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LOWER BOUNDS FOR BLOW-UP TIME IN A PARABOLIC PROBLEM WITH A GRADIENT TERM UNDER VARIOUS BOUNDARY CONDITIONS

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Abstract

This paper deals with the blow-up phenomena of the solution u of a nonlinear parabolic problem with a gradient nonlinearity and time dependent coefficients. By using techniques based on Sobolev type and differential inequalities, we derive explicit lower bounds for the blow-up time, if blow-up occurs, when different boundary conditions are taken into account.

1. Introduction

In recent years there has been considerable attention paid to the question of blow-up to solutions of nonlinear parabolic problems, whose source term depends on the gradient of the solution. We cite the book [11] (chapter IV) and the references therein.

In this paper we discuss the following problem

$$(1.1)\qquad \begin{cases} u_t = \Delta u + k_1(t)u^p - k_2(t)|\nabla u|^q, & \mathbf{x} \in \Omega,\ t \in (0, t^\star), \\ \alpha_1 u_\nu + \alpha_2 u = 0, & \mathbf{x} \in \partial\Omega,\ t \in (0, t^\star), \\ u = u_0(\mathbf{x}) \geq 0, & \mathbf{x} \in \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$, $N \geq 2$, whose boundary $\partial\Omega$ is sufficiently smooth. The coefficients $k_1(t)$ and $k_2(t)$, associated respectively to the source term and the dissipative gradient term, are positive and regular functions in $[0, t^\star)$, $t^\star$ being the blow-up time, $p > 1, q > 1$, α_1 and α_2 are nonnegative constants, and $u_0(\mathbf{x})$ is a nonnegative function in Ω satisfying the compatibility conditions on $\partial\Omega$. Moreover u_ν represents the normal derivative of u with respect the exterior unit vector $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_N)$ to $\partial\Omega$. It follows by the maximum principle that in the interval of existence $u(\mathbf{x}, t) \geq 0$. We remark that the gradient term in (1.1) has a damping effect, working against blow-up.

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We will focus our study on lower bounds for blow-up time of problem (1.1), which are of a great interest in several practical cases (see, for example, [10] and [11]), since an explicit value of $t^\star$ cannot be generally determined. More precisely, as we are interested in solutions blowing up at finite time $t^\star$, we will assume in the problem (1.1) $p > q$, since for $p \le q$ it is well known that the solution will not blow up in finite time (see [11]).

We also have to underline that only in the case $k_1(t) = k_2(t) = 1$, $\alpha_1 = 0$ and $\alpha_2 = 1$ in (1.1), a lower bound for blow-up time was obtained by Payne and Song in [9]. In the same way, [1]–[8] provide good references about upper and lower bounds of blow-up time for solutions of various parabolic problems.

In Section 2 we consider problem (1.1) under Dirichlet boundary conditions ($\alpha_1 = 0$ and $\alpha_2 = 1$) and in order to obtain an explicit lower bound for $t^\star$ we derive a first order differential inequality by using the Talenti-Sobolev inequality (see [4] and [12]) which is valid for a nonnegative function that vanishes on $\partial\Omega$ and for a bounded domain $\Omega \subset \mathbf{R}^3$.

Lower bounds under Neumann and Robin boundary conditions on $\partial\Omega$ are also derived in Sections 3 and 4, if the spatial domain $\Omega \subset \mathbf{R}^3$ is star-shaped, convex in two orthogonal directions and the origin inside, assumption due to the use of a Sobolev type inequality (see [7] and [8]).

Throughout the paper we will assume $p > q > 2$. Moreover the blow-up time of the solution u is considered in $L^{n(p-1)}$-norm ($n > 2$).

2. Lower bound under Dirichlet boundary condition

In this section we choose as parameters of problem (1.1) $\alpha_1 = 0$ and $\alpha_2 = 1$, i.e. we consider the homogeneous Dirichlet condition $u = 0$ on $\partial\Omega \times (0, t^\star)$. In order to derive a lower bound of $t^\star$, let us introduce the auxiliary function

$$\Psi(t) = k_1(t) \int_\Omega u^{n(p-1)}\, d\mathbf{x}, \quad n > 2, \tag{2.1}$$

with $\Psi(0) > 0$; if we set $s = p - 1$, by differentiation, we lead to

$$\begin{aligned}\Psi'(t) = k_1' \int_\Omega u^{ns}\, d\mathbf{x} + k_1 ns \int_\Omega u^{ns-1} \Delta u\, d\mathbf{x} \\ + nsk_1^2 \int_\Omega u^{s(n+1)}\, d\mathbf{x} - nsk_1k_2 \int_\Omega u^{ns-1} |\nabla u|^q\, d\mathbf{x}.\end{aligned} \tag{2.2}$$

Due to the divergence theorem and the boundary condition, from the identity

$$u^{ns-2}|\nabla u|^2 = \frac{4}{(ns)^2} |\nabla u^{ns/2}|^2, \tag{2.3}$$

we obtain

$$\int_\Omega u^{ns-1} \Delta u\, d\mathbf{x} = -4 \frac{ns-1}{(ns)^2} \int_\Omega |\nabla u^{ns/2}|^2\, d\mathbf{x}. \tag{2.4}$$

Next, arguing as in (2.3) and using inequality (2.10) in [6], we achieve

$$\int_\Omega u^{ns-1}|\nabla u|^q \, d\mathbf{x} \ge \left(\frac{2\sqrt{\lambda_1}}{ns+q-1}\right)^q \int_\Omega u^{ns+q-1} \, d\mathbf{x}, \tag{2.5}$$

λ_1 being the first positive eigenvalue of the fixed membrane problem

$$\begin{cases} \Delta w + \lambda w = 0, & \mathbf{x} \in \Omega, \\ w = 0, & \mathbf{x} \in \partial\Omega, \\ w > 0, & \mathbf{x} \in \Omega. \end{cases}$$

Let us assume k_1 such that $\frac{k_1'(t)}{k_1(t)} \le \beta$, with $\beta \ge 0$; replacing (2.1), (2.4) and (2.5) into (2.2) and setting $u^s = V$ we obtain

$$\begin{aligned} \Psi'(t) \le \beta\Psi - 4\frac{ns-1}{ns} k_1 \int_\Omega |\nabla V^{n/2}|^2 \, d\mathbf{x} \\ + nsk_1^2 \int_\Omega V^{n+1} \, d\mathbf{x} - nsk_2 k_1 m \int_\Omega V^{n+\alpha} \, d\mathbf{x}, \end{aligned} \tag{2.6}$$

where $m = \left(\frac{2\sqrt{\lambda_1}}{ns+q-1}\right)^q$ and $\alpha = \frac{q-1}{s} < 1$ (recall $p > q$). With the aim of reducing (2.6) to a differential inequality containing only powers of Ψ in its right hand side, let us analyze the term $\int_\Omega V^{n+1} \, d\mathbf{x}$. Since $n > 2$, Hölder inequality returns

$$\int_\Omega V^{n+1} \, d\mathbf{x} \le \left(\int_\Omega V^{n+\alpha} \, d\mathbf{x}\right)^{1/\mathfrak{p}} \left(\int_\Omega V^{(3/2)n} \, d\mathbf{x}\right)^{1/\mathfrak{q}}, \tag{2.7}$$

where $\frac{1}{\mathfrak{p}} = \frac{n-2}{n-2\alpha}$ and $\frac{1}{\mathfrak{q}} = \frac{2(1-\alpha)}{n-2\alpha}$; moreover due to the same inequality we obtain

$$\int_\Omega V^{(3/2)n} \, d\mathbf{x} \le \left(\int_\Omega V^n \, d\mathbf{x}\right)^{3/4} \left(\int_\Omega V^{3n} \, d\mathbf{x}\right)^{1/4}. \tag{2.8}$$

As $u = 0$ on $\partial\Omega$, the Talenti-Sobolev inequality provides

$$\left(\int_\Omega V^{3n} \, d\mathbf{x}\right)^{1/4} \le \Gamma^{3/2} \left(\int_\Omega |\nabla V^{n/2}|^2 \, d\mathbf{x}\right)^{3/4}, \tag{2.9}$$

$\Gamma = \left(\frac{2}{\pi}\right)^{2/3} 3^{-1/2}$ being the best Sobolev constant. Therefore, by replacing (2.9) and (2.8) into (2.7) we can write

$$\begin{aligned} \int_\Omega V^{n+1} \, d\mathbf{x} \le \Gamma^{3/2\mathfrak{q}} \left(\frac{1}{\mu^{\mathfrak{p}-1}} \int_\Omega V^{n+\alpha} \, d\mathbf{x}\right)^{1/\mathfrak{p}} \\ \times \left[\mu \left(\int_\Omega V^n \, d\mathbf{x}\right)^{3/4} \left(\int_\Omega |\nabla V^{n/2}|^2 \, d\mathbf{x}\right)^{3/4}\right]^{1/\mathfrak{q}}, \end{aligned} \tag{2.10}$$

where we have introduced a time dependent and positive function μ, to be successively chosen. Therefore, using

$$a^r b^{1-r} \le ra + (1-r)b, \tag{2.11}$$

valid for $a, b > 0$ and $0 < r < 1$, we have

$$\left[\left(\int_\Omega V^n \, d\mathbf{x}\right)^3\right]^{1/4} \left(\int_\Omega |\nabla V^{n/2}|^2 \, d\mathbf{x}\right)^{3/4} \le \frac{1}{4\nu^3}\left(\int_\Omega V^n \, d\mathbf{x}\right)^3 + \frac{3}{4}\nu \int_\Omega |\nabla V^{n/2}|^2 \, d\mathbf{x},$$

where also ν is a positive and time dependent function to be computed; (2.10) is so reduced to

$$\int_\Omega V^{n+1} \, d\mathbf{x} \le \Gamma^{3/2\mathfrak{q}} \left(\frac{1}{\tilde{\mu}} \int_\Omega V^{n+\alpha} \, d\mathbf{x}\right)^{1/\mathfrak{p}} \tilde{\mu} \times \left[\frac{1}{4\nu^3}\left(\int_\Omega V^n \, d\mathbf{x}\right)^3 + \frac{3}{4}\nu \int_\Omega |\nabla V^{n/2}|^2 \, d\mathbf{x}\right]^{1/\mathfrak{q}}, \tag{2.12}$$

being $\tilde{\mu} = \mu^{\mathfrak{p}-1}$. Now, arranging (2.12) by means of (2.11), we have

$$\int_\Omega V^{n+1} \, d\mathbf{x} \le \Gamma^{3/2\mathfrak{q}} \left[\frac{1}{\mathfrak{p}} \mu^{1-\mathfrak{p}} \int_\Omega V^{n+\alpha} \, d\mathbf{x} + \frac{1}{\mathfrak{q}}\left(\frac{\mu}{4\nu^3}\left(\int_\Omega V^n \, d\mathbf{x}\right)^3 + \frac{3}{4}\nu\mu \int_\Omega |\nabla V^{n/2}|^2 \, d\mathbf{x}\right)\right]. \tag{2.13}$$

Consequently, by replacing (2.13) into (2.6), we can write

$$\Psi'(t) \le \beta\Psi + d_1(t) \int_\Omega |\nabla V^{n/2}|^2 \, d\mathbf{x} + d_2(t) \int_\Omega V^{n+\alpha} \, d\mathbf{x} + d_3(t)\Psi^3,$$

with

$$\begin{cases} d_1(t) = k_1\left(\dfrac{3}{4}\Gamma^{3/2\mathfrak{q}} ns \dfrac{1}{\mathfrak{q}} \nu\mu k_1 - 4\dfrac{ns-1}{ns}\right), \\ d_2(t) = nsk_1\left(\Gamma^{3/2\mathfrak{q}} k_1 \dfrac{1}{\mathfrak{p}} \mu^{1-\mathfrak{p}} - k_2 m\right), \\ d_3(t) = nsk_1^{-1}\Gamma^{3/2\mathfrak{q}} \dfrac{\mu}{4\mathfrak{q}\nu^3}. \end{cases} \tag{2.14}$$

Choosing in (2.14) firstly μ to make $d_2 = 0$ and successively ν to make the coefficient $d_1 = 0$, we conclude

$$\Psi'(t) \le \beta\Psi + d_3(t)\Psi^3. \tag{2.15}$$

Since we have assumed $\Psi(t)$ blowing up at time $t^\star$, $\Psi(t)$ can be non decreasing, so that $\Psi(t) \ge \Psi(0)$ with $t \in [0, t^\star)$, or non increasing (possibly with some kind of oscillations), in which case there exists a time t_1 where $\Psi(t_1) = \Psi(0)$. As a consequence, $\Psi(t) \geq \Psi(0)$, $t \in [t_1, t^\star)$. It implies that

$$\Psi(t) \leq \Psi(0)^{-2}\Psi(t)^3, \quad t \in [t_1, t^\star), \tag{2.16}$$

so that (2.15) and (2.16) produce the desired differential inequality

$$\Psi'(t) \leq D(t)\Psi(t)^3, \quad t \in [t_1, t^\star), \tag{2.17}$$

with

$$D(t) = \beta\Psi(0)^{-2} + d_3(t). \tag{2.18}$$

Integrating (2.17) between t_1 and $t^\star$, the inequality

$$\frac{1}{2\Psi(0)^2} \leq \int_{t_1}^{t^\star} D(\tau)\, d\tau \leq \int_0^{t^\star} D(\tau)\, d\tau, \tag{2.19}$$

provides a lower bound for $t^\star$.

Therefore, we have proven the following

THEOREM 2.1. *Let* Ω *be a bounded domain in* $\mathbf{R}^3$; *assume* $\dfrac{k_1'(t)}{k_1(t)} \leq \beta$, $\beta \geq 0$. *If* u *is a classical solution of problem* (1.1), *with* $u = 0$ *on* $\partial\Omega$, *blowing up in* $L^{n(p-1)}$*-norm* $(n > 2)$, *then a lower bound of the blow-up time* $t^\star$ *is given by* (2.19).

3. Lower bound under Neumann boundary condition

In this section we will study problem (1.1) under homogeneous Neumann boundary conditions, $u_\nu = 0$ on $\partial\Omega \times (0, t^\star)$, corresponding to $\alpha_1 = 1$ and $\alpha_2 = 0$. As in the previous section, starting from equation (2.2), if we suppose $\dfrac{k_1'(t)}{k_1(t)} \leq \beta$, $\beta \geq 0$, thanks to the divergence theorem and the boundary condition, we lead to

$$\begin{aligned}\Psi'(t) \leq \beta\Psi - 4\frac{ns-1}{ns}k_1\int_\Omega |\nabla V^{n/2}|^2\, d\mathbf{x} \\ + nsk_1^2\int_\Omega V^{n+1}\, d\mathbf{x} - nsk_2k_1\bar{m}\int_\Omega V^{n+\alpha}\, d\mathbf{x},\end{aligned} \tag{3.1}$$

where $V = u^s$, $s = p-1$, $\bar{m} = \left(\dfrac{2\sqrt{\mu_2}}{ns+q-1}\right)^q$, μ_2 being the first positive eigenvalue of the free membrane problem

$$\begin{cases} \Delta w + \mu w = 0, & \mathbf{x} \in \Omega, \\ w_\nu = 0, & \mathbf{x} \in \partial\Omega, \\ w > 0, & \mathbf{x} \in \Omega. \end{cases}$$

As far as the term $\int_\Omega V^{n+1}\,d\mathbf{x}$ is concerned, by using Hölder inequality and (2.11) we obtain, since $n > 2$,

$$\int_\Omega V^{n+1}\,d\mathbf{x} \le \frac{1}{\mathfrak{p}}\frac{1}{\gamma^{\mathfrak{p}-1}}\int_\Omega V^{n+\alpha}\,d\mathbf{x} + \frac{1}{\mathfrak{q}}\gamma\int_\Omega V^{(3/2)n}\,d\mathbf{x}, \tag{3.2}$$

γ being a positive and time dependent function to be chosen, and where $\mathfrak{p}$ and $\mathfrak{q}$ are defined in (2.7). Now, by supposing Ω a bounded domain of $\mathbf{R}^3$ with the origin inside, star-shaped and convex in two orthogonal directions, the following Sobolev type inequality (see Lemma A.2 of [8])

$$\int_\Omega v^{(3/2)n}\,d\mathbf{x} \le \left\{\frac{3}{2\rho_0}\int_\Omega v^n\,d\mathbf{x} + \frac{n}{2}\left(1+\frac{d}{\rho_0}\right)\int_\Omega v^{n-1}|\nabla v|\,d\mathbf{x}\right\}^{3/2}, \tag{3.3}$$

valid for any nonnegative C^1-function $v(\mathbf{x})$ defined in Ω, with $n \ge 1$ and

$$\rho_0 = \min_{\partial\Omega}(\mathbf{x}\cdot\boldsymbol{\nu}) > 0 \quad \text{and} \quad d = \max_{\overline{\Omega}}|\mathbf{x}|, \tag{3.4}$$

holds. By choosing $v = V$ in (3.3) and by applying

$$(a+b)^{3/2} \le \sqrt{2}(a^{3/2}+b^{3/2}), \tag{3.5}$$

valid for $a, b > 0$, we obtain:

$$\int_\Omega V^{(3/2)n}\,d\mathbf{x} \le \sqrt{2}\left\{\left(\frac{3}{2\rho_0}\int_\Omega V^n\,d\mathbf{x}\right)^{3/2} + \left[\frac{n}{2}\left(1+\frac{d}{\rho_0}\right)\int_\Omega V^{n-1}|\nabla V|\,d\mathbf{x}\right]^{3/2}\right\}. \tag{3.6}$$

On the other hand, being $V^{n-1}|\nabla V| = \frac{2}{n}V^{n/2}|\nabla V^{n/2}|$, Hölder inequality produces

$$\begin{aligned}\left(\int_\Omega V^{n-1}|\nabla V|\,d\mathbf{x}\right)^{3/2} &\le \left(\frac{2}{n}\right)^{3/2}\left\{\left(\int_\Omega V^n\,d\mathbf{x}\right)^{3/4}\left(\int_\Omega |\nabla V^{n/2}|^2\,d\mathbf{x}\right)^{3/4}\right\}\\ &= \left(\frac{2}{n}\right)^{3/2}\left[\left(\int_\Omega V^n\,d\mathbf{x}\right)^3\right]^{1/4}\left(\int_\Omega |\nabla V^{n/2}|^2\,d\mathbf{x}\right)^{3/4}\\ &\le \left(\frac{2}{n}\right)^{3/2}\left[\frac{1}{4\zeta^3}\left(\int_\Omega V^n\,d\mathbf{x}\right)^3 + \frac{3}{4}\zeta\int_\Omega |\nabla V^{n/2}|^2\,d\mathbf{x}\right],\end{aligned} \tag{3.7}$$

ζ being another positive and time dependent function to be determined. Ultimately, (3.2), (3.6) and (3.7) into (3.1) return

$$\Psi'(t) \le \beta\Psi + n_1(t)\Psi^{3/2} + n_2(t)\Psi^3 + n_3(t)\int_\Omega |\nabla V^{n/2}|^2\,d\mathbf{x} + n_4(t)\int_\Omega V^{n+\alpha}\,d\mathbf{x}, \tag{3.8}$$

where

$$\begin{cases} n_1(t) = ns\gamma\sqrt{2k_1}\left(\dfrac{3}{2\rho_0}\right)^{3/2}\dfrac{1}{\mathfrak{q}}, \\ n_2(t) = \dfrac{ns\gamma\sqrt{2}}{4k_1\zeta^3}\left(1+\dfrac{d}{\rho_0}\right)^{3/2}\dfrac{1}{\mathfrak{q}}, \\ n_3(t) = \left[\dfrac{3\sqrt{2}}{4}nsk_1\gamma\zeta\left(1+\dfrac{d}{\rho_0}\right)^{3/2}\dfrac{1}{\mathfrak{q}} - 4\dfrac{ns-1}{ns}\right]k_1, \\ n_4(t) = nsk_1\left[\dfrac{\gamma^{1-\mathfrak{p}}}{\mathfrak{p}}k_1 - \overline{m}k_2\right]. \end{cases} \tag{3.9}$$

If in (3.9) γ is taken such that $n_4 = 0$ and successively ζ such that $n_3 = 0$, relation (3.8) is reduced to

$$\Psi'(t) \le \beta\Psi + n_1(t)\Psi^{3/2} + n_2(t)\Psi^3. \tag{3.10}$$

Since we have assumed $\Psi(t)$ blowing up at time $t^\star$, then reasoning as in Section 2 there exists a time $t_1 \in [0, t^\star)$ such that $\Psi(t) \ge \Psi(0)$, $t \in [t_1, t^\star)$. It implies that

$$\begin{cases} \Psi(t) \le \Psi(0)^{-2}\Psi(t)^3, & t \in [t_1, t^\star), \\ \Psi^{3/2}(t) \le \Psi(0)^{-3/2}\Psi(t)^3, & t \in [t_1, t^\star), \end{cases} \tag{3.11}$$

so that (3.10) and (3.11) produce the desired differential inequality

$$\Psi'(t) \le N(t)\Psi(t)^3, \quad t \in [t_1, t^\star), \tag{3.12}$$

with

$$N(t) = \Psi(0)^{-3/2}(n_1(t) + \beta\Psi(0)^{-1/2}) + n_2(t). \tag{3.13}$$

Integration (3.12) between t_1 and $t^\star$, the following inequality

$$\frac{1}{2\Psi(0)^2} \le \int_{t_1}^{t^\star} N(\tau)\,d\tau \le \int_0^{t^\star} N(\tau)\,d\tau, \tag{3.14}$$

provides a lower bound for $t^\star$.

These results are summarized in the following

THEOREM 3.1. *Let Ω be a bounded domain in $\mathbf{R}^3$, with the origin inside, star-shaped and convex in two orthogonal directions; assume* $\dfrac{k_1'(t)}{k_1(t)} \le \beta$, $\beta \ge 0$. *If u is a classical solution of problem* (1.1), *with $u_\nu = 0$ on $\partial\Omega$, blowing up in $L^{n(p-1)}$-norm* ($n > 2$), *then a lower bound of the blow up time $t^\star$ is given by* (3.14).

4. Lower bound under Robin boundary condition

In this section we will study problem (1.1) under Robin boundary conditions, $u_\nu = -\alpha_2 u$ on $\partial\Omega \times (0, t^\star)$, corresponding to $\alpha_1 = 1$ and $\alpha_2 > 0$. First we prove an inequality to be used in deriving a lower bound for $t^\star$.

LEMMA 4.1. *Let Ω be a bounded star-shaped domain of $\mathbf{R}^N$, $N \geq 2$, with the origin inside. If ξ_1 is the first positive eigenvalue of the free membrane problem*

$$\begin{cases} \Delta w + \xi w = 0, & \mathbf{x} \in \Omega, \\ w_\nu + \alpha_2 w = 0, & \mathbf{x} \in \partial\Omega,\ \alpha_2 > 0, \\ w > 0, & \mathbf{x} \in \Omega, \end{cases} \tag{4.1}$$

and the geometry of Ω is chosen such that

$$\frac{\xi_1}{\alpha_2} \geq \frac{N+d}{\rho_0}, \tag{4.2}$$

then

$$\int_\Omega |\nabla w|^2 \, d\mathbf{x} \geq \mathscr{A} \int_\Omega w^2 \, d\mathbf{x}, \tag{4.3}$$

valid for any nonnegative C^1-function $w(\mathbf{x})$ solving (4.1), *with d and ρ_0 defined in* (3.4) *and* $\mathscr{A} = \dfrac{\rho_0 \xi_1 - \alpha_2(N+d)}{\rho_0 + \alpha_2 d}$.

Proof. We have for the variational definition of ξ_1

$$\xi_1 \int_\Omega w^2 \, d\mathbf{x} \leq \int_\Omega |\nabla w|^2 \, d\mathbf{x} + \alpha_2 \int_{\partial\Omega} w^2 \, dS. \tag{4.4}$$

Regards the second term in (4.4), the following Sobolev type inequality (see Lemma A.1 of [8]) is considered:

$$\int_{\partial\Omega} u^n \, dS \leq \frac{N}{\rho_0} \int_\Omega u^n \, d\mathbf{x} + \frac{nd}{\rho_0} \int_\Omega u^{n-1} |\nabla u| \, d\mathbf{x}, \tag{4.5}$$

with d and ρ_0 given in (3.4). By using (4.5), with $u = w$ and $n = 2$, we obtain

$$\begin{aligned} \int_{\partial\Omega} w^2 \, dS &\leq \frac{N}{\rho_0} \int_\Omega w^2 \, d\mathbf{x} + \frac{2d}{\rho_0} \int_\Omega w |\nabla w| \, d\mathbf{x} \\ &\leq \frac{N}{\rho_0} \int_\Omega w^2 \, d\mathbf{x} + \frac{d}{\rho_0} \int_\Omega w^2 \, d\mathbf{x} + \frac{d}{\rho_0} \int_\Omega |\nabla w|^2 \, d\mathbf{x}, \end{aligned} \tag{4.6}$$

where in the last step both Schwarz and Young inequalities have been applied. Now, by replacing (4.6) into (4.4) we can write

$$\xi_1 \int_\Omega w^2 \, d\mathbf{x} \le \alpha_2 \frac{N+d}{\rho_0} \int_\Omega w^2 \, d\mathbf{x} + \left(1 + \frac{\alpha_2 d}{\rho_0}\right) \int_\Omega |\nabla w|^2 \, d\mathbf{x};$$

since $\alpha_2 > 0$ and (4.2) is verified, (4.3) is proven.

In order to estimate a lower bound for $t^\star$, we differentiate $\Psi(t)$ defined in (2.1), obtaining one more time (2.2). Let us set $s = p - 1$; by the identity (2.3), the divergence theorem and the boundary condition

$$\begin{aligned} \int_\Omega u^{ns-1} \Delta u \, d\mathbf{x} &= -\alpha_2 \int_{\partial\Omega} u^{ns} \, dS - (ns-1) \int_\Omega u^{ns-2} |\nabla u|^2 \, d\mathbf{x} \\ &\le -4 \frac{ns-1}{(ns)^2} \int_\Omega |\nabla u^{ns/2}|^2 \, d\mathbf{x}, \end{aligned} \tag{4.7}$$

where in the last step we have dropped the term $-\alpha_2 \int_{\partial\Omega} u^{ns} \, dS$ and set, as before, $V = u^s$. On the other hand, using Hölder and (2.11) inequalities, we achieve as before

$$\int_\Omega V^{n+1} \, d\mathbf{x} \le \frac{1}{\mathfrak{p}} \delta \int_\Omega V^{n+\alpha} \, d\mathbf{x} + \frac{1}{\mathfrak{q}} \frac{1}{\delta^{1/(\mathfrak{p}-1)}} \int_\Omega V^{(3/2)n} \, d\mathbf{x}, \tag{4.8}$$

δ being a positive and time dependent function to be chosen, and where $\mathfrak{p}$ and $\mathfrak{q}$ are defined in (2.7). If Ω is a bounded domain of $\mathbf{R}^3$ with the origin inside, star-shaped and convex in two orthogonal directions, relations (3.5), (3.6) and (3.7) return

$$\begin{aligned} \int_\Omega V^{(3/2)n} \, d\mathbf{x} \le{}& \sqrt{2} \left(\frac{3}{2\rho_0}\right)^{3/2} \left(\int_\Omega V^n \, d\mathbf{x}\right)^{3/2} + \frac{\sqrt{2}}{4\sigma^3} \left(1 + \frac{d}{\rho_0}\right)^{3/2} \left(\int_\Omega V^n \, d\mathbf{x}\right)^3 \\ &+ \frac{3\sqrt{2}}{4} \sigma \left(1 + \frac{d}{\rho_0}\right)^{3/2} \int_\Omega |\nabla V^{n/2}|^2 \, d\mathbf{x}, \end{aligned}$$

σ being another positive and time dependent function to be computed, and with ρ_0 and d as in (3.4). By replacing this inequality into (4.8), we can write

$$\begin{aligned} \int_\Omega V^{n+1} \, d\mathbf{x} \le{}& \frac{1}{\mathfrak{p}} \delta \int_\Omega V^{n+\alpha} \, d\mathbf{x} + \frac{1}{\mathfrak{q}} \frac{1}{\delta^{1/(\mathfrak{p}-1)}} \sqrt{2} \left(\frac{3}{2\rho_0}\right)^{3/2} \left(\int_\Omega V^n \, d\mathbf{x}\right)^{3/2} \\ &+ \frac{1}{\mathfrak{q}} \frac{1}{\delta^{1/(\mathfrak{p}-1)}} \frac{\sqrt{2}}{4\sigma^3} \left(1 + \frac{d}{\rho_0}\right)^{3/2} \left(\int_\Omega V^n \, d\mathbf{x}\right)^3 \\ &+ \frac{1}{\mathfrak{q}} \frac{1}{\delta^{1/(\mathfrak{p}-1)}} \frac{3\sqrt{2}}{4} \sigma \left(1 + \frac{d}{\rho_0}\right)^{3/2} \int_\Omega |\nabla V^{n/2}|^2 \, d\mathbf{x}. \end{aligned} \tag{4.9}$$

With reference to the term $\int_\Omega u^{ns-1}|\nabla u|^q\,d\mathbf{x}$, let us apply (4.3) of Lemma 4.1 with $w = u^{(ns+q-1)/2}$; by setting $z = ns + q - 1$, we obtain

$$\mathscr{A}\int_\Omega u^z\,d\mathbf{x} \le \int_\Omega |\nabla u^{z/2}|^2\,d\mathbf{x}. \tag{4.10}$$

On the other hand, $|\nabla u^{z/2}|^2 = \left(\frac{q}{2}\right)^2 u^{(q-2)(z/q)}|\nabla u^{z/q}|^2$, so that, using Hölder inequality for $q > 2$,

$$\int_\Omega |\nabla u^{z/2}|^2\,d\mathbf{x} \le \left(\frac{q}{2}\right)^2\left(\int_\Omega u^z\,d\mathbf{x}\right)^{(q-2)/q}\left(\int_\Omega |\nabla u^{z/q}|^q\,d\mathbf{x}\right)^{2/q}. \tag{4.11}$$

Therefore, (4.10) and (4.11) allow us to write

$$\mathscr{A}\int_\Omega u^z\,d\mathbf{x} \le \left(\frac{q}{2}\right)^2\left(\int_\Omega u^z\,d\mathbf{x}\right)^{(q-2)/q}\left(\int_\Omega |\nabla u^{z/q}|^q\,d\mathbf{x}\right)^{2/q},$$

i.e.,

$$\mathscr{A}^{q/2}\left(\frac{2}{q}\right)^q\int_\Omega u^z\,d\mathbf{x} \le \int_\Omega |\nabla u^{z/q}|^q\,d\mathbf{x}. \tag{4.12}$$

Now, arguing as in (2.3) and applying (4.12), we lead to

$$\begin{aligned}\int_\Omega u^{ns-1}|\nabla u|^q\,d\mathbf{x} &= \left(\frac{q}{z}\right)^q\int_\Omega |\nabla u^{z/q}|^q\,d\mathbf{x} \ge \left(\frac{q}{z}\right)^q\mathscr{A}^{q/2}\left(\frac{2}{q}\right)^q\int_\Omega u^z\,d\mathbf{x}\\ &= \tilde{m}\int_\Omega u^z\,d\mathbf{x} = \tilde{m}\int_\Omega V^{n+\alpha}\,d\mathbf{x},\end{aligned} \tag{4.13}$$

with $\tilde{m} = \left(\frac{2}{z}\right)^q\mathscr{A}^{q/2}$ and $\alpha = \frac{q-1}{s}$. If k_1 is such that $\frac{k_1'(t)}{k_1(t)} \le \beta$, with $\beta \ge 0$, using (4.7), (4.9) and (4.13) into (2.2) we obtain

$$\Psi' \le \beta\Psi + r_1(t)\Psi^{3/2} + r_2(t)\Psi^3 + r_3(t)\int_\Omega V^{n+\alpha}\,d\mathbf{x} + r_4(t)\int_\Omega |\nabla V^{n/2}|^2\,d\mathbf{x}, \tag{4.14}$$

where

$$\begin{cases} r_1(t) = \sqrt{2k_1}\left(\dfrac{3}{2\rho_0}\right)^{3/2} ns\dfrac{1}{\mathfrak{q}\delta^{1/(\mathfrak{p}-1)}},\\[2mm] r_2(t) = \dfrac{ns\sqrt{2}}{4k_1\mathfrak{q}\sigma^3}\,\dfrac{1}{\delta^{1/(\mathfrak{p}-1)}}\left(1+\dfrac{d}{\rho_0}\right)^{3/2},\\[2mm] r_3(t) = \left(\dfrac{\delta k_1}{\mathfrak{p}} - k_2\tilde{m}\right)nsk_1,\\[2mm] r_4(t) = \left[\dfrac{3\sqrt{2}}{4\mathfrak{q}}\,\dfrac{1}{\delta^{1/(\mathfrak{p}-1)}}\sigma\left(1+\dfrac{d}{\rho_0}\right)^{3/2}k_1 - 4\dfrac{ns-1}{(ns)^2}\right]nsk_1. \end{cases} \tag{4.15}$$

If in (4.15) δ is taken such that $r_3 = 0$ and successively σ such that $r_4 = 0$, (4.14) is reduced to

$$\Psi'(t) \le \beta\Psi + r_1(t)\Psi^{3/2} + r_2(t)\Psi^3. \tag{4.16}$$

Following the same steps used in the previous sections for the Dirichlet and Neumann problems, we obtain the desired differential inequality

$$\Psi'(t) \le R(t)\Psi(t)^3, \quad t \in [t_1, t^\star), \tag{4.17}$$

with

$$R(t) = \Psi(0)^{-3/2}(r_1(t) + \beta\Psi(0)^{-1/2}) + r_2(t). \tag{4.18}$$

Integrating (4.17) between t_1 and $t^\star$, the following inequality

$$\frac{1}{2\Psi(0)^2} \le \int_{t_1}^{t^\star} R(\tau)\, d\tau \le \int_0^{t^\star} R(\tau)\, d\tau, \tag{4.19}$$

provides a lower bound for $t^\star$.

Therefore the following theorem is proven:

THEOREM 4.1. *Let Ω be a bounded domain in $\mathbf{R}^3$, with the origin inside, star-shaped and convex in two orthogonal directions; assume $\dfrac{k_1'(t)}{k_1(t)} \le \beta$, $\beta \ge 0$. If u is a classical solution of problem* (1.1), *with $u_\nu + \alpha_2 u = 0$ on $\partial\Omega$ and $p > q > 2$, blowing up in $L^{n(p-1)}$-norm* ($n > 2$), *then a lower bound of the blow up time $t^\star$ is given by* (4.19).

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REFERENCES

[1] M. MARRAS AND S. VERNIER PIRO, Bounds for blow-up time in nonlinear parabolic system, Discrete Contin. Dyn. Syst., Suppl. (2011), 1025–1031.

[2] M. MARRAS AND S. VERNIER PIRO, Blow-up phenomena in reaction-diffusion systems, Discrete Contin. Dyn. Syst. **32** (2012), 4001–4014.

[3] M. MARRAS AND S. VERNIER PIRO, On global existence and bounds for blow-up time in nonlinear parabolic problems with time dependent coefficients, Discrete Contin. Dyn. Syst., Suppl. (2013), 535–544.

[4] L. E. PAYNE AND G. PHILIPPIN, Blow-up phenomena in parabolic problems with time dependent coefficients under Dirichlet boundary conditions, Proc. Amer. Math. Soc. **141** (2013), 2309–2318.

[5] L. E. PAYNE AND P. W. SCHAEFER, Lower bound for blow-up time in parabolic problems under Neumann conditions, Applicable Analysis **85** (2006), 1301–1311.

[6] L. E. Payne, G. A. Philippin and P. W. Schaefer, Blow-up phenomena for some nonlinear parabolic problems, Nonlinear Analysis **69** (2008), 3495–3502.
[7] L. E. Payne, G. A. Philippin and S. Vernier-Piro, Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition I, Z. Angew. Math. Phys. **61** (2010), 971–978.
[8] L. E. Payne, G. A. Philippin and S. Vernier-Piro, Blow-up phenomena for a semilinear heat equation with nonlinear boundary condition II, Nonlinear Analysis **73** (2010), 971–978.
[9] L. E. Payne and J. C. Song, Lower bound for blow-up time in a nonlinear parabolic problem, J. Math. Anal. Appl. **354** (2009), 394–396.
[10] Ph. Souplet, Finite time blowup for a nonlinear parabolic equation with a gradient term and applications, Math. Methods Appl. Sci. **19** (1996), 1317–1333.
[11] P. Quittner and Ph. Souplet, Superlinear parabolic problems, Blow-up, global existence and steady states, Birkhäuser advanced texts, Basler Lehrbücher, Birkhäuser Verlag, Basel, Switzerland, 2007.
[12] G. Talenti, Best constant in Sobolev inequality, Ann. Mat. Pura Appl. **110** (1976), 353–372.

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