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A NECESSARY CONDITION FOR TWO STRING LINKS TO HAVE THE SAME CLOSURE UP TO CONCORDANCE

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Abstract

To a link (obtained by closing a string link) we associate a certain diagram of groups. If two links are concordant, we show that there exists a certain type of isomorphism between the group diagrams.

1. Introduction

The concept of knot concordance (or knot cobordism) was introduced by Fox and Milnor [5]. Its study, as well as the study of link concordance, continued in the works of Cappell and Shaneson [3] Tristram [15], Levine ([9], [10]), Ko [8] and others.

In this paper we study link concordance via string links. Le Dimet [4] introduced the group of cobordism classes of k-string links (a string link is a generalization of a braid—see Definition 1 below). Besides their own interest, k-string links are naturally related to links because one obtains a k-link by simply closing a k-string link, like one does for a braid, with the advantage that the number of strings is preserved. The group of concordance classes of k-string links also have a kind of Artin representation (see [6]). This allows to break the problem of studying links in two problems: studying k-string links (with their natural multiplication) and studying when two k-string links have the same closure. This string-link approach was shown to be very useful in the case of link-homotopy allowing Habegger and Lin [7] to classify links up to link-homotopy. With a similar approach, Habegger and Lin [6] also obtained some advances in the case of link concordance, but, in their own words, "that was far from leading to a classification of links up to concordance".

Previously, Levine [11] had studied link-homotopy using an approach based on peripheral invariants, similar to that of Waldhausen's Theorem for links up to ambient isotopy (see [16]). In our paper [1] we provided a connection between the two approaches, obtaining a certain diagram of groups that distinguishes links

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if and only if they are not link-homotopic. The diagram had a small mistake that we correct in section 4 below.

In this paper we consider similar results for the case of link concordance. In this case, we show only that if two links are link concordant then there exists a certain type of isomorphism between the associated group diagrams, thus providing a necessary algebraic condition for two string links to have concordant closures.

This paper is divided as follows: in section 2, we deal with Habegger-Lin's actions for string-links (up to ambient isotopy), where we associate certain epimorphisms to elements in the stabilizer of 1 for both of Habegger-Lin's actions of $2k$-string links on k-string links (see Fig. 2 for the actions). In section 3, we study the case of link concordance, case in which both stabilizers coincide (see [6]) and we obtain a necessary condition for two string links to have the same closure up to concordance (Theorem 21 and Corollary 22). In section 4, we make a correction for the case of link-homotopy (as it appeared in our paper [1]). Our group diagram in [1] is a complete invariant, that is, it distinguishes links if and only if they are not link homotopic, but to show that we need a small change in the diagram that appeared in [1].

2. Ambient isotopy

We will use the following notation: I is the interval $[0,1]$, D is the unit disk $\{x \in \mathbf{R}^2 \mid \|x\| \leqslant 1\}$, $k \geqslant 1$ is an integer number, $\underline{k}$ is the set $\{1,2,\dots,k\}$, $(\forall i \in \underline{k})$ a_i is the point $\left(-1+\dfrac{2i}{k+1},0\right) \in D$ and $j_0 : \underline{k} \times I \to D \times I$ is the map defined by $(i,x)j_0 = (a_i,x)$. Note that, as above, if f is a map, we will usually write $(x)f$ instead of $f(x)$.

Definition 1. A *k-string link* is a (smooth or piecewise linear) proper embedding $f : \underline{k} \times I \to D \times I$ such that $f|_{\underline{k}\times\partial I} = j_0|_{\underline{k}\times\partial I}$ (see Fig. 1).

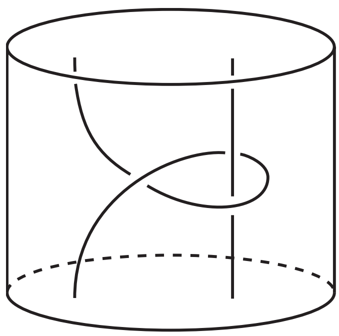


Figure 1. A 2-string link.

The product of two k-string links f and g, denoted by fg, is given by stacking f on the top of g and reparametrizing (see [1]). This product induces a monoid structure on the set $SL(k)$ of (ambient) isotopy classes of k-string links.

Habegger-Lin (see [6]) introduced a left and a right action of the monoid $SL(2k)$ on the set $SL(k)$ that we will call Habegger-Lin's actions. We use a slightly different notation (see Fig. 2 below).

DEFINITION 2. The *reflection* of a k-string link f is the k-string link f^R obtained by reflecting f in $D \times \frac{1}{2}$.

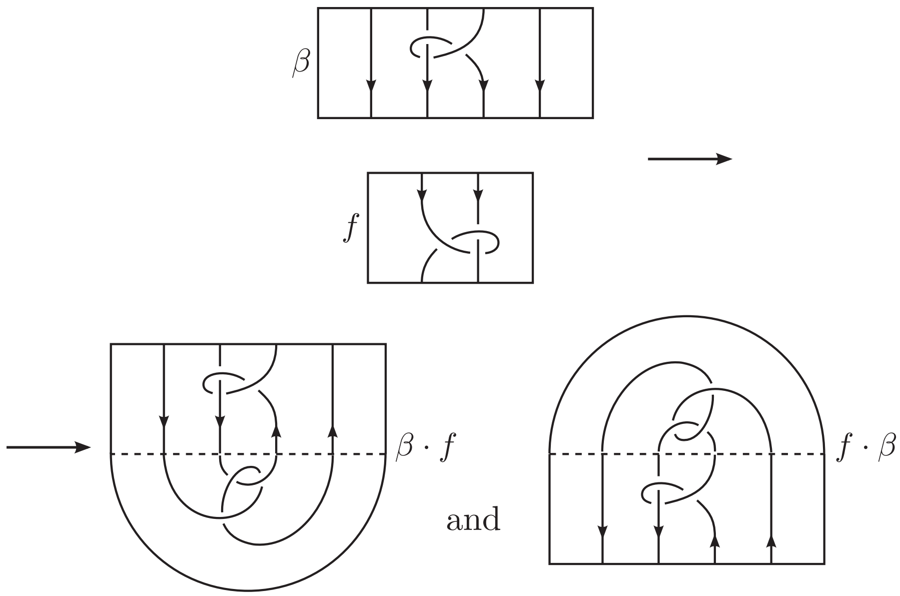


FIGURE 2. A 4-string link β acts from the left and from the right on a 2-string link f.

DEFINITION 3. A k-*link* (or a link of k components) is an embedding of a disjoint union of ordered oriented circles $\bigsqcup_{i=1}^{k} S^1$ into S^3.

To a k-string link f it is associated a k-link $\hat{f}$ called its *closure* (see [1]).

The fundamental group of the complement of a string link f is called the *group of* f and is denoted by $\pi(f)$.

For a group G, let $\{G_n\}$, $n \geqslant 1$, denote the lower central series of G, that is, $G_1 = G$ and inductively $G_{n+1} = [G, G_n]$ (where for sets $A, B \subseteq G$, $[A, B]$ denotes the group generated by all commutators $[a, b] = aba^{-1}b^{-1}$, $a \in A$, $b \in B$.)

Let $\tilde{G} = \varprojlim_n \frac{G}{G_n}$ be the nilpotent completion of G.

Let $F(k)$ denote the free group in k generators $\alpha_1, \alpha_2, \ldots, \alpha_k$.

Let f be a k-string link. We will denote by $x_i = x_i(f) \in \pi(f)$, for all $i \in \underline{k}$, the top meridians of f and by $y_i = y_i(f) \in \pi(f)$, for all $i \in \underline{k}$, the bottom meridians of f (see Fig. 3 and [2]).

For $j = 0, 1$, inclusions $i_j : D \times \{j\} \backslash \partial_j f \to D \times I \backslash f$ induce homomorphisms $\mu_0(f) : F(k) = F(\alpha_1, \alpha_2, \ldots, \alpha_k) \to \pi(f)$, $(\alpha_i)\mu_0(f) = x_i(f)$, and $\mu_1(f) : F(k) \to \pi(f)$, $(\alpha_i)\mu_1(f) = y_i(f)$, called respectively, the *top meridian map for f* and the *bottom meridian map for f*. By Stallings' Theorem [14] (see also [6]), they also induce isomorphisms on the lower central series quotients of fundamental groups:

$$\frac{F(k)}{F(k)_n} \xrightarrow[\cong]{(\mu_0(f))_n} \frac{\pi(f)}{\pi(f)_n} \xleftarrow[\cong]{(\mu_1(f))_n} \frac{F(k)}{F(k)_n}.$$

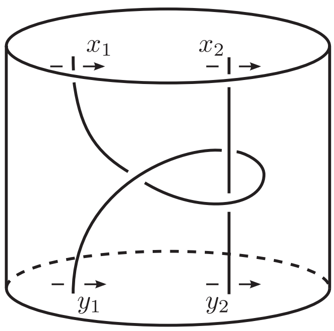


FIGURE 3. Top and bottom meridians of a 2-string link.

Therefore $(\mu_0(f))_n(\mu_1(f))_n^{-1}$ is an element $A_n(f) \in Aut\left(\frac{F(k)}{F(k)_n}\right)$, the group of automorphisms of $\frac{F(k)}{F(k)_n}$. Let us denote $A_n(f)$ also by $\bar{f}_n$.

$\mu_0(f)$ and $\mu_1(f)$ also induce isomorphisms (see [12]):

$$\widetilde{F(k)} \xrightarrow[\cong]{\widetilde{\mu_0(f)}} \widetilde{\pi(f)} \xleftarrow[\cong]{\widetilde{\mu_1(f)}} \widetilde{F(k)}.$$

Thus we have $A(f) = \tilde{f} = \widetilde{\mu_0(f)}\widetilde{\mu_1(f)}^{-1} \in Aut(\widetilde{F(k)})$.

The associations $f \mapsto \overline{f_n}$ and $f \mapsto \tilde{f}$ are monoid homomorphism from $SL(k)$ into $Aut\left(\frac{F(k)}{F(k)_n}\right)$ and $Aut(\widetilde{F(k)})$, respectively.

Note also that, since $\mu_0(f^R) = \mu_1(f)$ and $\mu_1(f^R) = \mu_0(f)$, we have $(\overline{f^R})_n = \bar{f}_n^{-1}$ and also $\widetilde{f^R} = \tilde{f}^{-1}$.

Let $F(2k)$ be the free group in $2k$ generators $\alpha_1, \alpha_2, \dots, \alpha_k, \widetilde{\alpha_k}, \dots, \widetilde{\alpha_2}, \widetilde{\alpha_1}$ and $\langle \alpha_i \widetilde{\alpha_i} \rangle^N = \langle \alpha_i \widetilde{\alpha_i} \mid i \in \underline{k} \rangle^N$ be the normal subgroup of $F(2k)$ generated by $\alpha_i \widetilde{\alpha_i}$, $i \in \underline{k}$.

Let $\langle \alpha_i \widetilde{\alpha}_i F(2k)_n \rangle^N$ be the normal subgroup of $\dfrac{F(2k)}{F(2k)_n}$ generated by $\alpha_i \widetilde{\alpha}_i F(2k)_n$, $i \in \underline{k}$.

Lemma 1. *There is an isomorphism*

$$\lambda : \frac{F(2k)}{\langle \alpha_i \widetilde{\alpha_i} \rangle^N F(2k)_n} \to \frac{\dfrac{F(2k)}{F(2k)_n}}{\langle \alpha_i \widetilde{\alpha}_i F(2k)_n \rangle^N}$$

defined by $(w \langle \alpha_i \widetilde{\alpha_i} \rangle^N F(2k)_n)\lambda = wF(2k)_n \langle \alpha_i \widetilde{\alpha}_i F(2k)_n \rangle^N$, *for any* $w \in F(2k)$.

Proof. First note that $w\langle \alpha_i \widetilde{\alpha_i} \rangle^N F(2k)_n = \langle \alpha_i \widetilde{\alpha_i} \rangle^N F(2k)_n \Leftrightarrow w \in \langle \alpha_i \widetilde{\alpha_i} \rangle^N \cdot F(2k)_n \Leftrightarrow w = uv$ with $u \in \langle \alpha_i \widetilde{\alpha_i} \rangle^N$ and $v \in F(2k)_n \Leftrightarrow wF(2k)_n = uvF(2k)_n = uF(2k)_n$ with $u \in \langle \alpha_i \widetilde{\alpha_i} \rangle^N \Leftrightarrow wF(2k)_n \in \langle \alpha_i \widetilde{\alpha}_i F(2k)_n \rangle^N$.

It follows that, for any $w, w' \in F(2k)$, we have

$$\begin{aligned} w\langle \alpha_i \widetilde{\alpha_i} \rangle^N F(2k)_n = w' \langle \alpha_i \widetilde{\alpha_i} \rangle^N F(2k)_n &\Leftrightarrow w^{-1} w' \in \langle \alpha_i \widetilde{\alpha_i} \rangle^N F(2k)_n \\ &\Leftrightarrow w^{-1} w' F(2k)_n \in \langle \alpha_i \tilde{\alpha}_i F(2k)_n \rangle^N \\ &\Leftrightarrow (w^{-1} F(2k)_n)(w' F(2k)_n) \in \langle \alpha_i \widetilde{\alpha}_i F(2k)_n \rangle^N \\ &\Leftrightarrow wF(2k)_n \langle \alpha_i \widetilde{\alpha}_i F(2k)_n \rangle^N = w' F(2k)_n \langle \alpha_i \widetilde{\alpha}_i F(2k)_n \rangle^N. \end{aligned}$$

It follows that λ is well-defined and injective. It is easy to see that it is also an onto homomorphism. □

Lemma 2. *Let β be a $2k$-string link with bottom meridians $y_1, y_2, \dots, y_k, \widetilde{y_k}, \dots, \widetilde{y_2}, \widetilde{y_1}$. There is an isomorphism*

$$\lambda' : \frac{\pi(\beta)}{\langle y_i \widetilde{y_i} \rangle^N \pi(\beta)_n} \to \frac{\dfrac{\pi(\beta)}{\pi(\beta)_n}}{\langle y_i \widetilde{y}_i \pi(\beta)_n \rangle^N}$$

defined by $(w\langle y_i \widetilde{y_i} \rangle^N \pi(\beta)_n)\lambda' = w\pi(\beta)_n \langle y_i \widetilde{y}_i \pi(\beta)_n \rangle^N$, *for any* $w \in \pi(\beta)$.

Proof. It can be proved as Lemma 1. □

Lemma 3. *If $f : G \to H$ is a group epimorphism, then f induces an isomorphism*

$$f' : \frac{G}{\ker(f) G_n} \to \frac{H}{H_n}.$$

Proof. If $L \subseteq G$ is a subgroup, we have $((L)f)f^{-1} = \ker(f)L$. In particular, if $L = G_n$, we have $((G_n)f)f^{-1} = \ker(f)G_n$, but $(G_n)f = H_n$, so $(H_n)f^{-1} = \ker(f)G_n$. Therefore f induces an isomorphism

$$f' : \frac{G}{\ker(f)G_n} = \frac{G}{(H_n)f^{-1}} \to \frac{H}{H_n}. \qquad \square$$

Let $F(k)$ be the free group in k generators $\alpha_1, \alpha_2, \dots, \alpha_k$ and $F(2k)$ be the free group in $2k$ generators $\alpha_1, \alpha_2, \dots, \alpha_k, \widetilde{\alpha_k}, \dots, \widetilde{\alpha_2}, \widetilde{\alpha_1}$. We will denote by ξ the epimorphism $\xi : F(2k) \to F(k)$ given by $(\alpha_i)\xi = \alpha_i$ and $(\widetilde{\alpha_i})\xi = \alpha_i^{-1}$ for any $i \in \underline{k}$. The kernel of ξ is $\langle \alpha_i \widetilde{\alpha_i} \rangle^N$, the normal subgroup of $F(2k)$ generated by $\{\alpha_i \widetilde{\alpha_i} \mid i \in \underline{k}\}$.

It follows from Lemma 3 that ξ induces an isomorphism

$$\xi' : \frac{F(2k)}{\langle \alpha_i \widetilde{\alpha_i} \rangle^N F(2k)_n} \to \frac{F(k)}{F(k)_n}.$$

Let $\overline{S_k(1)_n} = \left\{ \beta \in SL(2k) \mid \overline{\beta \cdot 1}_n \text{ is the identity automorphism of } \dfrac{F(k)}{F(k)_n} \right\}$. Clearly $\overline{S_k(1)_n}$ contains the stabilizer of 1 for Habegger-Lin's action $S_k(1) = \{\beta \in SL(2k) \mid \beta \cdot 1 = 1 \in SL(k)\}$.

Theorem 4. *Let $\beta \in SL(2k)$. If $\beta \in \overline{S_k(1)_n}$ then there exists an epimorphism $\bar{\bar{\beta}}_n : \dfrac{F(k)}{F(k)_n} \to \dfrac{F(k)}{F(k)_n}$ such that the diagram*

$$\begin{array}{ccc} \dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\bar{\beta}_n} & \dfrac{F(2k)}{F(2k)_n} \\ \Big\downarrow{\scriptstyle \xi_n} & & \Big\downarrow{\scriptstyle \xi_n} \\ \dfrac{F(k)}{F(k)_n} & \xrightarrow{\bar{\bar{\beta}}_n} & \dfrac{F(k)}{F(k)_n} \end{array}$$

is commutative, where ξ_n is induced from $\xi : F(2k) \to F(k)$.

Proof. Let $\beta \in SL(2k)$ have top meridians $x_1, x_2, \dots, x_k, \widetilde{x_k}, \dots, \widetilde{x_2}, \widetilde{x_1}$ and bottom meridians $y_1, y_2, \dots, y_k, \widetilde{y_k}, \dots, \widetilde{y_2}, \widetilde{y_1}$, and let $F(2k)$ have generators $\alpha_1, \alpha_2, \dots, \alpha_k, \widetilde{\alpha_k}, \dots, \widetilde{\alpha_2}, \widetilde{\alpha_1}$ as we saw earlier. We have isomorphisms

$$\frac{F(2k)}{F(2k)_n} \xrightarrow[\cong]{\mu_0(\beta)_n} \frac{\pi(\beta)}{\pi(\beta)_n} \xrightarrow[\cong]{(\mu_1(\beta)_n)^{-1}} \frac{F(2k)}{F(2k)_n}$$

where $(\mu_1(\beta)_n)^{-1}$ sends $y_i \pi(\beta)_n$ to $\alpha_i F(2k)_n$ and $\tilde{y}_i \pi(\beta)_n$ to $\tilde{\alpha}_i F(2k)_n$, therefore it induces an isomorphism

$$(\Gamma_1(\beta)_n)^{-1} : \frac{\dfrac{\pi(\beta)}{\pi(\beta)_n}}{\langle y_i \tilde{y}_i \pi(\beta)_n \rangle^N} \to \frac{\dfrac{F(2k)}{F(2k)_n}}{\langle \alpha_i \tilde{\alpha}_i F(2k)_n \rangle^N},$$

where $\langle u_i \rangle^N$ is the normal subgroup generated by u_i, $i \in \underline{k}$. By Lemma 2 there is an isomorphism between $\dfrac{\frac{\pi(\beta)}{\pi(\beta)_n}}{\langle y_i \tilde{y}_i \pi(\beta)_n \rangle^N}$ and $\dfrac{\pi(\beta)}{\langle y_i \tilde{y}_i \rangle^N \pi(\beta)_n}$. On the other side, $\beta \cdot 1$ is a k-string link and inclusion map induces an epimorphism $\sigma : \pi(\beta) \to \pi(\beta \cdot 1)$ sending y_i to y_i and $\tilde{y}_i$ to y_i^{-1}, for any $i \in \underline{k}$.

By Lemma 3, σ induces an isomorphism between

$$\frac{\pi(\beta)}{\langle y_i \tilde{y}_i \rangle^N \pi(\beta)_n} \quad \text{and} \quad \frac{\pi(\beta \cdot 1)}{\pi(\beta \cdot 1)_n}.$$

By Lemma 1, there is an isomorphism between

$$\frac{\frac{F(2k)}{F(2k)_n}}{\langle \alpha_i \tilde{\alpha}_i F(2k)_n \rangle^N} \quad \text{and} \quad \frac{F(2k)}{\langle \alpha_i \tilde{\alpha}_i \rangle^N F(2k)_n},$$

and Lemma 3 applied to ξ provides an isomorphism between

$$\frac{F(2k)}{\langle \alpha_i \tilde{\alpha}_i \rangle^N F(2k)_n} \quad \text{and} \quad \frac{F(k)}{F(k)_n}.$$

Putting all these data in a diagram, we have

$$\text{(I)} \qquad \begin{array}{ccccc}
\dfrac{F(2k)}{F(2k)_n} & \xrightarrow[\cong]{\mu_0(\beta)_n} & \dfrac{\pi(\beta)}{\pi(\beta)_n} & \xrightarrow[\cong]{(\mu_1(\beta)_n)^{-1}} & \dfrac{F(2k)}{F(2k)_n} \\
\downarrow & & \downarrow & & \downarrow \\
\dfrac{\frac{F(2k)}{F(2k)_n}}{\langle \alpha_i \tilde{\alpha}_i F(2k)_n \rangle^N} & & \dfrac{\frac{\pi(\beta)}{\pi(\beta)_n}}{\langle y_i \tilde{y}_i \pi(\beta)_n \rangle^N} & \xrightarrow[\cong]{(\Gamma_1(\beta)_n)^{-1}} & \dfrac{\frac{F(2k)}{F(2k)_n}}{\langle \alpha_i \tilde{\alpha}_i F(2k)_n \rangle^N} \\
\cong \downarrow & & \downarrow \cong & & \downarrow \cong \\
\dfrac{F(2k)}{\langle \alpha_i \tilde{\alpha}_i \rangle^N F(2k)_n} & & \dfrac{\pi(\beta)}{\langle y_i \tilde{y}_i \rangle^N \pi(\beta)_n} & & \dfrac{F(2k)}{\langle \alpha_i \tilde{\alpha}_i \rangle^N F(2k)_n} \\
\xi' \downarrow \cong & & \downarrow \cong & & \xi' \downarrow \cong \\
\dfrac{F(k)}{F(k)_n} & & \dfrac{\pi(\beta \cdot 1)}{\pi(\beta \cdot 1)_n} & & \dfrac{F(k)}{F(k)_n}
\end{array}$$

where $\cong$ represents isomorphisms and top vertical maps are quotient maps.

Note that the k-string link $\beta \cdot 1$ has x_i, $i \in \underline{k}$, as top meridians and $\tilde{x}_i^{-1}$, $i \in \underline{k}$, as bottom meridians. Therefore $[\alpha_i]\mu_0(\beta \cdot 1)_n = [x_i]$, where [ ] denotes the equivalence classes in the quotient groups $\dfrac{F(n)}{F(k)_n}$ and $\dfrac{\pi(\beta \cdot 1)}{\pi(\beta \cdot 1)_n}$. Similarly, $[\alpha_i]\mu_1(\beta \cdot 1)_n = [\tilde{x}_i^{-1}]$. Then

$$[\alpha_i]\mu_0(\beta \cdot 1)_n\mu_1(\beta \cdot 1)_n^{-1} = [x_i]\mu_1(\beta \cdot 1)_n^{-1}.$$

Thus we have the equivalence

$$\begin{aligned}[x_i]\mu_1(\beta \cdot 1)_n^{-1} &= [\tilde{x}_i^{-1}]\mu_1(\beta \cdot 1)_n^{-1}\\ &\Leftrightarrow [\alpha_i]\mu_0(\beta \cdot 1)_n\mu_1(\beta \cdot 1)_n^{-1} = [\alpha_i]\\ &\Leftrightarrow [\alpha_i](\overline{\beta \cdot 1})_n = [\alpha_i].\end{aligned}$$

Suppose now that $\beta \in \overline{S_k(1)_n}$. Then $\overline{\beta \cdot 1}_n$ is the identity map. Then $[x_i]\mu_1(\beta \cdot 1)_n^{-1} = [\tilde{x}_i^{-1}]\mu_1(\beta \cdot 1)_n^{-1}$, but $\mu_1(\beta \cdot 1)_n^{-1}$ is an isomorphism, so $[x_i] = [\tilde{x}_i^{-1}]$ in $\dfrac{\pi(\beta \cdot 1)}{\pi(\beta \cdot 1)_n}$, and we have seen that $\dfrac{\pi(\beta \cdot 1)}{\pi(\beta \cdot 1)_n}$ is isomorphic to

$$\frac{\dfrac{\pi(\beta)}{\pi(\beta)_n}}{\langle y_i\tilde{y}_i\pi(\beta)_n\rangle^N}.$$

It follows that, in this last group, the class of x_i is equal to the class of $\tilde{x}_i^{-1}$. Now observe that $\mu_0(\beta)_n : \dfrac{F(2k)}{F(2k)_n} \to \dfrac{\pi(\beta)}{\pi(\beta)_n}$ sends $\alpha_i\tilde{\alpha}_iF(2k)_n$ to $x_i\tilde{x}_i\pi(\beta)_n$, for any $i \in \underline{k}$. Therefore $\mu_0(\beta)_n$ induces a homomorphism

$$\Gamma_0(\beta)_n : \frac{\dfrac{F(2k)}{F(2k)_n}}{\langle \alpha_i\tilde{\alpha}_iF(2k)_n\rangle^N} \to \frac{\dfrac{\pi(\beta)}{\pi(\beta)_n}}{\langle y_i\tilde{y}_i\pi(\beta)_n\rangle^N}.$$

We have thus a commutative diagram

(II)
$$\begin{array}{ccccc}
\dfrac{F(2k)}{F(2k)_n} & \xrightarrow[\cong]{\mu_0(\beta)_n} & \dfrac{\pi(\beta)}{\pi(\beta)_n} & \xrightarrow[\cong]{(\mu_1(\beta)_n)^{-1}} & \dfrac{F(2k)}{F(2k)_n}\\
\downarrow & & \downarrow & & \downarrow\\
\dfrac{\dfrac{F(2k)}{F(2k)_n}}{\langle \alpha_i\tilde{\alpha}_iF(2k)_n\rangle^N} & \xrightarrow{\Gamma_0(\beta)_n} & \dfrac{\dfrac{\pi(\beta)}{\pi(\beta)_n}}{\langle y_i\tilde{y}_i\pi(\beta)_n\rangle^N} & \xrightarrow{(\Gamma_1(\beta)_n)^{-1}} & \dfrac{\dfrac{F(2k)}{F(2k)_n}}{\langle \alpha_i\tilde{\alpha}_iF(2k)_n\rangle^N}
\end{array}$$

where the vertical maps are quotient maps.

Diagrams (I) and (II) provide a commutative diagram

$$\text{(III)}\qquad \begin{array}{ccccc} \dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\mu_0(\beta)_n} & \dfrac{\pi(\beta)}{\pi(\beta)_n} & \xrightarrow{(\mu_1(\beta)_n)^{-1}} & \dfrac{F(2k)}{F(2k)_n} \\ \Big\downarrow{\scriptstyle \xi_n} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \xi_n} \\ \dfrac{F(k)}{F(k)_n} & \xrightarrow{\Gamma_0'(\beta)_n} & \dfrac{\pi(\beta\cdot 1)}{\pi(\beta\cdot 1)_n} & \xrightarrow{(\Gamma_1'(\beta)_n)^{-1}} & \dfrac{F(k)}{F(k)_n} \end{array}$$

where ξ_n is induced from $\xi : F(2k) \to F(k)$.

Let us recall that $\mu_0(\beta)_n(\mu_1(\beta)_n)^{-1} = \bar{\beta}_n$ and denote $\Gamma_0'(\beta)_n(\Gamma_1'(\beta)_n)^{-1}$ by $\bar{\bar{\beta}}_n$. Therefore we have a commutative diagram

$$(*)\qquad \begin{array}{ccc} \dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\bar{\beta}_n} & \dfrac{F(2k)}{F(2k)_n} \\ \Big\downarrow{\scriptstyle \xi_n} & & \Big\downarrow{\scriptstyle \xi_n} \\ \dfrac{F(k)}{F(k)_n} & \xrightarrow{\bar{\bar{\beta}}_n} & \dfrac{F(k)}{F(k)_n} \end{array}$$

Let $\overline{S_k(1)} = \{\beta \in SL(2k) \mid \widetilde{\beta\cdot 1}$ is the identity automorphism of $\widetilde{F(k)}\}$. Then $S_k(1) \subseteq \overline{S_k(1)} = \bigcap_n \overline{S_k(1)_n}$.

Corollary 5. *If* $\beta \in \overline{S_k(1)}$, *then there exists an epimorphism* $\tilde{\bar{\beta}} : \widetilde{F(k)} \to \widetilde{F(k)}$ *such that the diagram*

$$(**)\qquad \begin{array}{ccc} \widetilde{F(2k)} & \xrightarrow{\tilde{\beta}} & \widetilde{F(2k)} \\ \Big\downarrow{\scriptstyle \tilde{\xi}} & & \Big\downarrow{\scriptstyle \tilde{\xi}} \\ \widetilde{F(k)} & \xrightarrow{\tilde{\bar{\beta}}} & \widetilde{F(k)} \end{array}$$

is commutative, where $\tilde{\xi}$ *is induced from* ξ.

If $\beta \in SL(2k)$ and $f \in SL(k)$ we have also a previously defined Habegger-Lin's action $f \cdot \beta$. Thus we can consider ${}_k\overline{S(1)}_n = \left\{\beta \in SL(2k) \mid \overline{1\cdot\beta}_n \text{ is the identity automorphism of } \dfrac{F(k)}{F(k)_n}\right\}$. Then ${}_k\overline{S(1)}_n$ contains the stabilizer of 1 for Habegger-Lin's action ${}_kS(1) = \{\beta \in SL(2k) \mid 1\cdot\beta = 1 \in SL(k)\}$. Similarly we can define ${}_k\overline{S(1)} = \{\beta \in SL(2k) \mid \widetilde{1\cdot\beta}$ is the identity automorphism of $\widetilde{F(k)}\} = \bigcap_n {}_k\overline{S(1)}_n$.

THEOREM 6. *Let* $\beta \in SL(2k)$. $\beta \in \overline{S_k(1)}_n \cap {}_k\overline{S(1)}_n$ *if and only if there exists an automorphism* $\bar{\bar{\beta}}_n : \frac{F(k)}{F(k)_n} \to \frac{F(k)}{F(k)_n}$ *such that diagram (*) is commutative.*

Proof. If $\beta \in \overline{S_k(1)}_n \cap {}_k\overline{S(1)}_n$, then $\beta \in \overline{S_k(1)}_n$ and $\beta^R \in \overline{S_k(1)}_n$, so we have a commutative diagram

$$\begin{array}{ccccc}
\frac{F(2k)}{F(2k)_n} & \xrightarrow{\bar{\beta}_n} & \frac{F(2k)}{F(2k)_n} & \xrightarrow{\bar{\beta}_n^{-1}=\overline{\beta_n^R}} & \frac{F(2k)}{F(2k)_n} \\
\downarrow{\scriptstyle \xi_n} & & \downarrow{\scriptstyle \xi_n} & & \downarrow{\scriptstyle \xi_n} \\
\frac{F(k)}{F(k)_n} & \xrightarrow{\bar{\bar{\beta}}_n} & \frac{F(k)}{F(k)_n} & \xrightarrow{\overline{\overline{\beta_n^R}}} & \frac{F(k)}{F(k)_n}
\end{array}.$$

It follows that $\bar{\bar{\beta}}_n\ \overline{\overline{\beta_n^R}}$ is the identity map. Similarly $\overline{\overline{\beta_n^R}}\ \bar{\bar{\beta}}_n$ is the identity map.

Conversely suppose we have a commutative diagram (*) with $\overline{\overline{\beta_n}}$ an automorphism. Identifying $\frac{F(k)}{F(k)_n}$ with $\frac{\frac{F(2k)}{F(2k)_n}}{\langle \alpha_i \tilde{\alpha}_i F(2k)_n \rangle^N}$ as before, $\overline{\overline{\beta_n}}$ induces an automorphism

$$\bar{\bar{\beta}}'_n : \frac{\frac{F(2k)}{F(2k)_n}}{\langle \alpha_i \tilde{\alpha}_i F(2k)_n \rangle^N} \to \frac{\frac{F(2k)}{F(2k)_n}}{\langle \alpha_i \tilde{\alpha}_i F(2k)_n \rangle^N}$$

and we have a commutative diagram

$$\begin{array}{ccccc}
\frac{F(2k)}{F(2k)_n} & \xrightarrow{\mu_0(\beta)_n} & \frac{\pi(\beta)}{\pi(\beta)_n} & \xrightarrow{(\mu_1(\beta)_n)^{-1}} & \frac{F(2k)}{F(2k)_n} \\
\downarrow & & \downarrow & & \downarrow \\
\frac{\frac{F(2k)}{F(2k)_n}}{\langle \alpha_i \tilde{\alpha}_i F(2k)_n \rangle^N} & \xrightarrow{\bar{\bar{\beta}}'_n \Gamma_1(\beta)_n} & \frac{\frac{\pi(\beta)}{\pi(\beta)_n}}{\langle y_i \tilde{y}_i \pi(\beta)_n \rangle^N} & \xrightarrow{(\Gamma_1(\beta)_n)^{-1}} & \frac{\frac{F(2k)}{F(2k)_n}}{\langle \alpha_i \tilde{\alpha}_i F(2k)_n \rangle^N}
\end{array}$$

where the horizontal maps are isomorphisms and the vertical maps are quotient maps. Since $\mu_0(\beta)_n$ sends $\alpha_i \widetilde{\alpha}_i F(2k)_n$ to $x_i \widetilde{x}_i \pi(\beta)_n$, it follows that, in

$$\frac{\frac{\pi(\beta)}{\pi(\beta)_n}}{\langle y_i \tilde{y}_i \pi(\beta)_n \rangle^N} \cong \frac{\pi(\beta \cdot 1)}{\pi(\beta \cdot 1)_n},$$

the class of x_i is equals to the class of $\widetilde{x}_i^{\,-1}$. But the classes of x_i, $i \in \underline{k}$, are the top meridians of $\beta \cdot 1$ and the classes of $\widetilde{x}_i^{\,-1}$, $i \in \underline{k}$, are the bottom meridians of $\beta \cdot 1$, so $\overline{\beta \cdot 1}_n = \mu_0(\beta \cdot 1)_n (\mu_1(\beta \cdot 1)_n)^{-1}$ is the identity map. Therefore

$\beta \in \overline{S_k(1)}_n$. The same argument with β replaced by β^R and $\bar{\bar{\beta}}_n$ replaced by $\bar{\bar{\beta}}_n^{-1}$ shows that $\beta^R \in \overline{S_k(1)}_n$. □

COROLLARY 7. *$\beta \in \overline{S_k(1)} \cap {}_k\overline{S(1)}$ if and only if there exists an automorphism $\tilde{\bar{\bar{\beta}}} : \widetilde{F(k)} \to \widetilde{F(k)}$ such that diagram (**) is commutative.*

Proof. If $\beta \in \overline{S_k(1)}$ and $\beta^R \in \overline{S_k(1)}$, then they are in all $\overline{S_k(1)}_n$ and we have automorphisms $\overline{\overline{\beta_n}}$ that together provide automorphism $\tilde{\bar{\bar{\beta}}}$.

Conversely suppose we have automorphism $\tilde{\bar{\bar{\beta}}}$ making diagram (**) commutative.

The natural map $\dfrac{F}{F_n} \to \dfrac{\tilde{F}}{\tilde{F}_n}$, where $F = F(k)$ or $F(2k)$, is an isomorphism (see [9]). Therefore we have commutative diagrams (*) for every n, where $\bar{\bar{\beta}}_n$ is an automorphism. Then $\beta, \beta^R \in \overline{S_k(1)}_n$, for every n. Thus $\beta, \beta^R \in \overline{S_k(1)}$. □

THEOREM 8. *The intersection $\overline{S_k(1)}_n \cap {}_k\overline{S(1)}_n$ is closed in relation to the multiplication.*

Proof. By Theorem 6, $\beta \in \overline{S_k(1)}_n \cap {}_k\overline{S(1)}_n$ if and only if there exists automorphism $\bar{\bar{\beta}}_n$ commuting diagram (*). Thus if $\beta, \gamma \in \overline{S_k(1)}_n \cap {}_k\overline{S(1)}_n$, then there are automorphisms $\bar{\bar{\beta}}_n$ and $\bar{\bar{\gamma}}_n$ making diagrams below commutatives.

$$
\begin{array}{ccc}
\dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\bar{\beta}_n} & \dfrac{F(2k)}{F(2k)_n} \\
{\scriptstyle \xi_n}\big\downarrow & & \big\downarrow{\scriptstyle \xi_n} \\
\dfrac{F(k)}{F(k)_n} & \xrightarrow{\bar{\bar{\beta}}_n} & \dfrac{F(k)}{F(k)_n}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\bar{\gamma}_n} & \dfrac{F(2k)}{F(2k)_n} \\
{\scriptstyle \xi_n}\big\downarrow & & \big\downarrow{\scriptstyle \xi_n} \\
\dfrac{F(k)}{F(k)_n} & \xrightarrow{\bar{\bar{\gamma}}_n} & \dfrac{F(k)}{F(k)_n}
\end{array}
$$

Therefore the automorphism $\bar{\bar{\beta}}_n\bar{\bar{\gamma}}_n$ makes the diagram

$$
\begin{array}{ccc}
\dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\overline{(\beta\gamma)}_n} & \dfrac{F(2k)}{F(2k)_n} \\
{\scriptstyle \xi_n}\big\downarrow & & \big\downarrow{\scriptstyle \xi_n} \\
\dfrac{F(k)}{F(k)_n} & \xrightarrow{\bar{\bar{\beta}}_n\bar{\bar{\gamma}}_n} & \dfrac{F(k)}{F(k)_n}
\end{array}
$$

commutative. Therefore, by Theorem 6, $\beta\gamma \in \overline{S_k(1)}_n \cap {}_k\overline{S(1)}_n$. □

Similarly, using Corollary 7, we see that

COROLLARY 9. *$\overline{S_k(1)} \cap {}_k\overline{S(1)}$ is closed in relation to the multiplication.*

3. Concordance

Definition 4. k-string links $f, g : \underline{k} \times I \to D \times I$ are said to be *concordant* if there is an embedding $F : \underline{k} \times I \times I \to D \times I \times I$ such that $F|_{\underline{k} \times I \times \{0\}} = f$, $F|_{\underline{k} \times I \times \{1\}} = g$ and $F|_{\underline{k} \times \partial I \times I} = (j_0|_{\underline{k} \times \partial I}) \times id_I$ where $id_I : I \to I$ is the identity map.

The set of concordance classes of k-string links with the operation induced by the multiplication of k-string links is a group where the inverse of the class of f is the class of f^R (see [4]). This group will be denoted by $CSL(k)$.

Habegger-Lin's actions induce actions $CSL(2k) \times CSL(k) \to CSL(k)$ and $CSL(k) \times CSL(2k) \to CSL(k)$ (see [6]). The stabilizer of 1 for both of these actions is the same (see [6]) and will be denoted by $S_k^C(1) = \{\beta \in CSL(2k) \mid \beta \cdot 1 = 1 \in CSL(k)\} = \{\beta \in CSL(2k) \mid 1 \cdot \beta = 1 \in CSL(k)\}$.

Definition 5. Consider k-component links L_0 and L_1. A (link) concordance between L_0 and L_1 is an embedding:

$$H : \left(\bigsqcup_{i=1}^{k} S^1 \right) \times I \to S^3 \times I$$

such that $H(x, 0) = (L_0(x), 0)$ and $H(x, 1) = (L_1(x), 1)$.

If k-string links f and g are concordant then their closures $\hat{f}$ and $\hat{g}$ are (link) concordant. Actually, we have

Theorem 10 (Habegger-Lin). *Suppose $f, g \in CSL(k)$, then $\hat{f} = \hat{g}$ (that is, the closures of their representatives are link concordant) if and only if there exists $\beta \in S_k^C(1)$ such that $\beta \cdot f = g$.*

Proof. See [6]. □

If f and g are concordant k-string links, then $\bar{f}_n = A_n(f) = A_n(g) = \bar{g}_n$ (see [6]). Considering the nilpotent completion we also have $\tilde{f} = A(f) = A(g) = \tilde{g}$. Furthermore the $A_n(f)$ are braid-like automorphisms of $\dfrac{F(k)}{F(k)_n}$, that is (i) they send the class of each generator α_i into a conjugate of itself and (ii) they send the class of the product $\alpha_1 \alpha_2 \cdots \alpha_n$ into itself. Thus, if we denote the group of braid-like automorphisms of $\dfrac{F(k)}{F(k)_n}$ by $Aut_0\left(\dfrac{F(k)}{F(k)_n}\right)$, we have a homomorphism $A_n : CSL(k) \to Aut_0\left(\dfrac{F(k)}{F(k)_n}\right)$ called Artin representation. This homomorphism is actually an epimorphism (see [6]).

An automorphism that satisfies (i) above is called special.

Let $\overline{S_k^C(1)_n} = \left\{\beta \in CSL(2k) \mid \overline{\beta \cdot 1}_n \text{ is the identity automorphism of } \dfrac{F(k)}{F(k)_n}\right\}$ $= \{\beta \in CSL(2k) \mid \beta \cdot 1 \in \ker A_n\}$, ${}_k\overline{S^C(1)_n} = \{\beta \in CSL(2k) \mid 1 \cdot \beta \in \ker A_n\}$, $\overline{S_k^C(1)}$

$= \{\beta \in CSL(2k) \mid \beta \cdot 1 \in \ker A\}$ and $\overline{{}_kS^C(1)} = \{\beta \in CSL(2k) \mid 1 \cdot \beta \in \ker A\}$. Let $S_n(1) = \overline{S_k^C(1)_n} \cap \overline{{}_kS^C(1)_n}$ and $S(1) = \overline{S_k^C(1)} \cap \overline{{}_kS^C(1)}$.

Theorem 11. *$S_n(1)$ and $S(1)$ are groups.*

Proof. From Theorem 8 and Corollary 9 they are closed for multiplication. On the other side, if $\beta \in \overline{S_k^C(1)_n}$, then $\overline{\beta \cdot 1}_n = id$, the identity map. Then $id = (\overline{\beta \cdot 1}_n)^{-1} = \overline{(\beta \cdot 1)^{-1}}_n = \overline{(1 \cdot \beta^{-1})}_n$, so $\beta^{-1} \in \overline{{}_kS^C(1)_n}$. Therefore $S_n(1)$ is a group. Similarly $S(1)$ is a group. □

Clearly $S_n(1)$ and $S(1)$ contain $S_k^C(1)$, the stabilizer of 1 for Habegger-Lin's actions.

Theorem 6 and Corollary 7 provide the following results:

Theorem 12. *Let $\beta \in CSL(2k)$. $\beta \in S_n(1)$ if and only if there is a special automorphism $\bar{\bar{\beta}}_n : \dfrac{F(k)}{F(k)_n} \to \dfrac{F(k)}{F(k)_n}$ such that diagram (*) is commutative.*

Corollary 13. *$\beta \in S(1)$ if and only if there is a special automorphism $\tilde{\tilde{\beta}} : \widetilde{F(k)} \to \widetilde{F(k)}$ such that diagram (**) is commutative.*

In particular, if $\beta \in S_k^C(1)$, the stabilizer of 1 for Habegger-Lin's action, we have special automorphisms $\bar{\bar{\beta}}_n$ and $\tilde{\tilde{\beta}}$ as above.

Let $g \times 1$ represent the $2k$-string link obtained from a k-string link g by adding k straight strings at its end (see Fig. 4).

Lemma 14. *Let $f, g \in CSL(k)$ and $\gamma \in CSL(2k)$. $\gamma \cdot f = g$ if and only if γ is of the form $g \times 1\beta f^{-1} \times 1$, where $\beta \in S_k^C(1)$.*

Proof. See [1]—Lemma 4. □

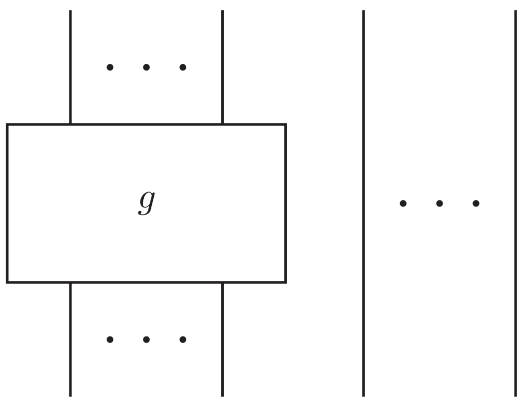


Figure 4. The string link $g \times 1$.

THEOREM 15. *If the closures of k-string links f and g are link concordant then, for every n, there exists a commutative diagram*

$$
\begin{array}{ccc}
\dfrac{F(k)}{F(k)_n} & \xrightarrow{\bar{\bar{\gamma}}_n} & \dfrac{F(k)}{F(k)_n} \\
\Big\uparrow{\scriptstyle \xi_n} & & \Big\uparrow{\scriptstyle \xi_n} \\
\dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\bar{\gamma}_n} & \dfrac{F(2k)}{F(2k)_n} \\
\Big\downarrow{\scriptstyle \overline{g\times 1}_n} & & \Big\downarrow{\scriptstyle \overline{f\times 1}_n} \\
\dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\bar{\beta}_n} & \dfrac{F(2k)}{F(2k)_n} \\
\Big\downarrow{\scriptstyle \xi_n} & & \Big\downarrow{\scriptstyle \xi_n} \\
\dfrac{F(k)}{F(k)_n} & \xrightarrow{\bar{\bar{\beta}}_n} & \dfrac{F(k)}{F(k)_n}
\end{array}
$$

where $\bar{\beta}_n$ *and* $\bar{\gamma}_n$ *are braid-like automorphisms and* $\bar{\bar{\beta}}_n$, $\bar{\bar{\gamma}}_n$ *are special automorphisms.*

Proof. If $\hat{f}$ and $\hat{g}$ are link concordant, then, by Habegger-Lin's Theorem (Theorem 10), there exists $\gamma \in S_k^C(1)$ such that $\gamma \cdot f = g$ (here we are using f and g to represent also the concordance classes of f and g).

By Lemma 14, $\gamma = g \times 1 \beta f^{-1} \times 1$, where $\beta \in S_k^C(1)$.

By Theorem 12, we have a commutative diagram as stated. □

COROLLARY 16. *If the closures of k-string links f and g are link concordant, then there is a commutative diagram*

$$
\begin{array}{ccc}
\widetilde{F(k)} & \xrightarrow{\tilde{\tilde{\gamma}}} & \widetilde{F(k)} \\
\Big\uparrow{\scriptstyle \tilde{\xi}} & & \Big\uparrow{\scriptstyle \tilde{\xi}} \\
\widetilde{F(2k)} & \xrightarrow{\tilde{\gamma}} & \widetilde{F(2k)} \\
\Big\downarrow{\scriptstyle \widetilde{g\times 1}} & & \Big\downarrow{\scriptstyle \widetilde{f\times 1}} \\
\widetilde{F(2k)} & \xrightarrow{\tilde{\beta}} & \widetilde{F(2k)} \\
\Big\downarrow{\scriptstyle \tilde{\xi}} & & \Big\downarrow{\scriptstyle \tilde{\xi}} \\
\widetilde{F(k)} & \xrightarrow{\tilde{\tilde{\beta}}} & \widetilde{F(k)}
\end{array}
$$

where $\tilde{\beta}$, $\tilde{\gamma}$ *are braid-like automorphisms and* $\tilde{\tilde{\beta}}$, $\tilde{\tilde{\gamma}}$ *are special automorphisms.*

Let $K_n = \ker \xi_n$, that is $K_n = \langle \alpha_i \widetilde{\alpha}_i F(2k)_n \mid i \in \underline{k} \rangle^N$, the normal subgroup of $\dfrac{F(2k)}{F(2k)_n}$ generated by $\alpha_i \widetilde{\alpha}_i F(2k)_n$, $i \in \underline{k}$, and let $K = \ker \tilde{\xi}$.

Proposition 17. *If the closures of k-string links f and g are concordant, then there is $\beta \in S_k^C(1)$ s.t., for all $n \geq 2$, $(K_n)\overline{\beta_n} \subseteq K_n$ and $((K_n)\overline{g \times 1_n})\overline{\beta_n} \subseteq (K_n)\overline{f \times 1_n}$.*

Proof. According to the proof of Theorem 15, if $\hat{f}$ and $\hat{g}$ are concordant, then there exists $\beta \in S_k^C(1)$ such that $\gamma = g \times 1 \beta f^{-1} \times 1 \in S_k^C(1)$. By Theorem 12 and the fact that $S_k^C(1) \subseteq S_n(1)$, we see that $\beta \in S_k^C(1)$ implies $(K_n)\bar{\beta}_n \subseteq K_n$ and $\gamma \in S_k^C(1)$ implies $(K_n)\overline{\gamma_n} \subseteq K_n$, so $(K_n)\overline{g \times 1_n}\,\bar{\beta}_n\,\overline{f^{-1} \times 1_n} \subseteq K_n$. Then $((K_n)\overline{g \times 1_n})\bar{\beta}_n \subseteq (K_n)\overline{f \times 1_n}$. □

Corollary 18. *If the closures of k-string links f and g are concordant, then there is $\beta \in S_k^C(1)$ s.t. $(\varprojlim_n K_n)\tilde{\beta} \subseteq \varprojlim_n K_n$ and $((\varprojlim_n K_n)\widetilde{g \times 1})\tilde{\beta} \subseteq (\varprojlim_n K_n)\widetilde{f \times 1}$.*

Corollary 19. *If the closures of k-string links f and g are concordant, then there is $\beta \in S_k^C(1)$ s.t. $(K_n)\bar{\beta}_n = K_n$ and $((K_n)\overline{g \times 1_n})\bar{\beta}_n = (K_n)\overline{f \times 1_n}$.*

Proof. It follows from Proposition 17 using β^{-1}. □

Corollary 20. *If the closures of k-string links f and g are concordant, then there is $\beta \in S_k^C(1)$ s.t. $(\varprojlim K_n)\tilde{\beta} = \varprojlim K_n$ and $((\varprojlim K_n)\widetilde{g \times 1})\tilde{\beta} = (\varprojlim K_n)\widetilde{f \times 1}$.*

Definition 6. If L is a link, the fundamental group of the complement of L is called the *group of L* and is denoted by $G(L)$.

By [12], the natural homomorphism $p_f : \pi(f) \to G(\hat{f})$ is onto with kernel normally generated by the commutators $[x_i(f), \lambda_i(f)]$, for any $i \in \underline{k}$, where $x_i(f)$ are the top meridians of f and $\lambda_i(f)$ are correspondent longitudes of f. Thus we have, for every $n \geqslant 2$, an induced epimorphism $(p_f)_n : \dfrac{\pi(f)}{\pi(f)_n} \to \dfrac{G(\hat{f})}{G(\hat{f})_n}$. Let $(q_f)_n = (\mu_0(f)_n)(p_f)_n : \dfrac{F(k)}{F(k)_n} \to \dfrac{G(\hat{f})}{G(\hat{f})_n}$. Then $(q_f)_n$ is an epimorphism that sends $\alpha_i F(k)_n$ to $u_i G(\hat{f})_n$, where $u_i \in G(\hat{f})$ is a meridian for the link $\hat{f}$ arising from $x_i(f)$.

Considering the commutative diagram

$$\begin{array}{ccc} \pi(f) & \xrightarrow{p_f} & G(\hat{f}) \\ \big\downarrow & & \big\downarrow \\ \dfrac{\pi(f)}{\pi(f)_n} & \xrightarrow[(p_f)_n]{} & \dfrac{G(\hat{f})}{G(\hat{f})_n} \end{array}$$

where the vertical maps are quocient maps, and the fact that p_f is onto, we see that $\ker((p_f)_n) = \frac{(G(\hat{f})_n)(p_f)^{-1}}{\pi(f)_n} = \frac{((\pi(f)_n)p_f)(p_f)^{-1}}{\pi(f)_n} = \frac{(\ker p_f)\pi(f)_n}{\pi(f)_n}$ is the normal subgroup of $\frac{\pi(f)}{\pi(f)_n}$ generated by the commutators of the form $[x_i(f)\pi(f)_n, \lambda_i(f)\pi(f)_n]$, for $i \in \underline{k}$.

Since $\bar{f}_n$ conjugates the classes of the meridians $\alpha_i F(k)_n$ by the classes of the correspondent longitudes (see [6]), and remembering that $\mu_0(f)_n$ is an isomorphism, we have that $\ker(q_f)_n = \langle (\alpha_i F(k)_n)\bar{f}_n \alpha_i^{-1} F(k)_n \mid i \in \underline{k} \rangle^N$. It follows that $\ker(\xi_n(q_f)_n) = \langle (\alpha_i F(2k)_n)\overline{f \times 1}_n \alpha_i^{-1} F(2k)_n, \alpha_i \tilde{\alpha}_i F(2k)_n \mid i \in \underline{k} \rangle^N$.

Let f be a k-string link. For each $n \geqslant 2$, consider the following diagram:

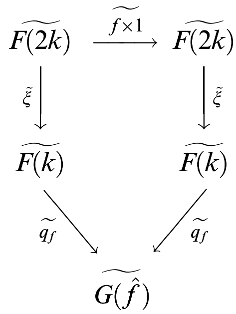
$$\begin{array}{ccccc}
\frac{F(2k)}{F(2k)_n} & & \xrightarrow{\overline{f\times 1}_n} & & \frac{F(2k)}{F(2k)_n} \\
\downarrow{\scriptstyle \xi_n} & & & & \downarrow{\scriptstyle \xi_n} \\
\frac{F(k)}{F(k)_n} & & & & \frac{F(k)}{F(k)_n} \\
{\scriptstyle (\mu_0(f))_n}\downarrow\cong & & & & \cong\downarrow{\scriptstyle (\mu_0(f))_n} \\
\frac{\pi(f)}{\pi(f)_n} & & & & \frac{\pi(f)}{\pi(f)_n} \\
& {\scriptstyle (p_f)_n}\searrow & & \swarrow{\scriptstyle (p_f)_n} & \\
& & \frac{G(\hat{f})}{G(\hat{f})_n} & &
\end{array}$$

It follows from [12]-Proposition 1 that the above diagram is commutative. We have already denoted $(\mu_0(f))_n(p_f)_n$ by $(q_f)_n$.

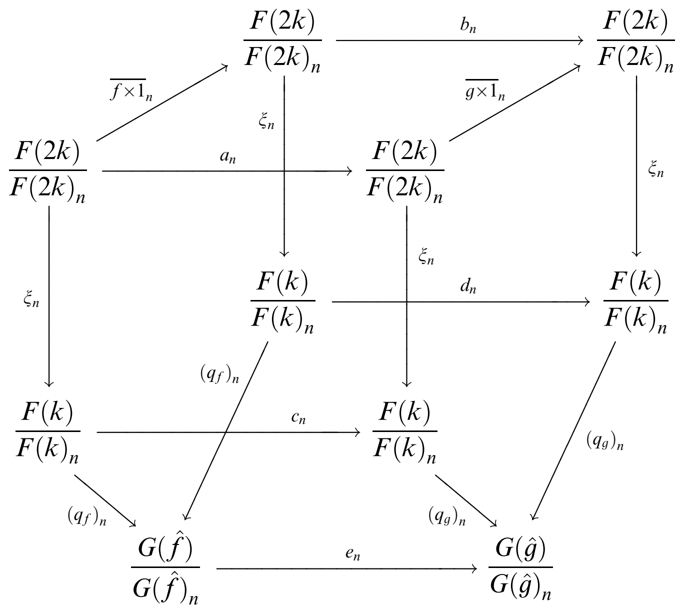
Definition 7. Given a k-string link f, the *n-level group diagram* for f, $n \geqslant 2$, is the commutative diagram:

$$\begin{array}{ccccc}
\frac{F(2k)}{F(2k)_n} & & \xrightarrow{\overline{f\times 1}_n} & & \frac{F(2k)}{F(2k)_n} \\
\downarrow{\scriptstyle \xi_n} & & & & \downarrow{\scriptstyle \xi_n} \\
\frac{F(k)}{F(k)_n} & & & & \frac{F(k)}{F(k)_n} \\
& {\scriptstyle (q_f)_n}\searrow & & \swarrow{\scriptstyle (q_f)_n} & \\
& & \frac{G(\hat{f})}{G(\hat{f})_n} & &
\end{array}$$

Given n-level group diagrams for all n and taking inverse limits, we have the *group diagram* for f:

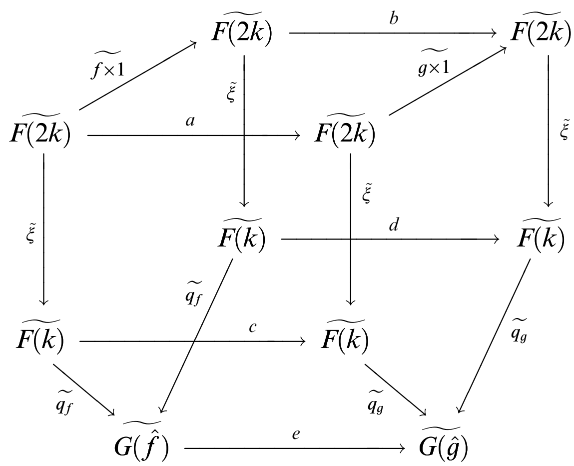


DEFINITION 8. Given n-level group diagrams for k-string links f and g, an n-level *braid-special isomorphism* between them is a commutative diagram:



where a_n, b_n are braid-like isomorphisms and c_n, d_n, e_n are special isomorphisms.

DEFINITION 9. Given group diagrams for k-string links f and g, a *braid-special isomorphism* between them is a commutative diagram:



where a, b are braid-like isomorphisms and c, d, e are special isomorphisms.

THEOREM 21. *Let L_1, L_2 be k-links and let f, g be k-string links with $\hat{f} = L_1$ and $\hat{g} = L_2$. If L_1 and L_2 are concordant, then, for each $n \geqslant 2$, there exists an n-level braid-special isomorphism between the n-level group diagram of f and the n-level group diagram of g.*

Proof. Since $\hat{f}$ and $\hat{g}$ are concordant, by Theorem 15, there exists, for each $n \geqslant 2$, a commutative diagram

$$\begin{array}{ccc}
\dfrac{F(k)}{F(k)_n} & \xrightarrow{\bar{\bar{\gamma}}_n} & \dfrac{F(k)}{F(k)_n} \\
{\scriptstyle \xi_n}\big\uparrow & & \big\uparrow{\scriptstyle \xi_n} \\
\dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\bar{\gamma}_n} & \dfrac{F(2k)}{F(2k)_n} \\
{\scriptstyle \overline{g\times 1}_n}\big\downarrow & & \big\downarrow{\scriptstyle \overline{f\times 1}_n} \\
\dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\bar{\beta}_n} & \dfrac{F(2k)}{F(2k)_n} \\
{\scriptstyle \xi_n}\big\downarrow & & \big\downarrow{\scriptstyle \xi_n} \\
\dfrac{F(k)}{F(k)_n} & \xrightarrow{\bar{\bar{\beta}}_n} & \dfrac{F(k)}{F(k)_n}
\end{array}$$

where $\bar{\beta}_n$ and $\bar{\gamma}_n$ are braid-like automorphisms and $\bar{\bar{\beta}}_n$, $\bar{\bar{\gamma}}_n$ are special automorphisms.

It follows from Corollary 19 and the fact that, as we saw,

$$\begin{aligned}\ker(\xi_n(q_f)_n) &= \langle (\alpha_i F(2k)_n)\overline{f\times 1}_n \alpha_i^{-1} F(2k)_n, \alpha_i\widetilde{\alpha}_i F(2k)_n \mid i\in \underline{k}\rangle^N \\ &= \langle (\alpha_i F(2k)_n)\overline{f\times 1}_n \tilde{\alpha}_i F(2k)_n, \alpha_i\tilde{\alpha}_i F(2k)_n \mid i\in \underline{k}\rangle^N \\ &= \langle (\alpha_i F(2k)_n)\overline{f\times 1}_n (\tilde{\alpha}_i F(2k)_n)\overline{f\times 1}_n, \alpha_i\tilde{\alpha}_i F(2k)_n \mid i\in \underline{k}\rangle^N \\ &= \langle (\alpha_i\tilde{\alpha}_i F(2k)_n)\overline{f\times 1}_n, \alpha_i\tilde{\alpha}_i F(2k)_n \mid i\in \underline{k}\rangle^N,\end{aligned}$$

that $(\ker(\xi_n(q_g)_n))\overline{\beta_n} = \ker(\xi_n(q_f)_n)$. Similarly $(\ker(\xi_n(q_{g^{-1}})_n))\overline{\gamma_n} = \ker(\xi_n(q_{f^{-1}})_n)$, but $\ker(q_{f^{-1}})_n = \ker(q_f)_n$ and $\ker(q_{g^{-1}})_n = \ker(q_g)_n$. Therefore there are special isomorphisms $\overline{\overline{\overline{\beta}}}_n, \overline{\overline{\overline{\gamma}}}_n : \dfrac{G(\hat{g})}{G(\hat{g})_n} \to \dfrac{G(\hat{f})}{G(\hat{f})_n}$ making the following diagram commutative

$$(1)\qquad \begin{array}{ccc}
\dfrac{G(\hat{g})}{G(\hat{g})_n} & \xrightarrow{\overline{\overline{\overline{\gamma}}}_n} & \dfrac{G(\hat{f})}{G(\hat{f})_n} \\
{\scriptstyle (q_g)_n}\uparrow & & \uparrow{\scriptstyle (q_f)_n} \\
\dfrac{F(k)}{F(k)_n} & \xrightarrow{\overline{\overline{\gamma}}_n} & \dfrac{F(k)}{F(k)_n} \\
{\scriptstyle \xi_n}\uparrow & & \uparrow{\scriptstyle \xi_n} \\
\dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\overline{\gamma}_n} & \dfrac{F(2k)}{F(2k)_n} \\
{\scriptstyle \overline{g\times 1}_n}\downarrow & & \downarrow{\scriptstyle \overline{f\times 1}_n} \\
\dfrac{F(2k)}{F(2k)_n} & \xrightarrow{\overline{\beta}_n} & \dfrac{F(2k)}{F(2k)_n} \\
{\scriptstyle \xi_n}\downarrow & & \downarrow{\scriptstyle \xi_n} \\
\dfrac{F(k)}{F(k)_n} & \xrightarrow{\overline{\overline{\beta}}_n} & \dfrac{F(k)}{F(k)_n} \\
{\scriptstyle (q_g)_n}\downarrow & & \downarrow{\scriptstyle (q_f)_n} \\
\dfrac{G(\hat{g})}{G(\hat{g})_n} & \xrightarrow{\overline{\overline{\overline{\beta}}}_n} & \dfrac{G(\hat{f})}{G(\hat{f})_n}
\end{array}$$

Now it is enough to show that $\overline{\overline{\overline{\beta}}}_n = \overline{\overline{\overline{\gamma}}}_n$.

For each $i\in \underline{k}$, let $\mu_i = (\alpha_i F(k)_n)(q_g)_n$. Then the μ_i, $i\in\underline{k}$, generate $\dfrac{G(\hat{g})}{G(\hat{g})_n}$. From diagram (1) we have

$$(\mu_i)\overline{\overline{\overline{\gamma}}}_n = (\alpha_i F(2k)_n)\xi_n(q_g)_n\overline{\overline{\overline{\gamma}}}_n = (\alpha_i F(2k)_n)\overline{\gamma}_n\xi_n(q_f)_n.$$

On the other side,

$$(\alpha_i F(k)_n)(q_g)_n = ((\alpha_i F(k)_n)\bar{g}_n)(q_g)_n,$$

so

$$\begin{aligned}(\mu_i)\bar{\bar{\bar{\beta}}}_n &= [(\alpha_i F(k)_n)(q_g)_n]\bar{\bar{\bar{\beta}}}_n = [(\alpha_i F(2k)_n)\xi_n(q_g)_n]\bar{\bar{\bar{\beta}}}_n \\ &= [(\alpha_i F(2k)_n)\overline{g\times 1}_n \xi_n(q_g)_n]\bar{\bar{\bar{\beta}}}_n = (\alpha_i F(2k)_n)\bar{\gamma}_n \overline{f\times 1}_n \xi_n(q_f)_n.\end{aligned}$$

To show that $(\mu_i)\bar{\bar{\bar{\gamma}}}_n = (\mu_i)\bar{\bar{\bar{\beta}}}_n$ it is enough to show that for the generators $w = \alpha_i F(2k)_n$ and $w = \tilde{\alpha}_i F(2k)_n$ we have $((w)\overline{f\times 1}_n w^{-1}) \in \ker \xi_n(q_f)_n$, and then to take $w = (\alpha_i F(2k)_n)\bar{\gamma}_n$.

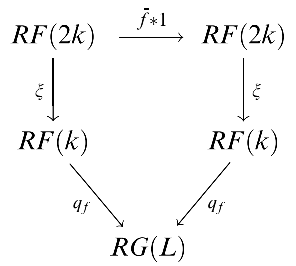
We have seen that $\ker(\xi_n(q_f)_n) = \langle (\alpha_i F(2k)_n)\overline{f\times 1}_n \alpha_i^{-1}F(2k)_n, \alpha_i\tilde{\alpha}_i F(2k)_n \mid i \in k\rangle^N$. If $w = \alpha_i F(2k)_n$, then $(w)\overline{f\times 1}_n w^{-1} = (\alpha_i F(2k)_n)\overline{f\times 1}_n \alpha_i^{-1}F(2k)_n \in \ker(\xi_n(q_f)_n)$. If $w = \tilde{\alpha}_i F(2k)_n$, then $(w)\overline{f\times 1}_n w^{-1} = (\tilde{\alpha}_i F(2k)_n)\overline{f\times 1}_n \tilde{\alpha}_i^{-1}F(2k)_n$, but $(\tilde{\alpha}_i F(2k)_n)\overline{f\times 1}_n = \tilde{\alpha}_i F(2k)_n$ since the last k longitudes of $f\times 1$ are trivial and $\overline{f\times 1}_n$ conjugates the classes of the meridians by the correspondent longitudes (see [6]). Thus $(\tilde{\alpha}_i F(2k)_n)\overline{f\times 1}_n \tilde{\alpha}_i^{-1}F(2k)_n = 1F(2k)n \in \ker(\xi_n(q_f)_n)$.

Therefore $\bar{\bar{\bar{\gamma}}}_n = \bar{\bar{\bar{\beta}}}_n$ and we have a n-level braid-special isomorphism as stated. □

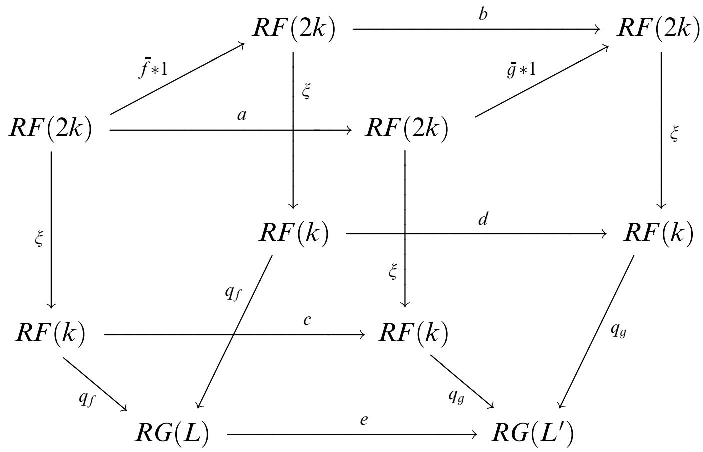
COROLLARY 22. *Let f and g be k-string links. If $\hat{f}$ and $\hat{g}$ are concordant, then there exists a braid-special isomorphism between the group diagram of f and the group diagram of g.*

4. Correction

We will take the opportunity to correct a mistake. In our paper [1] one should replace the diagram in the definition of a *group diagram for the homotopy class* [L] (Definition 12 there) by the commutative diagram



and the diagram in the definition of a *braid-special isomorphism* (Definition 13 there) by the commutative diagram:



where a, b are braid-like isomorphisms and c, d, e are special isomorphisms.

After making these changes we will have the converse in Theorem 8 there.

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