A. S. JANFADA AND S. SALAMI
KODAI MATH. J.
38 (2015), 352–364

ON θ-CONGRUENT NUMBERS ON REAL QUADRATIC NUMBER FIELDS

Ali S. Janfada and Sajad Salami

Abstract

Let $\mathbf{K} = \mathbf{Q}(\sqrt{m})$ be a real quadratic number field, where $m > 1$ is a squarefree integer. Suppose that $0 < \theta < \pi$ has rational cosine, say $\cos(\theta) = s/r$ with $0 < |s| < r$ and $\gcd(r, s) = 1$. A positive integer n is called a $(\mathbf{K}, \theta)$-congruent number if there is a triangle, called the $(\mathbf{K}, \theta, n)$-triangles, with sides in $\mathbf{K}$ having θ as an angle and $n\alpha_\theta$ as area, where $\alpha_\theta = \sqrt{r^2 - s^2}$. Consider the $(\mathbf{K}, \theta)$-congruent number elliptic curve $E_{n,\theta} : y^2 = x(x + (r + s)n)(x - (r - s)n)$ defined over $\mathbf{K}$. Denote the squarefree part of positive integer t by $\mathrm{sqf}(t)$. In this work, it is proved that if $m \neq \mathrm{sqf}(2r(r - s))$ and $mn \neq 2, 3, 6$, then n is a $(\mathbf{K}, \theta)$-congruent number if and only if the Mordell-Weil group $E_{n,\theta}(\mathbf{K})$ has positive rank, and all of the $(\mathbf{K}, \theta, n)$-triangles are classified in four types.

1. Introduction

A positive integer n is called a *congruent number* if it is the area of a right triangle with rational sides. Finding all congruent numbers is one of the classical problems in the modern number theory. We cite [8] for an exposition of the congruent number problem, and [4] to see the first study of θ-congruent numbers as a generalization of the classic one. Let $0 < \theta < \pi$ has rational cosine $\cos(\theta) = s/r$ with $0 < |s| < r$ and $\gcd(r, s) = 1$. Let $(U, V, W)_\theta$ denote a triangle with an angle θ between sides U and V. A positive integer n is called a *θ-congruent number* if there exists a triangle $(U, V, W)_\theta$ with sides in $\mathbf{Q}$ having area $n\alpha_\theta$, where $\alpha_\theta = \sqrt{r^2 - s^2}$. In other words, n is a θ-congruent number if it satisfies

$$2rn = UV, \quad W^2 = U^2 + V^2 - \frac{2s}{r}UV.$$

An ordinary congruent number is nothing but a $\pi/2$-congruent number. Clearly, if n is a θ-congruent number, then so is nt^2, for any positive integer t. We shall concentrate on squarefree numbers whenever θ-congruent numbers concerned. Let

$$E_{n,\theta} : y^2 = x(x + (r + s)n)(x - (r - s)n)$$

2010 *Mathematics Subject Classification*. Primary 11G05, Secondary 14H52.
Key words and phrases. θ-congruent number, elliptic curve, real quadratic number field.
Received August 15, 2014; revised November 11, 2014.

be the θ-congruent number elliptic curve, where r and s are as above. Theorem 2.4 gives an important connection between θ-congruent numbers and the Mordell-Weil group $E_{n,\theta}(\mathbf{Q})$. For more information and recent results about θ-congruent numbers see [5, 3, 14].

The notion θ-congruent number, which is defined over $\mathbf{Q}$, can be extended in a natural way over real quadratic number fields $\mathbf{K}$. In this case, we refer to n as a $(\mathbf{K},\theta)$-*congruent number* and to the triangle $(U,V,W)_\theta$ as a $(\mathbf{K},\theta,n)$-*triangle*. When n is not a θ-congruent number over $\mathbf{Q}$, a question proposed naturally: *Is n a $(\mathbf{K},\theta)$-congruent number for some real quadratic number field $\mathbf{K}$?* Tada [13] answered this question in the case $\theta=\pi/2$, by studying the structure of the $\mathbf{K}$-rational points on the elliptic curve $E_{n,\pi/2}: y^2=x(x^2-n^2)$. In this paper, we answer the above question for any $0<\theta<\pi$ and classify all $(\mathbf{K},\theta,n)$-triangles. Through the paper we shall consider $\mathbf{K}=\mathbf{Q}(\sqrt{m})$ to be a real quadratic field, where $m>1$ is squarefree. We denote the squarefree part of any positive integer N by $\mathrm{sqf}(N)$. The main results of this paper are the following theorems.

Theorem 1.1. *Let n be a positive squarefree integer with $\gcd(m,n)=1$ such that $mn\neq 2,3,6$ and $m\neq \mathrm{sqf}(2r(r-s))$, where m, r, s are as before. Then n is a $(\mathbf{K},\theta)$-congruent number if and only if $\mathrm{rank}(E_{n,\theta}(\mathbf{K}))>0$. Moreover, n is a $(\mathbf{K},\theta)$-congruent number if and only if either n or mn is a θ-congruent number over $\mathbf{Q}$.*

Theorem 1.1 is an extension of Part (2) of Theorem 2.4 in the following. Note that the non-equality conditions for mn and m in Theorem 1.1 are necessary. For a counterexample, when $n=1$ and $\theta=2\pi/3$, we have $r=2$, $s=-1$, $\alpha_\theta=\sqrt{3}$. Now taking $m=3=\mathrm{sqf}(2r(r-s))$, there is a $(\mathbf{Q}(\sqrt{3}),\theta,1)$-triangle with sides $(2,2,2\sqrt{3})$ and area $\sqrt{3}$ but using Theorem 2.1, $\mathrm{rank}(E_{1,\theta}(\mathbf{Q}(\sqrt{3})))=\mathrm{rank}(E_{1,\theta}(\mathbf{Q}))+\mathrm{rank}(E_{3,\theta}(\mathbf{Q}))=0$.

The following theorem classifies all types of $(\mathbf{K},\theta,n)$-triangles.

Theorem 1.2. *Assume that n is not a θ-congruent number over $\mathbf{Q}$ and let σ be the generator of $\mathrm{Gal}(\mathbf{K}/\mathbf{Q})$. Then any $(\mathbf{K},\theta,n)$-triangle with $(U,V,W)\in(\mathbf{K}^*)^3$ and $(0<U\leq V<W)$ is necessarily one of the following types*:

Type 1. $U\sqrt{m}, V\sqrt{m}, W\sqrt{m}\in\mathbf{Q}$;
Type 2. $U, V, W\sqrt{m}\in\mathbf{Q}$;
Type 3. $U, V\in\mathbf{K}\backslash\mathbf{Q}$ *such that* $\sigma(U)=V$, $W\in\mathbf{Q}$;
Type 4. $U, V\in\mathbf{K}\backslash\mathbf{Q}$ *such that* $\sigma(U)=-V$, $W\in\mathbf{Q}$.

Let $A=\mathrm{sqf}(r^2-s^2)$, $B=\mathrm{sqf}(2r(r-s))$ and $C=\mathrm{sqf}(2r(r+s))$. The following proposition shows when there is no $(\mathbf{K},\theta,n)$-triangle of Types 2, 3 and 4.

Proposition 1.3. *Let p be a prime number and the pair (m,A) (resp. (m,B) and (m,C)) can be written as $(p^\alpha a, p^\beta b)$, where $\alpha,\beta\in\{0,1\}$ and $gcd(p,ab)=1$. Then there is no $(\mathbf{K},\theta,n)$-triangle of Type 2 (resp. Type 3 and Type 4) whenever one of the following conditions hold.*

(1) $\underline{p = 2:}$ $(\alpha, \beta) = (0,0)$ *and* $(a,b) \overset{4}{\equiv} (3,3)$,
$(\alpha, \beta) = (0,1)$ *and* $(a,b) \overset{8}{\equiv} (3,1), (3,5), (7,5), (7,7)$,
$(\alpha, \beta) = (1,0)$ *and* $(a,b) \overset{8}{\equiv} (1,3), (1,5), (3,5), (3,7), (5,3), (5,7), (7,3), (7,7)$,
$(\alpha, \beta) = (1,1)$ *and* $(a,b) \overset{8}{\equiv} (1,3), (1,5), (3,1), (3,3), (5,1), (5,7), (7,5), (7,7)$;

(2) $\underline{p \overset{4}{\equiv} 1:}$ $(\alpha, \beta) = (0,1)$ *and* $\left(\frac{a}{p}\right) = -1$, $(\alpha, \beta) = (1,0)$ *and* $\left(\frac{b}{p}\right) = -1$,
$(\alpha, \beta) = (1,1)$ *and* $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = -1$;

(3) $\underline{p \overset{4}{\equiv} 3:}$ $(\alpha, \beta) = (0,1)$ *and* $\left(\frac{a}{p}\right) = -1$, $(\alpha, \beta) = (1,0)$ *and* $\left(\frac{b}{p}\right) = -1$,
$(\alpha, \beta) = (1,1)$ *and* $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = 1$.

The next result settles a condition on n and mn to be θ-congruent over $\mathbf{Q}$.

THEOREM 1.4. *Let n be a positive squarefree integer such that* $\gcd(m,n) = 1$ *and* $mn \neq 2, 3, 6$. *Then the following statements are equivalent.*

(1) *There is a* $(\mathbf{K}, \theta, n)$*-triangle* $(U, V, W)_\theta$ *with* $0 < U \leq V < W$, $W \notin \mathbf{Q}$ *and* $W\sqrt{m} \notin \mathbf{Q}$;
(2) *The integers n and mn are θ-congruent numbers over* $\mathbf{Q}$.

2. Preliminaries

Consider an elliptic curve $E : y^2 = x^3 + ax^2 + bx + c$ over $\mathbf{Q}$. Recall that the m-twist E^m of E is an elliptic curve over $\mathbf{Q}$ defined by $y^2 = x^3 + amx^2 + bm^2x + cm^3$. The next result establishes a fact about ranks [10].

THEOREM 2.1. *Let E be an elliptic curve over* $\mathbf{Q}$. *Then*

$$\operatorname{rank}(E(\mathbf{K})) = \operatorname{rank}(E(\mathbf{Q})) + \operatorname{rank}(E^m(\mathbf{Q})).$$

We denote the torsion subgroup of the groups $E(\mathbf{K})$ and $E^m(\mathbf{K})$ by $T(E, \mathbf{K})$ and $T(E^m, \mathbf{K})$, respectively. Also, we write $T_{n,\theta}(\mathbf{K})$ and $T^m_{n,\theta}(\mathbf{K})$, respectively, in the case $E = E_{n,\theta}$. The following proposition and theorem have essential roles in the proof of our results.

PROPOSITION 2.2 ([9, Proposition 1]). *Let E be an elliptic curve over* $\mathbf{K}$. *Then the map*

$$\phi : T(E, \mathbf{K})/T(E, \mathbf{Q}) \to T(E^m, \mathbf{Q}), \quad \phi(\tilde{P}) := P - \sigma(P)$$

is an injective map of abelian groups, where σ is the generator of $\mathrm{Gal}(\mathbf{K}/\mathbf{Q})$.

THEOREM 2.3 ([7, Theorem 4.2]). *Let* $\mathbf{F}$ *be an algebraic number field and* E *an elliptic curve over* $\mathbf{F}$ *defined by*

$$y^2 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3), \quad \alpha_1, \alpha_2, \alpha_3 \in \mathbf{F}.$$

Suppose that (x_0, y_0) *be an* $\mathbf{F}$*-rational point of* E. *Then, there exists an* $\mathbf{F}$*-rational point* (x_1, y_1) *with* $2(x_1, y_1) = (x_0, y_0)$ *if and only if* $x_0 - \alpha_1$, $x_0 - \alpha_2$, $x_0 - \alpha_3$ *are squares in* $\mathbf{F}$.

The next results give important information about θ-congruent numbers over $\mathbf{Q}$.

THEOREM 2.4 (Fujiwara, [4]). *Consider* $0 < \theta < \pi$ *with rational cosine.*
(1) *A positive integer* n *is a* θ*-congruent number if and only if* $E_{n,\theta}(\mathbf{Q})$ *has a point of order greater than* 2;
(2) *If* $n \neq 1, 2, 3, 6$, *then* n *is a* θ*-congruent number if and only if* $E_{n,\theta}(\mathbf{Q})$ *has positive rank.*

All possibilities for the torsion subgroup of $E_{n,\theta}(\mathbf{Q})$ can be found in the next result.

THEOREM 2.5 (Fujiwara, [5]). *Let* $T_{n,\theta}(\mathbf{Q})$ *be the torsion subgroup of the* θ*-congruent number elliptic curve* $E_{n,\theta}$ *over* $\mathbf{Q}$.
(1) $T_{n,\theta}(\mathbf{Q}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_8$ *if and only if there exist integers* $a, b > 0$ *such that* $\gcd(a,b) = 1$, a *and* b *have opposite parity and satisfy either of the following conditions.*
   (i) $n = 1,\ r = 8a^4b^4,\ r - s = (a-b)^4,\ (1+\sqrt{2})b > a > b$,
   (ii) $n = 2,\ r = (a^2-b^2)^4,\ r - s = 32a^4b^4,\ a > (1+\sqrt{2})b$;
(2) $T_{n,\theta}(\mathbf{Q}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_6$ *if and only if there exist integers* $u, v > 0$ *such that* $\gcd(u,v) = 1$, $u > 2v$ *and satisfy one of the following conditions*:
   (i) $n = 1,\ r = \frac{1}{2}(u-v)^3(u+v),\ r + s = u^3(u-2v)$,
   (ii) $n = 2,\ r = (u-v)^3(u+v),\ r + s = 2u^3(u-2v)$,
   (iii) $n = 3,\ r = \frac{1}{6}(u-v)^3(u+v),\ r + s = \frac{1}{3}u^3(u-2v)$,
   (iv) $n = 6,\ r = \frac{1}{3}(u-v)^3(u+v),\ r + s = \frac{2}{3}u^3(u-2v)$;
(3) $T_{n,\theta}(\mathbf{Q}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_4$ *if and only if either of the following holds.*
   (i) $n = 1$, $2r$ *and* $r - s$ *are squares but not satisfy* (i) *of Part* (1),
   (ii) $n = 2$, r *and* $2(r-s)$ *are squares but not satisfy* (ii) *of Part* (1);
(4) *Otherwise*, $T_{n,\theta}(\mathbf{Q}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Remark 2.6. For any squarefree integer $m > 1$, the m-twist $E^m_{n,\theta}$ of the elliptic curve $E_{n,\theta}$ is defined by $y^2 = x(x+(r+s)mn)(x-(r-s)mn)$ which is equal to $E_{mn,\theta}$, as seen. Therefore $E^m_{n,\theta}(\mathbf{Q}) = E_{mn,\theta}(\mathbf{Q})$, and hence $T^m_{n,\theta}(\mathbf{Q}) = T_{mn,\theta}(\mathbf{Q})$.

3. Proofs

Appealing to Proposition 2.2, we first settle all possibilities for the torsion subgroup of $E_{n,\theta}(\mathbf{K})$. Let h, k, and d be integers such that $2r = h^2 \operatorname{sqf}(2r)$, $r - s = k^2 \operatorname{sqf}(r - s)$ and $2r(r - s) = d^2 m$, where $m = \operatorname{sqf}(2r(r - s))$.

Proposition 3.1. *Assume that $m > 1$ and n are squarefree positive integers such that $\gcd(m, n) = 1$ and $mn \neq 2, 3, 6$. Let $T_{n,\theta}(\mathbf{K})$ be the torsion subgroup of $E_{n,\theta}(\mathbf{K})$.*

(1) *If $m = \operatorname{sqf}(2r(r - s))$ and $n = \operatorname{sqf}(2r)$, then*

$$T_{n,\theta}(\mathbf{K}) = \Bigg\{\infty, (0,0), (-(r+s)n, 0), ((r-s)n, 0), \left((nh)^2 - nd\sqrt{m}, \pm\left(\frac{d^2mn}{h} - n^2hd\sqrt{m}\right)\right), \left((nh)^2 + nd\sqrt{m}, \pm\left(\frac{d^2mn}{h} + n^2hd\sqrt{m}\right)\right)\Bigg\};$$

(2) *If $m = \operatorname{sqf}(2r(r - s))$ and $n = \operatorname{sqf}(r - s)$, then*

$$T_{n,\theta}(\mathbf{K}) = \Bigg\{\infty, (0,0), (-(r+s)n, 0), ((r-s)n, 0), \left((nk)^2 - nd\sqrt{m}, \pm\left(\frac{d^2mn}{k} - n^2kd\sqrt{m}\right)\right), \left((nk)^2 + nd\sqrt{m}, \pm\left(\frac{d^2mn}{k} + n^2kd\sqrt{m}\right)\right)\Bigg\};$$

(3) *Otherwise, $T_{n,\theta}(\mathbf{K}) = \{\infty, (0,0), (-(r+s)n, 0), ((r-s)n, 0)\}$.*

Proof. The 2-torsion subgroup of $E_{n,\theta}(\mathbf{K})$ is:

$$E_{n,\theta}[2](\mathbf{K}) = \{\infty, (0,0), (-(r+s)n, 0), ((r-s)n, 0)\}.$$

Therefore, we have $T_{n,\theta}(\mathbf{K}) \supset E_{n,\theta}[2](\mathbf{K}) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. By Remark 2.6 and Theorem 2.5, $T^m_{n,\theta}(\mathbf{Q}) = T_{mn,\theta}(\mathbf{Q}) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. Since $T_{n,\theta}(\mathbf{Q}) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$, by Proposition 2.2 and [9, Theorem 1] we have

$$T_{n,\theta}(\mathbf{K}) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \text{ or } \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}.$$

First let $T_{n,\theta}(\mathbf{K}) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z}$. Then there exists a point $P = (x_0, y_0)$ of order 4 in $T_{n,\theta}(\mathbf{K})$. Then $2P$ must be one of the points $(0,0)$, $(-(r+s)n, 0)$ and $((r-s)n, 0)$. If $2P = (0,0)$ then both $(r+s)n$ and $-(r-s)n$ are squares in $\mathbf{K}$, which is impossible since $\mathbf{K}$ is a real quadratic number field and hence -1 is not a square in $\mathbf{K}$. Similarly, if $2P = (-(r+s)n, 0)$, then $-(r+s)n$ and $-2rn$ are squares in $\mathbf{K}$, again a contradiction by the same reason. If $2P = ((r-s)n, 0)$, then $(r-s)n$ and $2rn$ are squares in $\mathbf{K}$. Since n is squarefree, these integers are squares in $\mathbf{K}$ if $m = \operatorname{sqf}(2r(r-s))$. By a simple computation using the duplication formula we obtain (1) and (2). Now, $T_{n,\theta}(\mathbf{K}) \cong \mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$ implies (3), and the proof is completed. □

Proof of Theorem 1.1. Consider the two sets

$$S = \{(U, V, W) \in (\mathbf{K}^*)^3 : 0 < U \le V < W, UV = 2rn \text{ and } U^2 + V^2 - 2sUV/r = W^2\},$$

$$T = \{(u, v) \in 2E_{n,\theta}(\mathbf{K}) \backslash \{\infty\} : v \ge 0\}.$$

There is a one to one correspondence between the two sets S and T via the two mutually inverse maps $\varphi : S \to T$ and $\psi : T \to S$ defined by

$$\varphi(U, V, W) := (W^2/4, W(V^2 - U^2)/8),$$

$$\psi(u, v) := (\sqrt{u + (r+s)n} - \sqrt{u - (r-s)n}, \sqrt{u + (r+s)n} + \sqrt{u - (r-s)n}, 2\sqrt{u}).$$

Clearly, $E_{n,\theta}(\mathbf{K}) \backslash E_{n,\theta}[2](\mathbf{K}) \neq \emptyset$ if and only if $S \neq \emptyset$.

Suppose that $m \neq \operatorname{sqf}(2r(r-s))$ and $mn \neq 2, 3, 6$. Then by proposition 3.1, we have $T_{n,\theta}(\mathbf{K}) = E_{n,\theta}[2](\mathbf{K})$. Therefore, $\operatorname{rank}(E_{n,\theta}(\mathbf{K})) > 0$ if and only if $E_{n,\theta}(\mathbf{K}) \backslash E_{n,\theta}[2](\mathbf{K}) \neq \emptyset$. So $\operatorname{rank}(E_{n,\theta}(\mathbf{K})) > 0$ if and only if either $\operatorname{rank}(E_{n,\theta}(\mathbf{Q})) > 0$ or $\operatorname{rank}(E^m_{n,\theta}(\mathbf{Q})) > 0$, by Theorem 2.1. The second part of the theorem follows from Remark 2.6. □

Proof of Theorem 1.2. Assume n is a $(\mathbf{K}, \theta)$-congruent number and $(U, V, W)_\theta$ is the corresponding $(\mathbf{K}, \theta, n)$-triangle with area $n\alpha_\theta$ such that $0 < U \le V < W$. As in the proof of the Theorem 1.1, there is a point $P = (x, y)$ in $E_{n,\theta}(\mathbf{K}) \backslash E_{n,\theta}[2](\mathbf{K})$ such that $\psi(P) = (U, V, W)$. Substituting P by $P + (0, 0)$, $P + (-(r+s)n, 0)$ or $P + ((r-s)n, 0)$, if necessary, we may assume that $x > [(r+s) + \sqrt{2r(r-s)}]n$. Putting $2P = (u, v)$ and using the map ψ in the proof of Theorem 1.1, we obtain

$$U = 2rnx/|y|, \quad V = x^2 + 2snx - (r^2 - s^2)n^2/|y|, \quad W = x^2 + (r^2 - s^2)n^2/|y|,$$

where $x, y \in \mathbf{K}$ and $|\cdot|$ is the usual absolute value induced from the embedding $\imath : \mathbf{K} \hookrightarrow \mathbf{R}$ with $\imath(\sqrt{m})$ positive. Suppose σ is a generator of $\operatorname{Gal}(\mathbf{K}/\mathbf{Q})$ and put $\sigma(P) = (\sigma(x), \sigma(y))$. Since $P + \sigma(P)$ is an element of $E_{n,\theta}(\mathbf{Q})$ and n is not a θ-congruent number, $P + \sigma(P) \in T_{n,\theta}(\mathbf{Q}) = \{\infty, (0, 0), (-(r+s)n, 0), ((r-s)n, 0)\}$. Hence, one of the following cases necessarily happens:

I. $\underline{P + \sigma(P) = \infty}$. In this case, $\sigma(x) = x$ and $\sigma(y) = -y$. So, $x, y\sqrt{m}$ and hence $U\sqrt{m}$, $V\sqrt{m}$ and $W\sqrt{m}$ are rational and we obtain a $(\mathbf{K}, \theta, n)$-triangle of Type 1.

II. $\underline{P + \sigma(P) = (0, 0)}$. We have $\sigma(x)/x = \sigma(y)/y$, which we denote by α. Then,

$$\sigma(y)^2 = \alpha^2 y^2 = \alpha^2 x^3 + 2sn\alpha^2 x^2 - (r^2 - s^2)n^2\alpha^2 x.$$

Since $\sigma(P)$ is a point on $E_{n,\theta}$, we get

$$\sigma(y)^2 = \sigma(x)^3 + 2sn\sigma(x)^2 - (r^2 - s^2)n^2\sigma(x)$$
$$= \alpha^3x^3 + 2sn\alpha^2x^2 - (r^2 - s^2)n^2\alpha x.$$

Clearly, $\alpha \neq 0, 1$ and $x \neq 0$, which implies $x\sigma(x) = \alpha x^2 = -(r^2 - s^2)n^2$. Therefore,

$$V = x(x + 2sn + \sigma(x))/|y|, \quad W\sqrt{m} = x(x - \sigma(x))\sqrt{m}/|y|.$$

Since $x/y = \sigma(x/y)$ and $x > [(r+s) + \sqrt{2r(r-s)}]n$, then $x/|y|$ is rational and hence $U = 2rnx/|y|$, V and $W\sqrt{m}$ are rational, which gives a $(\mathbf{K}, \theta, n)$-triangle of Type 2.

III. $\underline{P + \sigma(P) = ((r-s)n, 0)}$. We have $\sigma(x - (r-s)n)/(x - (r-s)n) = \sigma(y)/y$, which we denote by β. Put $z = x - (r-s)n$. Then,

$$\sigma(y)^2 = \beta^2[z^3 + (3r - s)nz^2 + 2r(r-s)n^2z].$$

Since $\sigma(P)$ is a point on $E_{n,\theta}$, we get

$$\sigma(y)^2 = \beta^3z^3 + (3r - s)n\beta^2z^2 + 2r(r-s)n^2\beta z.$$

Now $\beta \neq 0, 1$ and $z \neq 0$, which implies $\beta z^2 = 2r(r-s)n^2$. Substituting this equation and $x = z + (r-s)n$ in U, V and W, we obtain

$$U = \frac{z(\sigma(z) + 2rn)}{|y|}, \quad V = \frac{z(z + 2rn)}{|y|}, \quad W = \frac{z(z + 2(r-s)n + \sigma(z))}{|y|}.$$

Since $z/y = \sigma(z/y)$ and $z > 0$, then $z/|y|$ and hence W is rational and $\sigma(U) = V$. This time we obtain a $(\mathbf{K}, \theta, n)$-triangle of Type 3.

IV. $\underline{P + \sigma(P) = (-(r+s)n, 0)}$. Put $w = x + (r+s)n$. As in Case III, $w/|y|$ and

$$W = w(w - 2(r+s)n + \sigma(w))/|y|$$

are rational and $\sigma(U) = -V$, where

$$U = w(2rn - \sigma(w))/|y|, \quad V = w(w - 2rn)/|y|.$$

Therefore, we obtain a $(\mathbf{K}, \theta, n)$-triangle of Type 4. □

Proof of Proposition 1.3. If we suppose that there is a $(\mathbf{K}, \theta, n)$-triangle of Type 2, say $(U, V, W)_\theta = (u, v, w\sqrt{m})$ with $u, v, w \in \mathbf{Q}^+$, then $(x, y, z) = (ru - sv, v, mrw)$ is a non-zero solution of the equation

$$z^2 = mx^2 + m(r^2 - s^2)y^2. \tag{3.1}$$

And, if there is a $(\mathbf{K}, \theta, n)$-triangle of Type 3, say $(U, V, W)_\theta = (u - v\sqrt{m}, u + v\sqrt{m}, w)$ such that $\sigma(U) = V$, then $(x, y, z) = (u, v, rw)$ is a non-zero solution of

$$z^2 = 2r(r - s)x^2 + 2mr(r + s)y^2. \tag{3.2}$$

Similarly, if $(U, V, W)_\theta = (-u + v\sqrt{m}, u + v\sqrt{m}, w)$ is a $(\mathbf{K}, \theta, n)$-triangle of Type 4 such that $\sigma(U) = -V$, then $(x, y, z) = (u, v, rw)$ satisfies

$$z^2 = 2r(r + s)x^2 + 2mr(r - s)y^2. \tag{3.3}$$

By the Hasse local-global principle, the equations (3.1), (3.2) and (3.3) have solutions in $\mathbf{Q}$ if and only if they have a solution in $\mathbf{Q}_p$ for every prime p, where $\mathbf{Q}_p$ is the field of p-adic numbers. We assume that $A = \mathrm{sqf}(r^2 - s^2)$, and for a prime p the pair (m, A) ((m, B), and (m, C), resp.) can be written as $(p^\alpha a, p^\beta b)$, where $\alpha, \beta \in \{0, 1\}$ and $\gcd(p, a, b) = 1$. Then, using Hilbert symbols [11, Theorem 1, III], the equations (3.1), (3.2) and (3.3) have solutions in $\mathbf{Q}_2$ if and only if one of the following cases happens:

i) $(\alpha, \beta) = (0, 0)$ and $(a, b) \overset{4}{\not\equiv} (3, 3)$;

ii) $(\alpha, \beta) = (0, 1)$ and $(a, b) \overset{8}{\not\equiv} (3, 1), (3, 5), (7, 5), (7, 7)$;

iii) $(\alpha, \beta) = (1, 0)$ and $(a, b) \overset{8}{\not\equiv} (1, 3), (1, 5), (3, 5), (3, 7), (5, 3), (5, 7), (7, 3), (7, 7)$;

iv) $(\alpha, \beta) = (1, 1)$ and $(a, b) \overset{8}{\not\equiv} (1, 3), (1, 5), (3, 1), (3, 3), (5, 1), (5, 7), (7, 5), (7, 7)$.

Also, the equations (3.1), (3.2) and (3.3) have solutions in $\mathbf{Q}_p$ with $p \overset{4}{\equiv} 1$ if and only if one of the following happens:

i) $(\alpha, \beta) = (0, 1)$ and $\left(\dfrac{a}{p}\right) = 1$;

ii) $(\alpha, \beta) = (1, 0)$ and $\left(\dfrac{b}{p}\right) = 1$;

iii) $(\alpha, \beta) = (1, 1)$ and $\left(\dfrac{a}{p}\right)\left(\dfrac{b}{p}\right) = 1$.

Proof of Theorem 1.4. CASE 1. n and mn are $(\mathbf{Q}, \theta)$-congruent numbers. Consider the $(\mathbf{Q}, \theta, n)$-triangle $(U_1, V_1, W_1)_\theta$ and the $(\mathbf{Q}, \theta, mn)$-triangle $(U_2, V_2, W_2)_\theta$, where

$$0 < U_1 \le V_1 < W_1, \quad 2rn = U_1 V_1, \quad U_1^2 + V_1^2 - \frac{2sU_1V_1}{r} = W_1^2,$$

$$0 < U_2 \le V_2 < W_2, \quad 2rmn = U_2 V_2, \quad U_2^2 + V_2^2 - \frac{2sU_2V_2}{r} = W_2^2.$$

Hence, $(U_2/\sqrt{m}, V_2/\sqrt{m}, W_2/\sqrt{m})_\theta$ is a $(\mathbf{K}, \theta, n)$-triangle. Recall the maps φ and ψ in the proof of Theorem 1.2 and put

$$P = (u, v) = \varphi((U_1, V_1, W_1)) + \varphi((U_2/\sqrt{m}, V_2/\sqrt{m}, W_2/\sqrt{m})).$$

Then the additive law on $E_{n,\theta}(\mathbf{K})$ implies $u = a + b\sqrt{m}$, where

$$a = \frac{m^3 W_1^2 (V_1^2 - U_1^2)^2 + W_2^2 (V_2^2 - U_2^2)^2}{4m(W_2^2 - mW_1^2)^2} - \left(\frac{W_1^2}{4} + \frac{W_2^2}{4m} + 2sn\right) > 0,$$

$$b = -\frac{W_1 W_2 (V_1^2 - U_1^2)(V_2^2 - U_2^2)\sqrt{m}}{2(W_2^2 - mW_1^2)^2}.$$

We may assume $v \geq 0$. Since $(u, v) \in T$, then $\psi((u, v)) \in S$ which indicates the sides of a $(\mathbf{K}, \theta, n)$-triangle $(U, V, W)_\theta$. In fact, if we suppose $U = u_1 + u_2\sqrt{m}$, $V = v_1 + v_2\sqrt{m}$ and $W = w_1 + w_2\sqrt{m}$, where u_1, u_2, v_1, v_2, w_1, w_2 are rational, then

$$w_1 = \pm\sqrt{2(a \pm \sqrt{a^2 - mb^2})}, \quad w_2 = \frac{2b}{w_1},$$

and

$$U = (\alpha_1 - \beta_1) + (\alpha_2 - \beta_2)\sqrt{m}, \quad V = (\alpha_1 + \beta_1) + (\alpha_2 + \beta_2)\sqrt{m},$$

where

$$\alpha_1 = \pm\sqrt{\frac{(a + (r+s)n) \pm \sqrt{(a + (r+s)n)^2 - mb^2}}{2}}, \quad \alpha_2 = \frac{b}{2\alpha_1},$$

$$\beta_1 = \pm\sqrt{\frac{(a - (r-s)n) \pm \sqrt{(a - (r-s)n)^2 - mb^2}}{2}}, \quad \beta_2 = \frac{b}{2\beta_1}.$$

Conversely, suppose to the contrary that n or mn is not θ-congruent over $\mathbf{Q}$. First, assume n is not θ-congruent over $\mathbf{Q}$ but mn is θ-congruent over $\mathbf{Q}$. By Theorem 1.2 (1), there is no $(\mathbf{K}, \theta, n)$-triangle $(U, V, W)_\theta$ satisfying the conditions $0 < U \leq V < W$, $W \notin \mathbf{Q}$ and $W\sqrt{m} \notin \mathbf{Q}$.

Case 2. mn is not θ-congruent over $\mathbf{Q}$ but n is $(\mathbf{K}, \theta)$-congruent. Let $(U, V, W)_\theta$ denotes the sides of the corresponding $(\mathbf{K}, \theta, n)$-triangle. Multiplying the three sides by $\sqrt{m}$, we get the $(\mathbf{K}, \theta, mn)$-triangle $(U\sqrt{m}, V\sqrt{m}, W\sqrt{m})_\theta$. For the positive integer mn, we define the map φ' in the same way as φ. Put

$$2P' = \varphi'((U\sqrt{m}, V\sqrt{m}, W\sqrt{m}))$$

for some point $P' \in E_{mn,\theta}(\mathbf{K})$. For the generator σ of $\mathrm{Gal}(\mathbf{K}/\mathbf{Q})$, since $P' + \sigma(P')$ is an element in $E_{mn,\theta}(\mathbf{Q})$ and mn is not θ-congruent over $\mathbf{Q}$, we have

$$P' + \sigma(P') \in T_{mn,\theta}(\mathbf{Q}) = \{\infty, (0,0), (-(r+s)mn, 0), ((r-s)mn, 0)\}.$$

Therefore, by the same way as in the proof of Theorem 1.2, one of the following cases necessarily happens:

Type 1. $U, V, W \in \mathbf{Q}$;
Type 2. $U\sqrt{m}, V\sqrt{m}, W \in \mathbf{Q}$;
Type 3. $U, V \in K\backslash\mathbf{Q}$ such that $\sigma(U) = V$, $W\sqrt{m} \in \mathbf{Q}$;
Type 4. $U, V \in K\backslash\mathbf{Q}$ such that $\sigma(U) = -V$, $W\sqrt{m} \in \mathbf{Q}$.
Hence, there is no $(\mathbf{K}, \theta, n)$-triangle $(U, V, W)_\theta$ with $W \notin \mathbf{Q}$ and $W\sqrt{m} \notin \mathbf{Q}$.

Case 3. Both n and mn are not θ-congruent numbers over $\mathbf{Q}$, where $mn \neq 2, 3, 6$. If $m \neq \mathrm{sqf}(2r(r-s))$, by Theorem 1.1, n is not $(\mathbf{K}, \theta, n)$-congruent. If $m = \mathrm{sqf}(2r(r-s))$ and n is $(\mathbf{K}, \theta, n)$-congruent, we have $U = V$ for all $(\mathbf{K}, \theta, n)$-triangles $(U, V, W)_\theta$. Hence, there is no any $(\mathbf{K}, \theta, n)$-triangle $(U, V, W)_\theta$ with $W \notin \mathbf{Q}$ and $W\sqrt{m} \notin \mathbf{Q}$. We have completed the proof of Theorem 1.4. □

4. Examples

In this section, we give some examples of $(\mathbf{K}, \theta)$-congruent numbers and verify all four types of $(\mathbf{K}, \theta, n)$-triangles in Theorem 1.2 in the cases $\theta = \pi/3, 2\pi/3$. Given n, let $(U, V, W)_\theta$ be a $(\mathbf{K}, \theta, n)$-triangle. Then, we have

$$0 < U \leq V < W, \quad UV = 2rn, \quad W^2 = U^2 + V^2 - \frac{2s}{r}UV.$$

For any $(U, V, W)_\theta, \varphi((U, V, W)) = (W^2/4, W(V^2 - U^2)/8)$ is a point of $2E_{n,\theta}(\mathbf{K})\backslash\{\infty\}$. Also, for any point $(u, v) \in 2E_{n,\theta}(\mathbf{K})\backslash\{\infty\}$,

$$\psi((u, v)) = ((\sqrt{u + (r+s)n} - \sqrt{u - (r-s)n}, \sqrt{u + (r+s)n} + \sqrt{u - (r-s)n}, 2\sqrt{u})).$$

In our computations we have used Cremona's MWrank program [2] and the number theoretic Pari software [1].

I) Case $\theta = \pi/3$. In this case, we have $r = 2$, $s = 1$, and $\alpha_\theta = \sqrt{3}$, and hence the area of any $(\mathbf{K}, \pi/3, n)$-triangle is $n\sqrt{3}$.

Example 4.1. Take $n = 3$ and $m = 13$. We have the following $(\mathbf{Q}(\sqrt{13}), \pi/3, 3)$-triangles of types 1, 2, 3 and 4 in Theorem 1.1 and the corresponding points in the set $2E_{3,\pi/3}(\mathbf{Q}(\sqrt{13}))\backslash\{\infty\}$.

Type 1. An easy computing shows that the rank of $E_{39,\pi/3}(\mathbf{Q})$ is 2, and the generators of the group are $P_1 = [-9, -216]$ and $P_2 = [75, -720]$. We have

$$2P_1 = [1894/16, -91805/64] \in 2E_{39,\theta}(\mathbf{Q})\backslash\{\infty\}.$$

Now, using the map φ and ψ, defined in the proof of the Theorem 1.1 we get a rational $\pi/3$-triangle $(13/2, 24, 43/2)$ with area 39, which gives the following $(\mathbf{Q}(\sqrt{13}), \pi/3, 3)$-triangle of Type 1:

$$(U, V, W)_{\pi/3} = (\sqrt{13}/2, 24\sqrt{13}/13, 43\sqrt{13}/26)$$

which corresponds to the following point $Q = (1894/208, 91805\sqrt{13}/416)$.

Type 2. We have a $(\mathbf{Q}(\sqrt{13}), \pi/3, 3)$-triangle $(U, V, W)_{\pi/3} = (3, 4, \sqrt{13})$ of type 2 with the corresponding point $Q = (13/4, 7\sqrt{13}/8)$.

Type 3. Let $U = u - v\sqrt{13}$, $V = u + v\sqrt{13}$ and $W = w$, where $u, v, w \in \mathbf{Q} \backslash \{0\}$. Then the pair (u, v) satisfies the equation $u^2 - 13v^2 = 12$. An easy solution of this equation is $(u_0, v_0) = (5, 1)$. Parametrizing u and v in terms of $t \in \mathbf{Q}$ we obtain $u = -5t^2 + 26t - 65/t^2 - 13$ and $v = t^2 - 10t + 13/t^2 - 13$. By putting these into $w^2 = u^2 + 39v^2$ and taking $t = 13/4$ one can see that $w^2 = u^2 + 39v^2$ is a square in $\mathbf{Q}$. So, we obtain $(U, V, W)_{\pi/3} = (41 - 11\sqrt{13}/3, 41 + 11\sqrt{13}/3, 80/3)$, with $(\mathbf{Q}(\sqrt{13}), \pi/3, 3)$-triangle of type 3 with the corresponding point $Q = (1600/3, 18040\sqrt{13}/9)$.

Type 4. Let $U = -u + v\sqrt{13}$, $V = u + v\sqrt{13}$ and $W = w$, where $u, v, w \in \mathbf{Q} \backslash \{0\}$. Then the pair (u, v) satisfies $13v^2 - u^2 = 12$ with a solution $(u_0, v_0) = (1, 1)$. A similar discussion as in the previous step, taking $t = 8$, leads us to a $(\mathbf{Q}(\sqrt{13}), \pi/3, 3)$-triangle of Type 4, with the corresponding point $Q = (24964/51, 1002352\sqrt{13}/51)$.

Example 4.2. Let $n = 11$ and $m = 5$. One can see that n is $\pi/3$-congruent over $\mathbf{Q}$ and there is a $(\mathbf{Q}, \pi/3, 11)$-triangle $(U_1, V_1, W_1) = (55/12, 48/5, 499/60)$. Also, $nm = 55$ is $\pi/3$-congruent over $\mathbf{Q}$ and $(U_2, V_2, W_2) = (8, 55/2, 49/2)$ is a rational $\pi/3$-triangle with area $11\sqrt{3}$. Dividing its sides by $\sqrt{5}$, we obtain a $(\mathbf{Q}(\sqrt{5}), \pi/3, 11)$-triangle

$$(U_2/\sqrt{5}, V_2/\sqrt{5}, W_2/\sqrt{5}) = (8\sqrt{5}/5, 11\sqrt{5}/2, 49\sqrt{5}/10).$$

Now, a calculations as in the proof of Theorem 1.4 leads to a $(\mathbf{Q}(\sqrt{5}), \pi/3, 11)$-triangle

$$(U, V, W) = \Bigg(\frac{1}{310}(1470 + 499\sqrt{5}), \frac{88}{5909}(1470 - 499\sqrt{5}), \frac{1}{183179}(4145193 - 12554399\sqrt{5})\Bigg)$$

satisfying in Theorem 1.4.

II) Case $\theta = 2\pi/3$. In this case, we have $r = 2$, $s = -1$, and $\alpha_\theta = \sqrt{3}$. So, as in the case I, the area of any $(\mathbf{K}, 2\pi/3, n)$-triangle is $n\sqrt{3}$.

Example 4.3. Take $n = 17$ and $m = 13$. By a similar way as in Example 4.1, we find the following $(\mathbf{Q}(\sqrt{13}), 2\pi/3, 17)$-triangles with area $17\sqrt{13}$ of types 1, 2, 3 and 4 preceding by their corresponding points in $2E_{17, 2\pi/3}(\mathbf{Q}(\sqrt{13})) \backslash \{\infty\}$.

Type 1. $(U, V, W)_{2\pi/3} = (17\sqrt{13}/26, 8\sqrt{13}, 217\sqrt{13}/26)$, $Q = (47089/16, 9325575\sqrt{13}/10816)$;

Type 2. $(U, V, W)_{2\pi/3} = (1, 68, 19\sqrt{13})$, $Q = (13/4, 7\sqrt{13}/8)$;

Type 3. $(U, V, W)_{2\pi/3} = (9 - \sqrt{13}, 9 + \sqrt{13}, 16)$, $Q = (64, 72\sqrt{13})$;
Type 4. $(U, V, W)_{2\pi/3} = (-5 + 7\sqrt{13}/3, 5 + 7\sqrt{13}/3, 44/3)$, $Q = (484/9, 770\sqrt{13}/27)$.

Example 4.4. Let $n = 19$ and $m = 6$. Then 19 is a $2\pi/3$-congruent number over $\mathbf{Q}$ and there is a $(\mathbf{Q}, 2\pi/3, 6)$-triangle $(U_1, V_1, W_1) = (544/105, 1995/136, 254659/14280)$ with area $19\sqrt{3}$. Also, the integer $nm = 114$ is a $2\pi/3$-congruent number over $\mathbf{Q}$ and $(U_2, V_2, W_2) = (5, 912/10, 469/5)$ is a $2\pi/3$-triangle with area $114\sqrt{3}$ from which we obtain a $(\mathbf{Q}(\sqrt{6}), 2\pi/3, 19)$-triangle

$$(5\sqrt{6}/6, 76\sqrt{6}/5, 469\sqrt{6}/30).$$

By a similar methods as in Example 4.2, one can find a $(\mathbf{Q}(\sqrt{6}), 2\pi/3, 19)$-triangle

$$\begin{aligned}(U, V, W)_{2\pi/3} = ((&25449816 + 4838521\sqrt{6})/4683550,\\ &20(4145193 - 12554399\sqrt{6})/28499829,\\ &7(3589965612532 - 2573211605723\sqrt{6})/1170880474675),\end{aligned}$$

satisfying Theorem 1.4.

References

[1] C. Batue, K. Belabas, D. Bernardi, H. Cohen and M. Oliver, The computer algebra system Pari/Gp, Universite Bordeaux I, 1999, http://pari.math.u-bordeaux.fr/.

[2] John Cremona, MWrank program for elliptic curves over $\mathbf{Q}$, 2008, http://www.maths.nott.ac.uk/personal/jec/MWrank.

[3] A. S. Janfada, S. Salami, A. Dujella and C. J. Perel, On the high rank $\pi/3$ and $2\pi/3$-congruent number elliptic curves, Rocky Mountains Journal of Mathematics **44** (2014), 1867–1880.

[4] M. Fujiwara, θ-congruent numbers, Number theory (K. Györy, A. Pethö and V. Sós, eds.), de Gruyter, 1997, 235–241.

[5] M. Fujiwara, Some properties of θ-congruent numbers, Natur. Sci. Rep. Ochanomizu Univ. **52**, no. 2 (2001), 1–8.

[6] M. Kan, θ-congruent numbers and elliptic curves, Acta Arithmetica **94** (2000), 153–160.

[7] A. Knapp, Elliptic curves, Princeton University Press, 1992.

[8] N. Koblitz, Introduction to elliptic curves and modular forms, Grad. texts in Math. **97**, 2nd ed., Springer-Verlag, Berlin, 1993.

[9] S. Kwon, Torsion subgroups of elliptic curves over quadratic extensions, J. Number Theory **62** (1997), 144–162.

[10] P. Serf, The rank of elliptic curves over real quadratic number fields of class number 1, Ph.D. Thesis, Universität des Saarlandes, Saarbrücken, 1995.

[11] J. P. Serre, A course in arithmetic, GTM **7**, Springer-Verlag, 1973.

[12] J. H. Silverman, The arithmetic of elliptic curves, Grad. texts in Math. **106**, 2nd ed., Springer-Verlag, 2009.

[13] M. Tada, Congruent number over real quadratic fields, Hiroshima Math. J. **31** (2001), 331–343.

[14] S. Yoshida, Some variant of the congruent number problem, I, Kyushu J. Math. **55** (2001), 387–404.

Ali S. Janfada
Department of Mathematics
Urmia University
Urmia
Iran
E-mail: a.sjanfada@urmia.ac.ir
asjanfada@gmail.com

Sajad Salami
Instituto da Mathematica e Estatistica
UERJ
Brazil
E-mail: sajad.salami@ime.uerj.br