic $\{\alpha_i, \beta_j, \gamma\}_{1 \leq i, j \leq 3}$, the set of fixed points of σ_t consists of the isolated 64 points (see [17]). Let X be the holomorphic vector field generated by σ_t . Take the lift of X to $-K_M$ as section 2.3. Then higher Futaki invariants are calculated in [17] using the localization formula (25). We have

$$\begin{aligned} b_2 &= \frac{68}{45} \left(\sum_{i=1}^3 \alpha_i - \sum_{i=1}^3 \beta_i - 2\gamma \right), \\ b_3 &= \frac{68}{15} \left(\sum_{i=1}^3 \alpha_i - \sum_{i=1}^3 \beta_i - 2\gamma \right), \\ b_4 &= \frac{49}{9} \left(\sum_{i=1}^3 \alpha_i - \sum_{i=1}^3 \beta_i - 2\gamma \right), \\ b_5 &= \frac{10}{3} \left(\sum_{i=1}^3 \alpha_i - \sum_{i=1}^3 \beta_i - 2\gamma \right), \end{aligned}$$

$$b_6 = \frac{214}{315} \left(\sum_{i=1}^3 \alpha_i - \sum_{i=1}^3 \beta_i - 2\gamma \right),$$

$$b_7 = \frac{2}{15} \left(\sum_{i=1}^3 \alpha_i - \sum_{i=1}^3 \beta_i - 2\gamma \right).$$

The similar calculation gives

$$(28) \quad (-K_M)^7 = 13047715.$$

Finally, we obtain

$$F_2(\mathcal{X}, \mathcal{L}) = \frac{a_0 b_2 - b_0 a_2}{a_0^2}$$

$$= \frac{7616}{13047715} \left(\sum_{i=1}^3 \alpha_i - \sum_{i=1}^3 \beta_i - 2\gamma \right),$$

$$F_3(\mathcal{X}, \mathcal{L}) = \frac{22848}{13047715} \left(\sum_{i=1}^3 \alpha_i - \sum_{i=1}^3 \beta_i - 2\gamma \right),$$

$$F_4(\mathcal{X}, \mathcal{L}) = \frac{5488}{2609543} \left(\sum_{i=1}^3 \alpha_i - \sum_{i=1}^3 \beta_i - 2\gamma \right),$$

$$F_5(\mathcal{X}, \mathcal{L}) = \frac{3360}{2609543} \left(\sum_{i=1}^3 \alpha_i - \sum_{i=1}^3 \beta_i - 2\gamma \right),$$

$$F_6(\mathcal{X}, \mathcal{L}) = \frac{3424}{13047715} \left(\sum_{i=1}^3 \alpha_i - \sum_{i=1}^3 \beta_i - 2\gamma \right),$$

$$F_7(\mathcal{X}, \mathcal{L}) = \frac{672}{13047715} \left(\sum_{i=1}^3 \alpha_i - \sum_{i=1}^3 \beta_i - 2\gamma \right).$$

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