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HYERS–ULAM STABILITY OF A CLASS OF FRACTIONAL LINEAR DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, we investigate the Hyers–Ulam stability of a class of fractional linear differential equations. Applying the Laplace transform method, we prove that a class of fractional linear differential equations with Riemann–Liouville fractional derivatives is Hyers–Ulam stable. The results improve and extend some recent results.

1. Introduction and preliminaries

In 1940, the first stability problem concerning group homomorphisms was raised by Ulam [24]. Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given any $\epsilon > 0$, does there exists a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In the following years, Hyers affirmatively answered the question of Ulam for the case where G_1 and G_2 are Banach spaces (see [7]). Furthermore, the result of Hyers was generalized by Rassias (see [21]). Since then, the stability of many algebraic, differential, integral, operatorial, functional equations have been extensively investigated (see [1], [2], [3], [4], [6], [8], [9], [14], [17], [18], [19], [20], [23], [25], [29], [30], [31], [32], [33] and the references therein).

In recent years, many people have paid more and more attention to Hyers–Ulam stability of differential equations, and gained a series of results. S.-M. Jung investigated the Hyers–Ulam stability of some linear differential equations (see [11], [12], [13]), and in [22], H. Rezaei, S.-M. Jung and Th. M. Rassias discussed Hyers–Ulam stability of linear differential equations by applying Laplace transform method. D. Popa and I. Raşa proved the generalized Hyers–Ulam

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stability of the linear differential equation in a Banach space (see [19]). In [16], N. Lungu and D. Popa discussed Hyers–Ulam stability of a first order partial differential equation, and in [5], M. E. Gordji, Y. Cho, M. Ghaemi and B. Alizadeh investigated stability of the second order partial differential equations. J. Wang and Y. Zhou ([26], [27], [28]) proved the stability of fractional evolution equations and the stability of nonlinear differential equations with fractional integrable impulses, and they also introduced some new concepts about the stability of fractional differential equations. In [10], R. W. Ibrahim presented Hyers–Ulam stability of Cauchy differential equation of fractional order in the unit disk. However, the theory of Hyers–Ulam stability of fractional differential equations is still in the initial stages.

The main purpose of this paper is to prove the Hyers–Ulam stability of the following fractional linear differential equation with Riemann-Liouville fractional derivative by applying the Laplace transform method

$$\begin{cases} D^{\alpha}u(t) + du(t) = q(t), & t \in (0, T], \\ t^{1-\alpha}u(t)|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $0 < T < +\infty$, d is a constant in $\mathbf{C}$, $q \in C([0, T] \times \mathbf{C})$, and D^{α} is Riemann–Liouville fractional derivative of order $0 < \alpha < 1$ defined by

$$D^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-s)^{-\alpha} u(s)\, \mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\alpha} u(t), \tag{1.2}$$

here

$$I^{1-\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s)\, \mathrm{d}s \tag{1.3}$$

is Riemann-Liouville fractional integral of order $1-\alpha$ (see [15]).

For the sake of coherency we recall a few basic definitions, notions and properties about the Laplace transform of the fractional derivative. The Laplace transform of a function $u(t)$ of a real variable $t \in (0, \infty)$ is defined by

$$(\mathcal{L}u)(s) = \mathcal{L}[u(t)](s) := \int_0^{\infty} e^{-st} u(t)\, \mathrm{d}t \quad (s \in \mathbf{C}). \tag{1.4}$$

If the integral (1.4) is convergent at the point $s_0 \in \mathbf{C}$, then it converges absolutely for $s \in \mathbf{C}$ such that $\Re(s) > \Re(s_0)$. One of the most useful properties of the Laplace transform is the convolution property

$$\mathcal{L}\{u(t) * v(t)\} = \mathcal{L}\left\{\int_0^t u(t-\xi) v(\xi)\, \mathrm{d}\xi\right\} = \mathcal{L}\{u(t)\}\mathcal{L}\{v(t)\}. \tag{1.5}$$

The following results are some basic properties about the Laplace transform of the fractional derivatives.

LEMMA 1.1 ([15]). *If $\alpha > 0$, and let m be the smallest integer greater than or equal to α, then*

$$\mathcal{L}\{D^{\alpha}u(t)\} = s^{\alpha}\mathcal{L}\{u(t)\} - \Sigma_{k=0}^{m-1} s^{m-k-1} D^{k-m+\alpha}u(0). \tag{1.6}$$

Remark 1.2. In fact, if the initial conditions have the following form:

$$D^{k-m+\alpha}u(0+) = \lim_{t\to 0+} D^{k-m+\alpha}u(t) \quad (k = 0, 1, \dots, m-1) \tag{1.7}$$

exist, then

$$\mathcal{L}\{D^{\alpha}u(t)\} = s^{\alpha}\mathcal{L}\{u(t)\} - \Sigma_{k=0}^{m-1} s^{m-k-1} D^{k-m+\alpha}u(0+). \tag{1.8}$$

Remark 1.3. When $0 < \alpha < 1$, we have $m = 1$, one can get

$$\mathcal{L}\{D^{\alpha}u(t)\} = s^{\alpha}\mathcal{L}\{u(t)\} - D^{-(1-\alpha)}u(0+) = s^{\alpha}\mathcal{L}\{u(t)\} - I^{1-\alpha}u(0+), \tag{1.9}$$

here $D^{-(1-\alpha)}u(0+) = I^{1-\alpha}u(0+)$, $I^{1-\alpha}u(t)$ is defined in (1.3).

The Mittag–Leffler function $E_{\alpha,\beta}(z)$ is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (z, \beta \in \mathbf{C};\ \Re(\alpha) > 0), \tag{1.10}$$

when $\alpha = \beta = 1$, we can see that $E_{1,1}(z) = e^z$. More detailed information about the function can be found in [15].

LEMMA 1.4 ([15]). *If $\Re(s) > 0$, $\lambda \in \mathbf{C}$, $|\lambda s^{-\alpha}| < 1$, then*

$$\mathcal{L}\{t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\}(s) = \frac{s^{\alpha-\beta}}{s^{\alpha} - \lambda}, \tag{1.11}$$

where $E_{\alpha,\beta}(\lambda t^{\alpha})$ is the Mittag–Leffler function.

Remark 1.5. When $\alpha = \beta$, we have $\mathcal{L}\{t^{\alpha-1}E_{\alpha,\alpha}(\lambda t^{\alpha})\}(s) = \dfrac{1}{s^{\alpha} - \lambda}$.

The following result will play an important role in our next analysis.

LEMMA 1.6 ([15]). *Let $0 < \alpha < 1$ and let $u(t) \in C_{1-\alpha}[0, T] = \{u \in C(0, T];$ $t^{1-\alpha}u \in C[0, T]\}$.*

(a) *If $\lim_{t\to 0+}[t^{1-\alpha}u(t)] = c$, $c \in \mathbf{C}$, then*

$$I^{1-\alpha}u(0+) = c\Gamma(\alpha). \tag{1.12}$$

(b) *If* $I^{1-\alpha}u(0+) = b,\ b \in \mathbf{C}$, *and if there exists the limit* $\lim_{t\to 0+}[t^{1-\alpha}u(t)]$, *then*

$$\lim_{t\to 0+}[t^{1-\alpha}u(t)] = \frac{b}{\Gamma(\alpha)}. \tag{1.13}$$

2. Hyers–Ulam stability of differential equations of first order

The following definition can be found in [22].

Definition 2.1. The differential equation $\varphi(q,u,u',\dots,u^{(n)}) = 0$ has Hyers–Ulam stability if for given $\varepsilon > 0$ and a function u such that $|\varphi(q,u,u',\dots,u^{(n)})| \le \varepsilon$, there exists a solution u_a of the differential equation such that $|u(t) - u_a(t)| \le K(\varepsilon)$ and $\lim_{\varepsilon\to 0} K(\varepsilon) = 0$. If the preceding statement is also true when we replace ε and $K(\varepsilon)$ by $F(t)$ and $C(t)$, where F, C are appropriate functions not depending on u and u_a explicitly, then we say that the corresponding differential equation has the generalized Hyers–Ulam stability.

Theorem 2.2. *Let d be a scalar. If a function $u : (0,\infty) \to \mathbf{C}$ satisfies the inequality*

$$|u'(t) + du(t) - q(t)| \le \varepsilon \tag{2.1}$$

for all $t \in (0,\infty)$ and for some $\varepsilon > 0$, then there exists a solution $u_a : (0,\infty) \to \mathbf{C}$ of the differential equation

$$u'(t) + du(t) = q(t) \tag{2.2}$$

such that

$$|u(t) - u_a(t)| \le \varepsilon t E_{1,2}(|d|t) \tag{2.3}$$

for all $t \in (0,\infty)$, where $E_{1,2}(|d|t)$ is the Mittag–Leffler function.

Proof. Let $v(t) = u'(t) + du(t) - q(t)$, for $t \in (0,\infty)$, we get

$$\mathcal{L}\{v(t)\} = s\mathcal{L}\{u(t)\} - u(0) + d\mathcal{L}\{u(t)\} - \mathcal{L}\{q(t)\}, \tag{2.4}$$

and so

$$\mathcal{L}\{u(t)\} = \frac{\mathcal{L}\{v(t)\}}{s+d} + \frac{u(0) + \mathcal{L}\{q(t)\}}{s+d}. \tag{2.5}$$

Setting

$$u_a(t) = u(0)e^{-dt} + (E_{-d} * q)(t), \tag{2.6}$$

where $E_{-d}(t) = e^{-dt}$, one can check that u_a is a solution of (2.2). Since

$$\mathcal{L}\{E_{-d} * v\} = \mathcal{L}\{u - u_a\}, \tag{2.7}$$

so we have $u(t) - u_a(t) = (E_{-d} * v)(t)$.

By the condition (2.1), it follows that

$$
\begin{aligned}
|u(t) - u_a(t)| &= |(E_{-d} * v)(t)| \\
&= \left| \int_0^t e^{-d(t-s)} v(s)\, \mathrm{d}s \right| \\
&= \left| \sum_{n=0}^{\infty} \int_0^t \frac{(-d)^n (t-s)^n}{n!} v(s)\, \mathrm{d}s \right| \\
&\le \sum_{n=0}^{\infty} \int_0^t \left| \frac{(-d)^n (t-s)^n}{n!} v(s) \right| \mathrm{d}s \\
&\le \varepsilon \sum_{n=0}^{\infty} \frac{|d|^n}{n!} \int_0^t (t-s)^n\, \mathrm{d}s \\
&= \varepsilon \sum_{n=0}^{\infty} \frac{|d|^n t^{n+1}}{\Gamma(n+2)} \\
&= \varepsilon t E_{1,2}(|d|t),
\end{aligned}
\tag{2.8}
$$

which completes the proof.

Remark 2.3. In a recent result ([22], Theorem 3.3), the following control function of the equation (2.2) was obtained

$$
\begin{cases} \varepsilon t & (\text{for } \Re(d) = 0), \\ \dfrac{\varepsilon(1 - e^{-\Re(d)t})}{\Re(d)} & (\text{for } \Re(d) \neq 0). \end{cases}
\tag{2.9}
$$

In Theorem 2.2, we replace the control function by an expression related to Mittag–Leffler function. When $d = 0$, by a simple calculation, we have $E_{1,2}(0) = 1$, so in this case the result coincides with (2.9). When $\Re(d) < 0$, one can get

$$
\begin{aligned}
\varepsilon t E_{1,2}(|d|t) &= \varepsilon \sum_{k=0}^{\infty} \frac{|d|^k t^{k+1}}{(k+1)!} = \frac{\varepsilon(e^{|d|t} - 1)}{|d|} \le \frac{\varepsilon(e^{|d|t} - 1)}{|\Re(d)|} \\
&= \frac{\varepsilon(1 - e^{|d|t})}{\Re(d)} \le \frac{\varepsilon(1 - e^{-\Re(d)t})}{\Re(d)},
\end{aligned}
\tag{2.10}
$$

here $\dfrac{\varepsilon(1 - e^{-\Re(d)t})}{\Re(d)}$ is the control function in (2.9) when $\Re(d) \neq 0$.

COROLLARY 2.4. *Let d be a scalar. If a function $u : (0, \infty) \to \mathbf{C}$ satisfies the inequality*

$$
|u'(t) + du(t) - q(t)| \le F(t)
\tag{2.11}
$$

for all $t \in (0, \infty)$ *and for some* $F(t) > 0$, *then there exists a solution* $u_a : (0, \infty) \to \mathbf{C}$ *of the differential equation* (2.2) *such that*

$$|u(t) - u_a(t)| \le F(t)tE_{1,2}(|d|t) \tag{2.12}$$

for all $t \in (0, \infty)$, *where* $E_{1,2}(|d|t)$ *is the Mittag–Leffler function.*

Example 2.5. Consider the following differential equation

$$u'(t) + 2u(t) = e^{-2t}. \tag{2.13}$$

The function $u_1(t) = -e^{-3t}$ satisfies

$$|u_1'(t) + 2u_1(t) - e^{-2t}| \le \frac{1}{e^{2t}} - \frac{1}{e^{3t}}, \tag{2.14}$$

and the initial value is $u_1(0) = -1$.

By (2.6) and initial value $u_1(0) = -1$, we obtain an exact solution of the equation (2.13)

$$u_a(t) = -e^{-2t} + te^{-2t} \tag{2.15}$$

with $u_a(0) = -1 = u_1(0)$. By Corollary 2.4, the control function of $u_1(t)$ is $\frac{1}{2}\left(\frac{1}{e^{2t}} - \frac{1}{e^{3t}}\right)(e^{2t} - 1)$. A simple calculation shows that

$$|u_1(t) - u_a(t)| = \left(\frac{1}{e^{2t}} - \frac{1}{e^{3t}}\right)\left(\frac{t}{1 - e^{-t}} - 1\right) \le \frac{1}{2}\left(\frac{1}{e^{2t}} - \frac{1}{e^{3t}}\right)(e^{2t} - 1) \tag{2.16}$$

for all $t > 0$, thus, the error of the approximate solution $u_1(t)$ can be estimated.

3. Hyers–Ulam stability of fractional linear differential equations

In this section, we will extend Theorem 2.2 to the case of fractional linear differential equations.

Definition 3.1. The fractional differential equation $\varphi(q, u, D^{\alpha_1}u, \ldots, D^{\alpha_n}u) = 0$ has Hyers–Ulam stability if for given $\varepsilon > 0$ and a function u such that $|\varphi(q, u, D^{\alpha_1}u, \ldots, D^{\alpha_n}u)| \le \varepsilon$, there exists a solution u_a of the differential equation such that $|u(t) - u_a(t)| \le K(\varepsilon)$ and $\lim_{\varepsilon \to 0} K(\varepsilon) = 0$. If the preceding statement is also true when we replace ε and $K(\varepsilon)$ by $F(t)$ and $C(t)$, where F, C are appropriate functions not depending on u and u_a explicitly, then we say that the corresponding differential equation has the generalized Hyers–Ulam stability.

Some other concepts about stability of fractional differential equations can be found in [27] and [28].

THEOREM 3.2. *Let d be a scalar, $0 < \alpha < 1$, $0 < T < +\infty$. If a function $u : (0, T] \to \mathbf{C}$ satisfies the inequality*

$$|D^{\alpha}u(t) + du(t) - q(t)| \leq \varepsilon \tag{3.1}$$

for all $t \in (0, T]$ and for some $\varepsilon > 0$, then there exists a solution $u_a : (0, T] \to \mathbf{C}$ of the fractional differential equation

$$D^{\alpha}u(t) + du(t) = q(t) \tag{3.2}$$

such that

$$|u(t) - u_a(t)| \leq \varepsilon t^{\alpha} E_{\alpha,\alpha+1}(|d|t^{\alpha}) \tag{3.3}$$

for all $t \in (0, T]$, where $E_{\alpha,\alpha+1}(|d|t^{\alpha})$ is the Mittag–Leffler function.

Proof. Let $v(t) = D^{\alpha}u(t) + du(t) - q(t)$ for $t \in (0, T]$, by the initial value condition of (1.1), Lemmas 1.1 and 1.6, we have

$$\begin{aligned} \mathcal{L}\{v(t)\} &= s^{\alpha}\mathcal{L}\{u(t)\} - D^{\alpha-1}u(0+) + d\mathcal{L}\{u(t)\} - \mathcal{L}\{q(t)\} \\ &= s^{\alpha}\mathcal{L}\{u(t)\} - u_0\Gamma(\alpha) + d\mathcal{L}\{u(t)\} - \mathcal{L}\{q(t)\} \\ &= (s^{\alpha} + d)\mathcal{L}\{u(t)\} - u_0\Gamma(\alpha) - \mathcal{L}\{q(t)\}. \end{aligned} \tag{3.4}$$

Thus

$$\mathcal{L}\{u(t)\} = \frac{\mathcal{L}\{v(t)\}}{s^{\alpha} + d} + \frac{u_0\Gamma(\alpha) + \mathcal{L}\{q(t)\}}{s^{\alpha} + d}. \tag{3.5}$$

Setting

$$u_a(t) = u_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-dt^{\alpha}) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-d(t-s)^{\alpha})q(s)\,\mathrm{d}s, \tag{3.6}$$

then

$$t^{1-\alpha}u_a(t) = u_0\Gamma(\alpha)E_{\alpha,\alpha}(-dt^{\alpha}) + t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-d(t-s)^{\alpha})q(s)\,\mathrm{d}s, \tag{3.7}$$

when $t \to 0$ in (3.7), we obtain

$$t^{1-\alpha}u_a(t)|_{t=0} = u_0, \tag{3.8}$$

so $u_a(t)$ satisfies the initial value condition.

By the property of convolution and Lemma 1.4, we get

$$\begin{aligned} \mathcal{L}\{u_a(t)\} &= \mathcal{L}\{u_0\Gamma(\alpha)t^{\alpha-1}E_{\alpha,\alpha}(-dt^{\alpha})\} \\ &\quad + \mathcal{L}\left\{\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-d(t-s)^{\alpha})q(s)\,\mathrm{d}s\right\} \\ &= u_0\Gamma(\alpha)\mathcal{L}\{t^{\alpha-1}E_{\alpha,\alpha}(-dt^{\alpha})\} + \mathcal{L}\{t^{\alpha-1}E_{\alpha,\alpha}(-dt^{\alpha})\}\mathcal{L}\{q(t)\} \\ &= \frac{u_0\Gamma(\alpha) + \mathcal{L}\{q(t)\}}{s^{\alpha} + d}. \end{aligned} \tag{3.9}$$

By (3.9) and Lemma 1.1 we obtain

$$\mathcal{L}\{D^{\alpha}u_a(t) + du_a(t)\} = \mathcal{L}\{q(t)\}. \tag{3.10}$$

Since $\mathcal{L}$ is one-to-one, it follows that $D^{\alpha}u_a(t) + du_a(t) = q(t)$, so $u_a(t)$ is a solution of (3.2). Applying (3.5) and (3.9), we get

$$\mathcal{L}\{u(t)\} - \mathcal{L}\{u_a(t)\} = \frac{\mathcal{L}\{v(t)\}}{s^{\alpha} + d}. \tag{3.11}$$

By Lemma 1.4, we have

$$\mathcal{L}\{(t^{\alpha-1}E_{\alpha,\alpha}(-dt^{\alpha})) * v(t)\}(s) = \mathcal{L}\{t^{\alpha-1}E_{\alpha,\alpha}(-dt^{\alpha})\}\mathcal{L}\{v(t)\} = \frac{\mathcal{L}\{v(t)\}}{s^{\alpha} + d}. \tag{3.12}$$

Hence

$$\mathcal{L}\{u(t) - u_a(t)\} = \mathcal{L}\{(t^{\alpha-1}E_{\alpha,\alpha}(-dt^{\alpha})) * v(t)\}, \tag{3.13}$$

so,

$$u(t) - u_a(t) = (t^{\alpha-1}E_{\alpha,\alpha}(-dt^{\alpha})) * v(t). \tag{3.14}$$

Therefore, from (3.1), it follows that

$$\begin{aligned} |u(t) - u_a(t)| &= |(t^{\alpha-1}E_{\alpha,\alpha}(-dt^{\alpha})) * v(t)| \\ &= \left|\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-d(t-s)^{\alpha})v(s)\,\mathrm{d}s\right| \\ &= \left|\int_0^t \sum_{k=0}^{\infty} \frac{(-d)^k (t-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} v(s)\,\mathrm{d}s\right| \\ &= \left|\sum_{k=0}^{\infty} \int_0^t \frac{(-d)^k (t-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} v(s)\,\mathrm{d}s\right| \\ &\leq \sum_{k=0}^{\infty} \left|\int_0^t \frac{(-d)^k (t-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} v(s)\,\mathrm{d}s\right| \\ &\leq \sum_{k=0}^{\infty} \int_0^t \left|\frac{(-d)^k (t-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} v(s)\right| \mathrm{d}s \\ &\leq \varepsilon \sum_{k=0}^{\infty} \frac{|d|^k}{\Gamma(\alpha k+\alpha)} \int_0^t (t-s)^{\alpha k+\alpha-1}\,\mathrm{d}s \\ &= \varepsilon \sum_{k=0}^{\infty} \frac{|d|^k t^{\alpha k+\alpha}}{\Gamma(\alpha k+\alpha+1)} \\ &= \varepsilon t^{\alpha} E_{\alpha,\alpha+1}(|d|t^{\alpha}), \end{aligned} \tag{3.15}$$

which completes the proof.

Remark 3.3. When $\alpha = 1$, we can see that Theorem 3.2 coincides with Theorem 2.2, so Theorem 3.2 generalizes Theorem 2.2. In fact, the control function which at the right side of the inequality (3.3) includes more information about Hyers–Ulam stability of fractional linear differential equations.

COROLLARY 3.4. *Let d be a scalar, $0 < \alpha < 1$, $0 < T < +\infty$. If a function $u : (0, T] \to \mathbf{C}$ satisfies the inequality*

$$|D^\alpha u(t) + du(t) - q(t)| \le F(t) \tag{3.16}$$

for all $t \in (0, T]$ and for some $F(t) > 0$, then there exists a solution $u_a : (0, T] \to \mathbf{C}$ of the fractional differential equation (3.2) *such that*

$$|u(t) - u_a(t)| \le F(t) t^\alpha E_{\alpha,\alpha+1}(|d|t^\alpha) \tag{3.17}$$

for all $t \in (0, T]$, where $E_{\alpha,\alpha+1}(|d|t^\alpha)$ is the Mittag–Leffler function.

Example 3.5. Consider the following fractional differential equation

$$D^{1/2}u(t) + 7u(t) = \frac{16}{5\sqrt{\pi}} t^{5/2} + 7t^3 + \frac{1}{20}, \tag{3.18}$$

where $\alpha = \frac{1}{2}$, $d = 7$, $q(t) = \frac{16}{5\sqrt{\pi}} t^{5/2} + 7t^3 + \frac{1}{20}$.

For $\varepsilon = \frac{1}{10}$, the function $u_1(t) = t^3$ satisfies

$$\left| D^{1/2}u_1(t) + 7u_1(t) - \frac{16}{5\sqrt{\pi}} t^{5/2} - 7t^3 - \frac{1}{20} \right| < \frac{1}{10}, \tag{3.19}$$

and initial value of $u_1(t)$ is $t^{1/2}u_1(t)|_{t=0} = 0$.

By (3.6) and $t^{1/2}u_1(t)|_{t=0} = 0$, we can construct an exact solution of equation (3.18)

$$u_a(t) = \int_0^t (t-s)^{-1/2} E_{1/2,1/2}(-7(t-s)^{1/2}) \left(\frac{16}{5\sqrt{\pi}} s^{5/2} + 7s^3 + \frac{1}{20} \right) \mathrm{d}s. \tag{3.20}$$

By Theorem 3.2, the control function of $u_1(t)$ is $\frac{1}{10}\sqrt{t}E_{1/2,3/2}(7\sqrt{t})$, thus

$$|u_1(t) - u_a(t)| < \frac{1}{10}\sqrt{t}E_{1/2,3/2}(7\sqrt{t}), \tag{3.21}$$

and the error of the approximate solution $u_1(t)$ can be estimated.

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