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INTERACTION BETWEEN FAST DIFFUSION AND GEOMETRY OF DOMAIN*

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Abstract

Let Ω be a domain in $\mathbf{R}^N$, where $N \geq 2$ and $\partial\Omega$ is not necessarily bounded. We consider two fast diffusion equations $\partial_t u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $\partial_t u = \Delta u^m$, where $1 < p < 2$ and $0 < m < 1$. Let $u = u(x,t)$ be the solution of either the initial-boundary value problem over Ω, where the initial value equals zero and the boundary value is a positive continuous function, or the Cauchy problem where the initial datum equals a nonnegative continuous function multiplied by the characteristic function of the set $\mathbf{R}^N \setminus \Omega$. Choose an open ball B in Ω whose closure intersects $\partial\Omega$ only at one point, and let $\alpha > \dfrac{(N+1)(2-p)}{2p}$ or $\alpha > \dfrac{(N+1)(1-m)}{4}$. Then, we derive asymptotic estimates for the integral of u^α over B for short times in terms of principal curvatures of $\partial\Omega$ at the point, which tells us about the interaction between fast diffusion and geometry of domain.

1. Introduction

Let Ω be a domain in $\mathbf{R}^N$, where $N \geq 2$ and $\partial\Omega$ is not necessarily bounded. We consider two fast diffusion equations of the forms $\partial_t u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $\partial_t u = \Delta u^m$, where $1 < p < 2$ and $0 < m < 1$. Let $f \in C^0(\partial\Omega)$ be a function satisfying

$$0 < c_1 \leq f(x) \leq c_2 \quad (x \in \partial\Omega) \tag{1.1}$$

for two positive constants c_1 and c_2, and let $g \in C^0(\mathbf{R}^N)$ be a function satisfying

$$0 \leq g(x) \leq c_3 \quad (x \in \mathbf{R}^N) \tag{1.2}$$

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for a positive constant c_3. Consider the bounded solution $u = u(x,t)$ of either the initial-boundary value problem:

$$\partial_t u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{in } \Omega \times (0,\infty), \tag{1.3}$$

$$u = f \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.4}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}, \tag{1.5}$$

or the Cauchy problem:

$$\partial_t u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{in } \mathbf{R}^N \times (0,\infty) \quad \text{and} \quad u = g\mathscr{X}_{\Omega^c} \quad \text{on } \mathbf{R}^N \times \{0\}, \tag{1.6}$$

where $\mathscr{X}_{\Omega^c}$ is the characteristic function of the set $\Omega^c = \mathbf{R}^N \backslash \Omega$. The first theorem tells us about the interaction between fast diffusion and geometry of domain for $\partial_t u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

THEOREM 1.1. *Let u be the solution of either problem* (1.3)–(1.5) *or problem* (1.6). *Let* $\alpha > \dfrac{(N+1)(2-p)}{2p}$ *and* $x_0 \in \Omega$. *Assume that the open ball $B_R(x_0)$ centered at x_0 and with radius $R > 0$ is contained in Ω and such that $\overline{B_R(x_0)} \cap \partial\Omega = \{y_0\}$ for some $y_0 \in \partial\Omega$ and $\partial\Omega \cap B_\delta(y_0)$ is of class C^2 for some $\delta > 0$. Suppose that $g(y_0) > 0$ for problem* (1.6). *Then we have*:

$$\lim_{t\to 0^+} t^{-(N+1)/2p} \int_{B_R(x_0)} (u(x,t))^\alpha \, dx = c\left\{\prod_{j=1}^{N-1}\left[\frac{1}{R} - \kappa_j(y_0)\right]\right\}^{-1/2}. \tag{1.7}$$

Here, $\kappa_1(y_0), \dots, \kappa_{N-1}(y_0)$ denote the principal curvatures of $\partial\Omega$ at y_0 with respect to the inward normal direction to $\partial\Omega$ and c is a positive constant depending only on p, α, N, and either $f(y_0)$ or $g(y_0)$. When $\kappa_j(y_0) = \dfrac{1}{R}$ for some $j \in \{1,\dots,N-1\}$, the formula (1.7) *holds by setting the right-hand side to ∞* (*notice that $\kappa_j(y_0) \le \dfrac{1}{R}$ for every $j \in \{1,\dots,N-1\}$*).

Concerning $\partial_t u = \Delta u^m$ with $0 < m < 1$, let $u = u(x,t)$ be the bounded nonnegative solution of either the initial-boundary value problem:

$$\partial_t u = \Delta u^m \quad \text{in } \Omega \times (0,\infty), \tag{1.8}$$

$$u = f \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.9}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}, \tag{1.10}$$

or the Cauchy problem:

$$\partial_t u = \Delta u^m \quad \text{in } \mathbf{R}^N \times (0,\infty) \quad \text{and} \quad u = g\mathscr{X}_{\Omega^c} \quad \text{on } \mathbf{R}^N \times \{0\}. \tag{1.11}$$

The second theorem tells us about the interaction between fast diffusion and geometry of domain for $\partial_t u = \Delta u^m$.

THEOREM 1.2. *Let u be the solution of either problem* (1.8)–(1.10) *or problem* (1.11). *Let* $\alpha > \dfrac{(N+1)(1-m)}{4}$ *and* $x_0 \in \Omega$. *Assume that the open ball* $B_R(x_0)$ *centered at* x_0 *and with radius* $R > 0$ *is contained in* Ω *and such that* $\overline{B_R(x_0)} \cap \partial\Omega = \{y_0\}$ *for some* $y_0 \in \partial\Omega$ *and* $\partial\Omega \cap B_\delta(y_0)$ *is of class* C^2 *for some* $\delta > 0$. *Suppose that* $g(y_0) > 0$ *for problem* (1.11). *Then we have*:

$$\lim_{t\to 0^+} t^{-(N+1)/4} \int_{B_R(x_0)} (u(x,t))^\alpha \, dx = c\left\{\prod_{j=1}^{N-1}\left[\frac{1}{R} - \kappa_j(y_0)\right]\right\}^{-1/2}. \tag{1.12}$$

Here, $\kappa_1(y_0), \dots, \kappa_{N-1}(y_0)$ *denote the principal curvatures of* $\partial\Omega$ *at* y_0 *with respect to the inward normal direction to* $\partial\Omega$ *and* c *is a positive constant depending only on* m, α, N, *and either* $f(y_0)$ *or* $g(y_0)$. *When* $\kappa_j(y_0) = \dfrac{1}{R}$ *for some* $j \in \{1, \dots, N-1\}$, *the formula* (1.12) *holds by setting the right-hand side to* ∞.

When $p > 2$, $m > 1$, $\alpha = 1$, and $f \equiv g \equiv 1$, the same formulas (1.7) and (1.12) were obtained for problems (1.3)–(1.5) and (1.8)–(1.10) in [MS1]. With the aid of the techniques employed in [MS3], one can easily see that the formulas (1.7) and (1.12) also hold true for problems (1.6) and (1.11). Moreover, in [MS3], the nonlinear diffusion equation of the form $\partial_t u = \Delta\phi(u)$ where $\delta_1 \le \phi'(s) \le \delta_2$ $(s \in \mathbf{R})$ for some positive constants δ_1 and δ_2 was also dealt with. By a little more observation, we see that any $\alpha > 0$ is OK for these cases.

In Theorems 1.1 and 1.2, if p is close to 1 or if $N \ge 4$ and m is close to 0, then $\alpha = 1$ can not be chosen. Indeed, when $\alpha = \dfrac{(N+1)(2-p)}{2p}$ or $\alpha = \dfrac{(N+1)(1-m)}{4}$, $c = \infty$.

The main ingredients of the proofs of the formulas (1.7) and (1.12) consist of two steps. One is the reduction to the case where $\partial\Omega$ is bounded and of class C^2, and where both f and g are constant, with the aid of the comparison principle. The other is the construction of appropriate super- and subsolutions to the problems near $\partial\Omega$ in a short time. In fact, in [MS1], such barriers were constructed in a set $\Omega_\rho \times (0, \tau]$, with

$$\Omega_\rho = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \rho\}, \tag{1.13}$$

where ρ and τ were chosen sufficiently small. When $p > 2$ or $m > 1$, the property of finite speed of propagation of disturbances from rest yields that both the solution u and the barriers equal zero on $\Gamma_\rho \times (0, \tau]$, where

$$\Gamma_\rho = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) = \rho\}. \tag{1.14}$$

This property does not occur when $1 < p < 2$ or $0 < m < 1$, because of the property of *infinite* speed of propagation of disturbances from rest. Also in

[MS3], the equation $\partial_t u = \Delta\phi(u)$ has the property of infinite speed of propagation of disturbances from rest. To compare the solution with the barriers on $\Gamma_\rho \times (0, \tau]$, in [MS3], the result of Atkinson and Peletier [AP] concerning the asymptotic behavior of one-dimensional similarity solutions and the following short time behavior of u obtained by [MS2] play a key role:

$$\lim_{t\to 0^+} -4t\Phi(u) = \operatorname{dist}(x, \partial\Omega)^2 \text{ uniformly on every compact subset of } \Omega, \tag{1.15}$$

where the function Φ is defined by

$$\Phi(s) = \int_1^s \frac{\phi'(\xi)}{\xi}\, d\xi \quad \text{for } s > 0. \tag{1.16}$$

However, when $1 < p < 2$ or $0 < m < 1$, the short time behavior of u is not controlled by the distance function in such a way. To overcome this difficulty in the proofs of Theorems 1.1 and 1.2, we use the fact that the short time behavior of the solution u is described by the boundary blow-up solutions given in [M, BM]. The results of the present paper in the case where $f \equiv g \equiv 1$ were announced in [S].

The present paper is organized as follows. Section 2 is devoted to some preliminaries; the definitions of bounded solutions are mentioned, the regularity results for the solutions are quoted from the references, and we refer to the references for the comparison principles. Throughout the following four sections the comparison principles, which are mentioned in Section 2, play a key role. In Section 3, it is shown that the short time behavior of the solutions is described by the boundary blow-up solutions given in [M, BM] in the case where $\partial\Omega$ is bounded and of class C^2 and where both f and g are positive constants. In Section 4, the problems are reduced to the case where $\partial\Omega$ is bounded and of class C^2 and where both f and g are positive constants. Sections 5 and 6 are devoted to the construction of super- and subsolutions near the boundary $\partial\Omega$ for short times in the p-Laplace case and in the porous medium type case, respectively. In Section 7 we prove Theorems 1.1 and 1.2.

2. Prelimiaries: bounded solutions, regularity and comparison principles

Let us first consider the equation $\partial_t u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $1 < p < 2$. By a bounded solution u of problem (1.3)–(1.5) we mean that $u \in C^0(\overline{\Omega} \times (0, \infty)) \cap L^p_{loc}(0, \infty; W^{1,p}_{loc}(\Omega)) \cap L^\infty(\Omega \times (0, \infty))$ satisfies (1.3) in the weak sense and $u(\cdot, t) \to 0$ in $L^1_{loc}(\Omega)$ as $t \to 0^+$, and by a bounded solution u of problem (1.6) we mean that $u \in C_{loc}(0, \infty; L^2_{loc}(\mathbf{R}^N)) \cap L^p_{loc}(0, \infty; W^{1,p}_{loc}(\mathbf{R}^N)) \cap L^\infty(\mathbf{R}^N \times (0, \infty))$ satisfies the differential equation in the weak sense and $u(\cdot, t) \to g(\cdot)\mathscr{X}_{\Omega^c}(\cdot)$ in $L^1_{loc}(\mathbf{R}^N)$ as $t \to 0^+$.

It is known that such bounded solutions u together with ∇u are locally Hölder continuous, and both boundary and initial regularity of such solutions are known. See [DiB, DiBGV, L]. Moreover, it is shown in [BIV, Corollary 2.1, p. 2159] that such solutions are local strong ones, more precisely $\partial_t u \in L^2_{loc}$.

The comparison principle for such strong solutions is obtained by Kurta [K1, K2] for both the initial-boundary value problem and the Cauchy problem. Furthermore, note that one can easily prove Kurta's comparison principle also for bounded weak solutions by taking his testing function modulo a Steklov time averaging process. See [DiB, DiBGV] for the process, and see also [DiBGV, Corollary 1.1, p. 189] for the comparison principle for weak solutions of the initial-boundary value problem over bounded domains.

Let us next consider the porous medium type equation $\partial_t u = \Delta u^m$ with $0 < m < 1$. By a bounded nonnegative solution u of problem (1.8)–(1.10) we mean that $u \in C^0(\overline{\Omega} \times (0, \infty)) \cap L^\infty(\Omega \times (0, \infty))$ is nonnegative and satisfies (1.8) in the weak sense and $u(\cdot, t) \to 0$ in $L^1_{loc}(\Omega)$ as $t \to 0^+$, and by a bounded nonnegative solution u of problem (1.11) we mean that $u \in C_{loc}(0, \infty; L^2_{loc}(\mathbf{R}^N)) \cap L^\infty(\mathbf{R}^N \times (0, \infty))$ is nonnegative and satisfies the differential equation in the weak sense and $u(\cdot, t) \to g(\cdot)\mathscr{X}_{\Omega^c}(\cdot)$ in $L^1_{loc}(\mathbf{R}^N)$ as $t \to 0^+$.

It is known that such bounded solutions u are locally Hölder continuous, and both boundary and initial regularity of such solutions are known. See [DiB, DiBGV].

The comparison principle for such solutions of both the initial-boundary value problem and the Cauchy problem can be easily proved by modifying the proofs of [MS3, Theorem A.1, pp. 253–257] and [BKP, Proposition A, pp. 1006–1008], with the aid of an idea of Dahlberg and Kenig [DaK, Lemma 2.3, pp. 271–273] which circumvents the singularity coming from u^m with $0 < m < 1$ at $u = 0$. See also [DiBGV, Corollary 5.1, p. 201] for the comparison principle for weak solutions of the initial-boundary value problem over bounded domains.

3. Initial behavior and boundary blow-up solutions

Let Ω be a domain in $\mathbf{R}^N$ where $\partial\Omega$ is bounded and of class C^2. Then it is known that there exists a unique solution $v \in W^{1,p}_{loc}(\Omega)$ of

$$\text{(3.1)} \qquad \operatorname{div}(|\nabla v|^{p-2}\nabla v) = \frac{1}{2-p} v \quad \text{and } v > 0 \text{ in } \Omega,$$

$$\text{(3.2)} \qquad v(x) \to \infty \qquad \text{as } x \to \partial\Omega,$$

$$\text{(3.3)} \qquad v(x) \to 0 \qquad \text{as } |x| \to \infty \text{ provided } \Omega \text{ is unbounded.}$$

Here, v belongs to $C^1(\Omega)$ and ∇v is locally Hölder continuous in Ω, and moreover

$$\text{(3.4)} \qquad \frac{v(x)}{d(x)^{-p/(2-p)}} \to c(p) \quad \text{as } d(x) \to 0 \text{ uniformly in } \Omega,$$

where

$$\text{(3.5)} \qquad d(x) = \operatorname{dist}(x, \partial\Omega) \quad \text{for } x \in \Omega \quad \text{and} \quad c(p) = \frac{2-p}{p}\left(\frac{2-p}{2p(p-1)}\right)^{-1/(2-p)}.$$

The case where Ω is bounded was proved in [M, Theorem 6.4 and Corollary 4.5, p. 245 and p. 231] and the case where Ω is unbounded, that is, Ω is an exterior domain, the existence of v can be obtained with the aid of the argument in [BM, 1.6, p. 12], and the uniqueness also follows by virtue of (3.3).

Also, it is known by [BM, Theorem 2.7, pp. 18–19] that there exists a unique solution $w \in C^2(\Omega)$ of

$$\Delta w^m = \frac{1}{1-m} w \quad \text{and } w > 0 \text{ in } \Omega, \tag{3.6}$$

$$w(x) \to \infty \quad \text{as } x \to \partial\Omega, \tag{3.7}$$

$$w(x) \to 0 \quad \text{as } |x| \to \infty \text{ provided } \Omega \text{ is unbounded.} \tag{3.8}$$

Note that in [BM] the function $w(x)^m$ is dealt with instead of $w(x)$. Moreover,

$$\frac{w(x)}{d(x)^{-2/(1-m)}} \to c(m) \quad \text{as } d(x) \to 0 \text{ uniformly in } \Omega, \tag{3.9}$$

where

$$c(m) = \left(\frac{2m(1+m)}{1-m}\right)^{1/(1-m)}. \tag{3.10}$$

See [BM, Theorem 2.3, p. 17] or [M, Corollary 4.5, p. 231] for (3.9).

Proposition 3.1. *Assume that $\partial\Omega$ is bounded and of class C^2. Let u be the solution of either problem* (1.3)–(1.5) *or problem* (1.6) *where both f and g are positive constants. Then*

$$t^{-1/(2-p)} u(x,t) \to v(x) \quad \text{as } t \to 0^+ \text{ uniformly on compact sets in } \Omega, \tag{3.11}$$

and moreover

$$u(x,t) \le t^{1/(2-p)} v(x) \quad \text{in } \Omega \times (0,\infty), \tag{3.12}$$

where v is the solution of problem (3.1)–(3.3).

Proof. Define the function $V = V(x,t)$ for $(x,t) \in \Omega \times (0,\infty)$ by

$$V(x,t) = t^{1/(2-p)} v(x).$$

Then V solves

$$V > 0 \quad \text{and} \quad \partial_t V = \operatorname{div}(|\nabla V|^{p-2} \nabla V) \quad \text{in } \Omega \times (0,\infty), \tag{3.13}$$

$$V = \infty \quad \text{on } \partial\Omega \times (0,\infty). \tag{3.14}$$

Therefore it follows from the comparison principle that

$$u \le V \quad \text{in } \Omega \times (0,\infty),$$

which gives (3.12).

Since $\partial\Omega$ is bounded and of class C^2, there exists a number $\varepsilon_0 > 0$ such that, for each $\varepsilon \in (0, \varepsilon_0]$, the set Ω^ε defined by

$$\Omega^\varepsilon = \{x \in \mathbf{R}^N : \mathrm{dist}(x, \overline{\Omega}) < \varepsilon\} \tag{3.15}$$

is also a domain with bounded C^2 boundary $\partial\Omega^\varepsilon$. To distinguish the notation Ω^ε from the complement $\Omega^c = \mathbf{R}^N \backslash \Omega$, hereafter we never use the letter "c" for this definition (3.15). For each $\varepsilon \in (0, \varepsilon_0)$, consider the boundary blow-up solution $v_\varepsilon \in C^1(\Omega^\varepsilon)$ of

$$\mathrm{div}(|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon) = \frac{1}{2-p} v_\varepsilon \quad \text{and } v_\varepsilon > 0 \text{ in } \Omega^\varepsilon, \tag{3.16}$$

$$v_\varepsilon(x) \to \infty \qquad \text{as } x \to \partial\Omega^\varepsilon, \tag{3.17}$$

$$v_\varepsilon(x) \to 0 \qquad \text{as } |x| \to \infty \text{ provided } \Omega^\varepsilon \text{ is unbounded.} \tag{3.18}$$

In view of the argument in [M, Proof of Theorem 4.4, pp. 239–240], we observe that $\mathrm{dist}(x, \partial\Omega^\varepsilon) = \mathrm{dist}(x, \partial\Omega) + \varepsilon$ for $x \in \Omega$ and there exists $r > 0$ independent of ε such that Ω^ε satisfies the uniform interior and exterior ball condition with radius r for all $\varepsilon \in (0, \varepsilon_0]$, and we see that

$$v_\varepsilon \to v \quad \text{as } \varepsilon \to 0^+ \text{ uniformly on compact sets in } \Omega. \tag{3.19}$$

Define the function $V_\varepsilon = V_\varepsilon(x, t)$ for $(x, t) \in \Omega^\varepsilon \times (0, \infty)$ by

$$V_\varepsilon(x, t) = t^{1/(2-p)} v_\varepsilon(x).$$

Then, for each $\varepsilon \in (0, \varepsilon_0]$, V_ε solves

$$V_\varepsilon > 0 \quad \text{and} \quad \partial_t V_\varepsilon = \mathrm{div}(|\nabla V_\varepsilon|^{p-2}\nabla V_\varepsilon) \quad \text{in } \Omega^\varepsilon \times (0, \infty), \tag{3.20}$$

$$V_\varepsilon = \infty \qquad \text{on } \partial\Omega^\varepsilon \times (0, \infty), \tag{3.21}$$

$$V_\varepsilon = 0 \qquad \text{in } \overline{\Omega^{\varepsilon/2}}. \tag{3.22}$$

Hence, for each $\varepsilon \in (0, \varepsilon_0]$, there exists $t_\varepsilon > 0$ such that

$$V_\varepsilon \le u \begin{cases} \text{on } \partial\Omega \times (0, t_\varepsilon] \text{ if } u \text{ solves problem (1.3)–(1.5)}, \\ \text{on } \partial\Omega^{\varepsilon/2} \times (0, t_\varepsilon] \text{ if } u \text{ solves problem (1.6)}, \end{cases}$$

since both f and g are positive constants and both $\partial\Omega$ and $\partial\Omega^{\varepsilon/2}$ are compact sets in Ω^ε. Thus, we have from the comparison principle

$$V_\varepsilon \le u \quad \text{in } \Omega \times (0, t_\varepsilon],$$

which together with (3.12) concludes that

$$v_\varepsilon(x) \le t^{-1/(2-p)} u(x, t) \le v(x) \quad \text{for every } (x, t) \in \Omega \times (0, t_\varepsilon].$$

Therefore (3.11) follows from (3.19). $\square$

PROPOSITION 3.2. *Assume that $\partial\Omega$ is bounded and of class C^2. Let u be the solution of either problem* (1.8)–(1.10) *or problem* (1.11) *where both f and g are positive constants. Then*

$$t^{-1/(1-m)}u(x,t) \to w(x) \quad \text{as } t \to 0^+ \text{ uniformly on compact sets in } \Omega, \tag{3.23}$$

and moreover

$$u(x,t) \le t^{1/(1-m)}w(x) \quad \text{in } \Omega \times (0,\infty), \tag{3.24}$$

where w is the solution of problem (3.6)–(3.8).

Proof. This follows from the same argument as in the proof of Proposition 3.1. □

4. Reduction to the case where $\partial\Omega$ is bounded and of class C^2 and where both f and g are positive constants

Let us first consider the solution u of problem (1.3)–(1.5). Let $\alpha > \dfrac{(N+1)(2-p)}{2p}$, $x_0 \in \Omega$, and assume that $B_R(x_0)$ is contained in Ω and such that $\overline{B_R(x_0)} \cap \partial\Omega = \{y_0\}$ for some $y_0 \in \partial\Omega$ and $\partial\Omega \cap B_\delta(y_0)$ is of class C^2 for some $\delta > 0$. We find a bounded C^2 domain Ω_* satisfying

$$B_R(x_0) \subset \Omega_* \subset \Omega, \quad \overline{B_R(x_0)} \cap \partial\Omega_* = \{y_0\}, \quad \text{and}$$
$$B_{\delta/2}(y_0) \cap \partial\Omega \subset \partial\Omega_* \cap \partial\Omega \subset B_\delta(y_0) \cap \partial\Omega.$$

Let $\hat{u} = \hat{u}(x,t)$ be the bounded solution of the initial-boundary value problem:

$$\partial_t \hat{u} = \operatorname{div}(|\nabla\hat{u}|^{p-2}\nabla\hat{u}) \quad \text{in } \Omega_* \times (0,\infty), \tag{4.1}$$

$$\hat{u} = \max f \quad \text{on } \partial\Omega_* \times (0,\infty), \tag{4.2}$$

$$\hat{u} = 0 \quad \text{on } \Omega_* \times \{0\}. \tag{4.3}$$

Then by the comparison principle we have

$$u \le \hat{u} \quad \text{in } \Omega_* \times (0,\infty). \tag{4.4}$$

Take a small $\varepsilon > 0$ arbitrarily. Choose a function $\hat{f}_\varepsilon \in C^2(\partial\Omega_*)$ satisfying

$$\hat{f}_\varepsilon(y_0) = f(y_0) + \frac{\varepsilon}{2}, \quad \hat{f}_\varepsilon = \max f + \frac{\varepsilon}{2} \quad \text{on } \Omega \cap \partial\Omega_*, \quad \text{and} \tag{4.5}$$
$$\hat{f}_\varepsilon \ge f \quad \text{on } \partial\Omega \cap \partial\Omega_*.$$

Let $\hat{v}_\varepsilon \in C^1(\overline{\Omega_*})$ solve

$$0 = \operatorname{div}(|\nabla\hat{v}_\varepsilon|^{p-2}\nabla\hat{v}_\varepsilon) \quad \text{in } \Omega_*, \tag{4.6}$$

$$\hat{v}_\varepsilon = \hat{f}_\varepsilon \quad \text{on } \partial\Omega_*. \tag{4.7}$$

Then by the comparison principle we have

$$u \le \hat{v}_\varepsilon \quad \text{in } \Omega_* \times (0, \infty). \tag{4.8}$$

Moreover, we can find a small number $\delta_\varepsilon \in (0, \delta/9)$ and two C^2 domains $\Omega_{+,\varepsilon}$ and $\Omega_{-,\varepsilon}$ having bounded C^2 boundaries with the following properties: both $\Omega_{+,\varepsilon}$ and $\mathbf{R}^N \backslash \overline{\Omega_{-,\varepsilon}}$ are bounded; $\mathbf{R}^N \backslash \overline{\Omega_{-,\varepsilon}} \subset B_{3\delta_\varepsilon}(y_0)$; $B_R(x_0) \subset \Omega_{+,\varepsilon} \subset \Omega_* \subset \Omega \subset \Omega_{-,\varepsilon}$; $\overline{B_R(x_0)} \cap \partial\Omega_{+,\varepsilon} = \overline{B_R(x_0)} \cap \partial\Omega_{-,\varepsilon} = \{y_0\}$; $\partial\Omega_{+,\varepsilon} \cap \partial\Omega_* \subset B_{2\delta_\varepsilon}(y_0) \cap \partial\Omega$;

$$B_{\delta_\varepsilon}(y_0) \cap \partial\Omega \subset \partial\Omega_{\pm,\varepsilon} \cap \partial\Omega \subset B_{2\delta_\varepsilon}(y_0) \cap \partial\Omega (\subset \partial\Omega_* \cap \partial\Omega);$$

$$f(y_0) - \varepsilon \le f \quad \text{on } \overline{B_{4\delta_\varepsilon}(y_0)} \cap \partial\Omega \quad \text{and} \quad \hat{v}_\varepsilon \le f(y_0) + \varepsilon \quad \text{on } \overline{B_{4\delta_\varepsilon}(y_0)} \cap \overline{\Omega_*}. \tag{4.9}$$

Let $u_\pm^\varepsilon = u_\pm^\varepsilon(x, t)$ be the two bounded solutions of the initial-boundary value problems:

$$\partial_t u_\pm^\varepsilon = \operatorname{div}(|\nabla u_\pm^\varepsilon|^{p-2} \nabla u_\pm^\varepsilon) \quad \text{in } \Omega_{\pm,\varepsilon} \times (0, \infty), \tag{4.10}$$

$$u_\pm^\varepsilon = f(y_0) \pm \varepsilon \quad \text{on } \partial\Omega_{\pm,\varepsilon} \times (0, \infty), \tag{4.11}$$

$$u_\pm^\varepsilon = 0 \quad \text{on } \Omega_{\pm,\varepsilon} \times \{0\}. \tag{4.12}$$

Here we obtain

Proposition 4.1. *Let u be the solution of problem* (1.3)–(1.5). *For every small $\varepsilon > 0$ there exists $\tau_\varepsilon > 0$ satisfying*

$$u_-^\varepsilon \le u \le u_+^\varepsilon \quad \textit{in } B_R(x_0) \times (0, \tau_\varepsilon],$$

where $u_\pm^\varepsilon$ are the solutions of problems (4.10)–(4.12).

Proof. By combining (4.8) and the second inequality of (4.9) with (4.11), we see that

$$u \le u_+^\varepsilon \quad \text{on } (\partial\Omega_{+,\varepsilon} \cap \overline{B_{4\delta_\varepsilon}(y_0)}) \times (0, \infty). \tag{4.13}$$

Since $\partial\Omega_{+,\varepsilon} \backslash B_{4\delta_\varepsilon}(y_0)$ is a compact set contained in Ω_*, by applying Proposition 3.1 to the bounded C^2 domain Ω_* and the solution $\hat{u}$ of problem (4.1)–(4.3), we have from the corresponding estimate (3.12) and (4.4) that there exists $\tau_{1,\varepsilon} > 0$ satisfying

$$u \le u_+^\varepsilon \quad \text{on } (\partial\Omega_{+,\varepsilon} \backslash B_{4\delta_\varepsilon}(y_0)) \times (0, \tau_{1,\varepsilon}]. \tag{4.14}$$

Hence with the aid of (4.13) and (4.14) we have from the comparison principle that

$$u \le u_+^\varepsilon \quad \text{in } \Omega_{+,\varepsilon} \times (0, \tau_{1,\varepsilon}]. \tag{4.15}$$

On the other hand, the first inequality of (4.9) gives

$$u_-^\varepsilon \le u \quad \text{on } (\partial\Omega \cap \overline{B_{4\delta_\varepsilon}(y_0)}) \times (0, \infty). \tag{4.16}$$

Since $\partial\Omega_{-,\varepsilon} \subset \overline{B_{3\delta_\varepsilon}(y_0)}$, by applying Proposition 3.1 to the domain $\Omega_{-,\varepsilon}$ with bounded C^2 boundary and the solution u_-^ε of problem (4.10)–(4.12), we have from the corresponding estimate (3.12) and (1.1) that there exists $\tau_{2,\varepsilon} > 0$ satisfying

$$u_-^\varepsilon \le u \quad \text{on } (\partial\Omega \backslash \overline{B_{4\delta_\varepsilon}(y_0)}) \times (0, \tau_{2,\varepsilon}]. \tag{4.17}$$

Therefore with the aid of (4.16) and (4.17) we have from the comparison principle that

$$u_-^\varepsilon \le u \quad \text{in } \Omega \times (0, \tau_{2,\varepsilon}]. \tag{4.18}$$

In conclusion, (4.15) and (4.18) complete the proof if we set $\tau_\varepsilon = \min\{\tau_{1,\varepsilon}, \tau_{2,\varepsilon}\}$. □

Let us next consider the solution u of problem (1.6). Take a small $\varepsilon > 0$ arbitrarily. Since $g(y_0) > 0$ and $g \in C^0(\mathbf{R}^N)$, there exists a small number $\delta_\varepsilon \in (0, \delta/9)$ such that

$$g(y_0) - \frac{1}{2}\varepsilon \le g \le g(y_0) + \frac{1}{2}\varepsilon \quad \text{in } \overline{B_{4\delta_\varepsilon}(y_0)}. \tag{4.19}$$

Moreover we find a small number $\gamma_\varepsilon \in (0, \delta_\varepsilon)$ and two C^2 domains $\Omega_{+,\varepsilon}$ and $\Omega_{-,\varepsilon}$ having bounded C^2 boundaries with the following properties: both $\Omega_{+,\varepsilon}$ and $\mathbf{R}^N \backslash \overline{\Omega_{-,\varepsilon}}$ are bounded; $\mathbf{R}^N \backslash \overline{\Omega_{-,\varepsilon}} \subset B_{3\delta_\varepsilon}(y_0)$; $B_R(x_0) \subset \Omega_{+,\varepsilon} \subset \Omega \subset \Omega_{-,\varepsilon}$; $\overline{B_R(x_0)} \cap \partial\Omega_{+,\varepsilon} = \overline{B_R(x_0)} \cap \partial\Omega_{-,\varepsilon} = \{y_0\}$; $B_{\delta_\varepsilon}(y_0) \cap \partial\Omega \subset \partial\Omega_{\pm,\varepsilon} \cap \partial\Omega \subset B_{2\delta_\varepsilon}(y_0) \cap \partial\Omega$;

$$\overline{(\Omega_{+,\varepsilon})^{\gamma_\varepsilon}} \cap (\mathbf{R}^N \backslash \Omega) \subset B_{4\delta_\varepsilon}(y_0), \tag{4.20}$$

where $(\Omega_{+,\varepsilon})^{\gamma_\varepsilon}$ is the domain defined by (3.15), that is,

$$(\Omega_{+,\varepsilon})^{\gamma_\varepsilon} = \{x \in \mathbf{R}^N : \mathrm{dist}(x, \overline{\Omega_{+,\varepsilon}}) < \gamma_\varepsilon\}.$$

Let $u_\pm^\varepsilon = u_\pm^\varepsilon(x,t)$ be the two bounded solutions of the Cauchy problems (1.6) where the initial data $g\mathscr{X}_{\Omega^c}$ is replaced by $(g(y_0) \pm \varepsilon)\mathscr{X}_{(\Omega_{\pm,\varepsilon})^c}$, respectively. Hence we have

Proposition 4.2. *Let u be the solution of problem* (1.6). *For every small $\varepsilon > 0$ there exists $\tau_\varepsilon > 0$ satisfying*

$$u_-^\varepsilon \le u \le u_+^\varepsilon \quad \text{in } B_R(x_0) \times (0, \tau_\varepsilon],$$

where $u_\pm^\varepsilon$ are the solutions of problems (1.6) *where the initial data $g\mathscr{X}_{\Omega^c}$ is replaced by $(g(y_0) \pm \varepsilon)\mathscr{X}_{(\Omega_{\pm,\varepsilon})^c}$, respectively.*

Proof. In view of (4.19) and the fact that $\mathbf{R}^N \backslash \overline{\Omega_{-,\varepsilon}} \subset B_{3\delta_\varepsilon}(y_0)$, we notice that

$$(g(y_0) - \varepsilon)\mathscr{X}_{(\Omega_{-,\varepsilon})^c} \le g\mathscr{X}_{\Omega^c} \quad \text{in } \mathbf{R}^N.$$

Hence it follows from the comparison principle that

$$u_-^\varepsilon \le u \quad \text{in } \mathbf{R}^N \times (0, \infty). \tag{4.21}$$

On the other hand, (4.19) and (4.20) yield that

$$g\mathscr{X}_{\Omega^c} \le g(y_0) + \frac{1}{2}\varepsilon < g(y_0) + \varepsilon = (g(y_0) + \varepsilon)\mathscr{X}_{(\Omega_{+,\varepsilon})^c} \quad \text{in } (\Omega_{+,\varepsilon})^{\gamma_\varepsilon} \backslash \Omega_{+,\varepsilon}.$$

Therefore by the initial behavior of the solutions there exists $\tau_\varepsilon > 0$ such that

$$u \le u_+^\varepsilon \quad \text{on } \partial(\Omega_{+,\varepsilon})^{\gamma_\varepsilon/2} \times (0, \tau_\varepsilon],$$

which together with the comparison principle yields that

$$u \le u_+^\varepsilon \quad \text{in } (\Omega_{+,\varepsilon})^{\gamma_\varepsilon/2} \times (0, \tau_\varepsilon]. \tag{4.22}$$

Thus, combining (4.21) with (4.22) completes the proof. $\square$

Finally, Propositions 4.1 and 4.2 yield

$$\begin{aligned}\int_{B_R(x_0)} (u_-^\varepsilon(x,t))^\alpha \, dx &\le \int_{B_R(x_0)} (u(x,t))^\alpha \, dx \\ &\le \int_{B_R(x_0)} (u_+^\varepsilon(x,t))^\alpha \, dx \quad \text{for every } t \in (0, \tau_\varepsilon].\end{aligned}$$

These two inequalities show that the proofs of Theorem 1.1 for the equation $\partial_t u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ are reduced to the case where $\partial\Omega$ is bounded and of class C^2 and where f and g are positive constants, since we later know that the positive constants c in formula (1.7) are continuous with respect to positive constants f and g, respectively. Also, the proofs for the equation $\partial_t u = \Delta u^m$ follow from the same arguments as in those for the equation $\partial_t u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$.

5. Super- and subsolutions near the boundary for short times: the p-Laplace case

By virtue of section 4, we can assume that $\partial\Omega$ is bounded and of class C^2 and $f \equiv g \equiv \beta$ for some positive constant $\beta > 0$.

Let us first consider the solution u of problem (1.3)–(1.5). Namely, we consider the bounded solution $u = u(x,t)$ of the initial-boundary value problem:

$$\begin{aligned} &\partial_t u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) && \text{in } \Omega \times (0, \infty), \\ &u = \beta && \text{on } \partial\Omega \times (0, \infty), \\ &u = 0 && \text{on } \Omega \times \{0\}. \end{aligned}$$

For $\xi \ge 0$, define $\varphi = \varphi(\xi)$ by

$$\varphi(\xi) = \beta - \left(\frac{2-p}{2p(p-1)}\right)^{-1/(2-p)} \int_0^\xi (\eta^2 + \lambda)^{-1/(2-p)} \, d\eta, \tag{5.1}$$

where $\lambda > 0$ is determined uniquely by the equation $\varphi(\infty) = 0$. Then $\varphi = \varphi(\xi)$ satisfies

$$(p-1)|\varphi'|^{p-2}\varphi'' + \frac{1}{p}\varphi'\xi = 0 \quad \text{for } \xi > 0, \tag{5.2}$$

$$\varphi(0) = \beta, \quad \varphi' < 0 \quad \text{in } [0, \infty), \quad \text{and} \quad \varphi(\infty) = 0. \tag{5.3}$$

l'Hospital's rule gives

$$\lim_{\xi \to \infty} \frac{\varphi(\xi)}{\xi^{-p/(2-p)}} = c(p), \tag{5.4}$$

where $c(p)$ is the constant given by (3.5). Note that, if we set $h(s,t) = \varphi(t^{-1/p}s)$ for $s \geq 0$ and $t > 0$, then h satisfies the one-dimensional problem:

$$\partial_t h = \partial_s(|\partial_s h|^{p-2}\partial_s h) \quad \text{in } (0,\infty)^2, \qquad h = \beta \quad \text{on } \{0\} \times (0,\infty), \qquad \text{and}$$
$$h = 0 \quad \text{on } (0,\infty) \times \{0\}.$$

For small $\varepsilon > 0$, define $\varphi_\pm = \varphi_\pm(\xi)$ $(\xi > 0)$ by

$$\varphi_\pm(\xi) = \beta \pm \varepsilon - \left(\frac{2-p}{2p(p-1)}\right)^{-1/(2-p)} \times \int_0^\xi \left(\eta^2 \mp 2p\varepsilon \int_0^\eta \sqrt{1+s^2}\, ds + \lambda_\pm\right)^{-1/(2-p)} d\eta, \tag{5.5}$$

where each $\lambda_\pm > 0$ is determined uniquely by the equation $\varphi_\pm(\infty) = 0$. Notice that

$$\varphi_\pm \to \varphi \quad \text{as } \varepsilon \to 0^+ \text{ uniformly on } [0,\infty), \tag{5.6}$$

$$\lambda_\pm \to \lambda \quad \text{as } \varepsilon \to 0^+, \tag{5.7}$$

where λ is given in (5.1). Then $\varphi_\pm = \varphi_\pm(\xi)$ satisfies

$$(p-1)|\varphi_\pm'|^{p-2}\varphi_\pm'' + \frac{1}{p}\varphi_\pm'[\xi \mp p\varepsilon\sqrt{1+\xi^2}] = 0 \quad \text{for } \xi > 0, \tag{5.8}$$

$$\varphi_\pm(0) = \beta \pm \varepsilon, \quad \varphi_\pm' < 0 \quad \text{in } [0,\infty), \quad \text{and} \quad \varphi_\pm(\infty) = 0. \tag{5.9}$$

l'Hospital's rule gives

$$\lim_{\xi \to \infty} \frac{\varphi_\pm(\xi)}{\xi^{-p/(2-p)}} = c(p)(1 \mp p\varepsilon)^{-1/(2-p)}. \tag{5.10}$$

Since $\partial\Omega$ is bounded and of class C^2, there exists $\rho_0 > 0$ such that the distance function $d = d(x)$ of $x \in \overline{\Omega}$ to the boundary $\partial\Omega$ is C^2-smooth on $\overline{\Omega_{\rho_0}}$, where Ω_{ρ_0} is defined by (1.13) with $\rho = \rho_0$.

By setting

$$w_\pm(x,t) = \varphi_\pm(t^{-1/p}d(x)) \quad \text{for } (x,t) \in \Omega \times (0,\infty), \tag{5.11}$$

we obtain

PROPOSITION 5.1. *Let u be the solution of problem* (1.3)–(1.5) *where $\partial\Omega$ is bounded and of class C^2 and $f \equiv \beta$ for some positive constant $\beta > 0$. For every small $\varepsilon > 0$ there exist $\rho_\varepsilon \in (0,\rho_0)$ and $\tau_\varepsilon > 0$ satisfying*

$$w_- \le u \le w_+ \quad \text{in } \Omega_{\rho_\varepsilon} \times (0,\tau_\varepsilon], \tag{5.12}$$

where $w_\pm$ are given by (5.11) *and $\Omega_{\rho_\varepsilon}$ is defined by* (1.13) *with $\rho = \rho_\varepsilon$.*

Proof. Take a small $\varepsilon > 0$. For $x \in \Omega_{\rho_0}$ and $t > 0$, a straightforward computation gives

$$\partial_t w_\pm - \operatorname{div}(|\nabla w_\pm|^{p-2}\nabla w_\pm) = -t^{-1}\varphi_\pm'[\pm\varepsilon\sqrt{1+\xi^2} + t^{1/p}|\varphi_\pm'|^{p-2}\Delta d],$$

where $\xi = t^{-1/p}d(x)$ and

$$|\varphi_\pm'|^{p-2} = (-\varphi_\pm')^{p-2} = \left(\frac{2-p}{2p(p-1)}\right)\left[\xi^2 \mp 2p\varepsilon\int_0^\xi \sqrt{1+s^2}\,ds + \lambda_\pm\right].$$

Therefore, by using (5.7) and observing that

$$t^{1/p}\xi^2 \le |\xi|d(x) \quad \text{and} \quad t^{1/p}\left|\int_0^\xi \sqrt{1+s^2}\,ds\right| \le t^{1/p}(|\xi| + \xi^2),$$

we notice that there exist $\rho_{1,\varepsilon} \in (0,\rho_0)$ and $\tau_{1,\varepsilon} > 0$ satisfying

$$(\pm 1)(\partial_t w_\pm - \operatorname{div}(|\nabla w_\pm|^{p-2}\nabla w_\pm)) > 0 \quad \text{in } \Omega_{\rho_{1,\varepsilon}} \times (0,\tau_{1,\varepsilon}], \tag{5.13}$$

where $w_\pm$ are given by (5.11) and $\Omega_{\rho_{1,\varepsilon}}$ is defined by (1.13) with $\rho = \rho_{1,\varepsilon}$.

By (3.4), there exists $\rho_\varepsilon \in (0,\rho_{1,\varepsilon})$ satisfying

$$c(p)\left(1+\frac{p\varepsilon}{4}\right)^{-1/(2-p)} d(x)^{-p/(2-p)} \le v(x) \le c(p)\left(1-\frac{p\varepsilon}{4}\right)^{-1/(2-p)} d(x)^{-p/(2-p)} \quad \text{for } x \in \Omega_{\rho_\varepsilon}.$$

Hence by (3.11) of Proposition 3.1 there exists $\tau_{2,\varepsilon} \in (0,\tau_{1,\varepsilon}]$ such that for $(x,t) \in \Gamma_{\rho_\varepsilon} \times (0,\tau_{2,\varepsilon}]$

$$c(p)\left(1+\frac{p\varepsilon}{2}\right)^{-1/(2-p)} (\rho_\varepsilon)^{-p/(2-p)} \le t^{-1/(2-p)}u(x,t) \le c(p)\left(1-\frac{p\varepsilon}{2}\right)^{-1/(2-p)} (\rho_\varepsilon)^{-p/(2-p)}, \tag{5.14}$$

where $\Gamma_{\rho_\varepsilon}$ is defined by (1.14) with $\rho = \rho_\varepsilon$.

Moreover, by (5.10), there exists $\tau_\varepsilon \in (0, \tau_{2,\varepsilon}]$ such that for $(x,t) \in \Gamma_{\rho_\varepsilon} \times (0, \tau_\varepsilon]$

$$t^{-1/(2-p)}(\rho_\varepsilon)^{p/(2-p)} w_+(x,t) \geq c(p)\left(1 - \frac{p\varepsilon}{2}\right)^{-1/(2-p)},$$

$$t^{-1/(2-p)}(\rho_\varepsilon)^{p/(2-p)} w_-(x,t) \leq c(p)\left(1 + \frac{p\varepsilon}{2}\right)^{-1/(2-p)}.$$

Thus combining these inequalities with (5.14) yields that

$$w_- \leq u \leq w_+ \quad \text{on } \Gamma_{\rho_\varepsilon} \times (0, \tau_\varepsilon]. \tag{5.15}$$

Observe that

$$w_- = \beta - \varepsilon < \beta = u < \beta + \varepsilon = w_+ \quad \text{on } \partial\Omega \times (0, \tau_\varepsilon], \tag{5.16}$$

$$w_- = u = w_+ = 0 \qquad \text{on } \Omega_{\rho_\varepsilon} \times \{0\}. \tag{5.17}$$

Therefore, by combining these with (5.15) and (5.13), we get the conclusion (5.12) from the comparison principle. □

Let us next consider the solution u of problem (1.6). Namely, we consider the bounded solution $u = u(x,t)$ of the Cauchy problem:

$$\partial_t u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{in } \mathbf{R}^N \times (0,\infty) \qquad \text{and} \qquad u = \beta \mathscr{X}_{\Omega^c} \quad \text{on } \mathbf{R}^N \times \{0\},$$

where $\mathscr{X}_{\Omega^c}$ is the characteristic function of the set $\Omega^c = \mathbf{R}^N \backslash \Omega$. For $\xi \in \mathbf{R}$, define $\psi = \psi(\xi)$ by

$$\psi(\xi) = \beta - \left(\frac{2-p}{2p(p-1)}\right)^{-1/(2-p)} \int_{-\infty}^{\xi} (\eta^2 + \lambda)^{-1/(2-p)}\, d\eta, \tag{5.18}$$

where $\lambda > 0$ is determined uniquely by the equation $\psi(\infty) = 0$. Then $\psi = \psi(\xi)$ satisfies

$$(p-1)|\psi'|^{p-2}\psi'' + \frac{1}{p}\psi'\xi = 0 \quad \text{for } \xi \in \mathbf{R}, \tag{5.19}$$

$$\psi(-\infty) = \beta, \quad \psi' < 0 \quad \text{in } \mathbf{R}, \quad \text{and} \quad \psi(\infty) = 0. \tag{5.20}$$

l'Hospital's rule gives

$$\lim_{\xi \to \infty} \frac{\psi(\xi)}{\xi^{-p/(2-p)}} = c(p), \tag{5.21}$$

where $c(p)$ is the constant given by (3.5). Note that, if we set $h(s,t) = \psi(t^{-1/p}s)$ for $s \in \mathbf{R}$ and $t > 0$, then h satisfies the one-dimensional problem:

$$\partial_t h = \partial_s(|\partial_s h|^{p-2}\partial_s h) \quad \text{in } \mathbf{R} \times (0,\infty) \qquad \text{and} \qquad h = \beta \mathscr{X}_{(-\infty,0]} \quad \text{on } \mathbf{R} \times \{0\}.$$

For small $\varepsilon > 0$, define $\psi_\pm = \psi_\pm(\xi)$ $(\xi \in \mathbf{R})$ by

$$\psi_\pm(\xi) = \beta \pm \varepsilon - \left(\frac{2-p}{2p(p-1)}\right)^{-1/(2-p)} \times \int_{-\infty}^{\xi} \left(\eta^2 \mp 2p\varepsilon \int_0^\eta \sqrt{1+s^2}\,ds + \lambda_\pm\right)^{-1/(2-p)} d\eta, \tag{5.22}$$

where each $\lambda_\pm > 0$ is determined uniquely by the equation $\psi_\pm(\infty) = 0$. Notice that

$$\psi_\pm \to \psi \quad \text{as } \varepsilon \to 0^+ \text{ uniformly on } \mathbf{R}, \tag{5.23}$$

$$\lambda_\pm \to \lambda \quad \text{as } \varepsilon \to 0^+, \tag{5.24}$$

where λ is given in (5.18). Then $\psi_\pm = \psi_\pm(\xi)$ satisfies

$$(p-1)|\psi_\pm'|^{p-2}\psi_\pm'' + \frac{1}{p}\psi_\pm'[\xi \mp p\varepsilon\sqrt{1+\xi^2}] = 0 \quad \text{for } \xi \in \mathbf{R}, \tag{5.25}$$

$$\psi_\pm(-\infty) = \beta \pm \varepsilon, \quad \psi_\pm' < 0 \quad \text{in } \mathbf{R}, \quad \text{and} \quad \psi_\pm(\infty) = 0. \tag{5.26}$$

l'Hospital's rule gives

$$\lim_{\xi\to\infty} \frac{\psi_\pm(\xi)}{\xi^{-p/(2-p)}} = c(p)(1 \mp p\varepsilon)^{-1/(2-p)}. \tag{5.27}$$

As in [MS3], let us introduce the signed distance function $d^* = d^*(x)$ of $x \in \mathbf{R}^N$ to the boundary $\partial\Omega$ defined by

$$d^*(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ -\operatorname{dist}(x, \partial\Omega) & \text{if } x \notin \Omega. \end{cases}$$

For every $\rho > 0$, let $\mathcal{N}_\rho$ be a compact neighborhood of $\partial\Omega$ in $\mathbf{R}^N$ defined by

$$\mathcal{N}_\rho = \{x \in \mathbf{R}^N : -\rho \le d^*(x) \le \rho\}. \tag{5.28}$$

If $\partial\Omega$ is bounded and of class C^2, there exists a number $\rho_0 > 0$ such that $d^*(x)$ is C^2-smooth on $\mathcal{N}_{\rho_0}$. For simplicity we have used the same letter $\rho_0 > 0$ as in the previous case for problem (1.3)–(1.5).

By setting

$$w_\pm(x,t) = \psi_\pm(t^{-1/p}d^*(x)) \quad \text{for } (x,t) \in \mathbf{R}^N \times (0,\infty), \tag{5.29}$$

we obtain

Proposition 5.2. *Let u be the solution of problem* (1.6) *where $\partial\Omega$ is bounded and of class C^2 and $g \equiv \beta$ for some positive constant $\beta > 0$. For every small $\varepsilon > 0$ there exist $\rho_\varepsilon \in (0, \rho_0)$ and $\tau_\varepsilon > 0$ satisfying*

$$w_- \le u \le w_+ \quad \text{in } \mathcal{N}_{\rho_\varepsilon} \times (0, \tau_\varepsilon], \tag{5.30}$$

where $w_\pm$ are given by (5.29) *and $\mathcal{N}_{\rho_\varepsilon}$ is defined by* (5.28) *with $\rho = \rho_\varepsilon$.*

Proof. The proof is similar to that of Proposition 5.1. The ingredients (5.7), (5.10), and (5.16) are replaced by (5.24), (5.27), and the corresponding inequalities on $\{x \in \mathbf{R}^N : d^*(x) = -\rho_\varepsilon\} \times (0, \tau_\varepsilon]$, respectively. □

6. Super- and subsolutions near the boundary for short times: the porous medium type case

By virtue of section 4, we can assume that $\partial\Omega$ is bounded and of class C^2 and $f \equiv g \equiv \beta$ for some positive constant $\beta > 0$.

Concerning $\partial_t u = \Delta u^m$ with $0 < m < 1$, the same constructions of super- and subsolutions as in [MS3] work. Let $u = u(x,t)$ be the bounded solution of problem (1.8)–(1.10) where $f \equiv \beta$. Namely, we consider the bounded solution $u = u(x,t)$ of the initial-boundary value problem:

$$\begin{aligned} &\partial_t u = \Delta u^m && \text{in } \Omega \times (0,\infty), \\ &u = \beta && \text{on } \partial\Omega \times (0,\infty), \\ &u = 0 && \text{on } \Omega \times \{0\}. \end{aligned}$$

Let us set $\phi(s) = s^m$ for $s \geq 0$. We use a result from Atkinson and Peletier [AP]: for every $\gamma > 0$, there exists a unique C^2 solution $f_\gamma = f_\gamma(\xi)$ of the problem:

$$(\phi'(f_\gamma)f_\gamma')' + \frac{1}{2}\xi f_\gamma' = 0 \quad \text{in } [0,\infty), \tag{6.1}$$

$$f_\gamma(0) = \gamma, \quad f_\gamma(\infty) = 0, \tag{6.2}$$

$$f_\gamma' < 0 \quad \text{in } [0,\infty). \tag{6.3}$$

Moreover, [AP, Theorem 5 and its example 3, p. 388 and p. 390] gives

$$\lim_{\xi \to \infty} \frac{f_\gamma(\xi)}{\xi^{-2/(1-m)}} = c(m), \tag{6.4}$$

where $c(m)$ is the constant given by (3.10). This behavior comes from the structure of the equation $\partial_t u = \Delta u^m$ with $0 < m < 1$, and it is different from that of the equation of the form $\partial_t u = \Delta\phi(u)$ with $\delta_1 \leq \phi'(s) \leq \delta_2$ $(s \in \mathbf{R})$ for two positive constants δ_1, δ_2, which is treated in [MS3, (3.15), p. 243]. Note that, if we put $h(s,t) = f_\gamma(t^{-1/2}s)$ for $s \geq 0$ and $t > 0$, then h satisfies the one-dimensional problem:

$$\partial_t h = \partial_s^2 \phi(h) \quad \text{in } (0,\infty)^2, \qquad h = \gamma \quad \text{on } \{0\} \times (0,\infty), \qquad \text{and}$$
$$h = 0 \quad \text{on } (0,\infty) \times \{0\}.$$

Let $0 < \varepsilon < \frac{1}{4}$. Then, as in [MS3, Proof of Lemma 3.1, pp. 242–244], by continuity we can find a sufficiently small $0 < \eta_\varepsilon \ll \varepsilon$ and two C^2 functions $f_\pm = f_\pm(\xi)$ for $\xi \geq 0$ satisfying:

$$f_\pm(\xi) = f_{\beta\pm\varepsilon}(\sqrt{1 \mp 2\eta_\varepsilon}\xi) \quad \text{if } \xi \ge \eta_\varepsilon;$$
$$f_\pm' < 0 \quad \text{in } [0,\infty);$$
$$f_- < f_\beta < f_+ \quad \text{in } [0,\infty);$$
$$(\phi'(f_\pm)f_\pm')' + \frac{1}{2}\xi f_\pm' = h_\pm(\xi)f_\pm' \quad \text{in } [0,\infty),$$

where $h_\pm = h_\pm(\xi)$ are defined by

$$h_\pm(\xi) = \begin{cases} \pm\eta_\varepsilon\xi & \text{if } \xi \ge \eta_\varepsilon, \\ \pm\eta_\varepsilon^2 & \text{if } \xi \le \eta_\varepsilon. \end{cases} \tag{6.5}$$

(Here, in order to use the functions $h_\pm$ also for problem (1.11) later, we defined $h_\pm(\xi)$ for all $\xi \in \mathbf{R}$.) The above construction of $f_\pm$ directly implies that

$$f_\pm \to f_\beta \quad \text{as } \varepsilon \to 0^+ \text{ uniformly on } [0,\infty). \tag{6.6}$$

Moreover, by (6.4) we have

$$\lim_{\xi\to\infty} \frac{f_\pm(\xi)}{\xi^{-2/(1-m)}} = c(m)(1 \mp 2\eta_\varepsilon)^{-1/(1-m)}. \tag{6.7}$$

By setting

$$w_\pm(x,t) = f_\pm(t^{-1/2}d(x)) \quad \text{for } (x,t) \in \Omega \times (0,\infty), \tag{6.8}$$

we obtain

Proposition 6.1. *Let u be the solution of problem* (1.8)–(1.10) *where $\partial\Omega$ is bounded and of class C^2 and $f \equiv \beta$ for some positive constant $\beta > 0$. For every small $\varepsilon > 0$ there exist $\rho_\varepsilon \in (0,\rho_0)$ and $\tau_\varepsilon > 0$ satisfying*

$$w_- \le u \le w_+ \quad \text{in } \Omega_{\rho_\varepsilon} \times (0,\tau_\varepsilon], \tag{6.9}$$

where $w_\pm$ are given by (6.8) *and $\Omega_{\rho_\varepsilon}$ is defined by* (1.13) *with $\rho = \rho_\varepsilon$.*

Proof. Take a small $\varepsilon > 0$. For $x \in \Omega_{\rho_0}$ and $t > 0$, a straightforward computation gives

$$\partial_t w_\pm - \Delta(w_\pm)^m = -t^{-1}f_\pm'[h_\pm(\xi) + t^{1/2}m(f_\pm)^{-(1-m)}\Delta d],$$

where $\xi = t^{-1/2}d(x)$. In view of (6.7), we observe that there exists a constant $C_\varepsilon > 0$ satisfying

$$t^{1/2}m(f_\pm)^{-(1-m)} \le \begin{cases} t^{1/2}C_\varepsilon\xi^2 = C_\varepsilon\xi d(x) & \text{if } \xi \ge \eta_\varepsilon, \\ t^{1/2}m(f_\pm(\sqrt{1 \mp 2\eta_\varepsilon}\eta_\varepsilon))^{-(1-m)} \le t^{1/2}C_\varepsilon\eta_\varepsilon^2 & \text{if } \xi \le \eta_\varepsilon. \end{cases}$$

Therefore, with the aid of the definition (6.5) of $h_\pm(\xi)$, we notice that there exist $\rho_{1,\varepsilon} \in (0,\rho_0)$ and $\tau_{1,\varepsilon} > 0$ satisfying

$$(\pm 1)(\partial_t w_\pm - \Delta(w_\pm)^m) > 0 \quad \text{in } \Omega_{\rho_{1,\varepsilon}} \times (0,\tau_{1,\varepsilon}], \tag{6.10}$$

where $w_\pm$ are given by (6.8) and $\Omega_{\rho_{1,\varepsilon}}$ is defined by (1.13) with $\rho = \rho_{1,\varepsilon}$.

By (3.9), there exists $\rho_\varepsilon \in (0, \rho_{1,\varepsilon})$ satisfying

$$c(m)\left(1+\frac{\eta_\varepsilon}{2}\right)^{-1/(1-m)} d(x)^{-2/(1-m)} \\ \le w(x) \le c(m)\left(1-\frac{\eta_\varepsilon}{2}\right)^{-1/(1-m)} d(x)^{-2/(1-m)} \quad \text{for } x \in \Omega_{\rho_\varepsilon}.$$

Hence by (3.23) of Proposition 3.2 there exists $\tau_{2,\varepsilon} \in (0, \tau_{1,\varepsilon}]$ such that for $(x,t) \in \Gamma_{\rho_\varepsilon} \times (0, \tau_{2,\varepsilon}]$

$$c(m)(1+\eta_\varepsilon)^{-1/(1-m)}(\rho_\varepsilon)^{-2/(1-m)} \le t^{-1/(1-m)} u(x,t) \\ \le c(m)(1-\eta_\varepsilon)^{-1/(1-m)}(\rho_\varepsilon)^{-2/(1-m)}, \tag{6.11}$$

where $\Gamma_{\rho_\varepsilon}$ is defined by (1.14) with $\rho = \rho_\varepsilon$.

Moreover, by (6.7), there exists $\tau_\varepsilon \in (0, \tau_{2,\varepsilon}]$ such that for $(x,t) \in \Gamma_{\rho_\varepsilon} \times (0, \tau_\varepsilon]$

$$t^{-1/(1-m)}(\rho_\varepsilon)^{2/(1-m)} w_+(x,t) \ge c(m)(1-\eta_\varepsilon)^{-1/(1-m)},$$
$$t^{-1/(1-m)}(\rho_\varepsilon)^{2/(1-m)} w_-(x,t) \le c(m)(1+\eta_\varepsilon)^{-1/(1-m)}.$$

Thus combining these inequalities with (6.11) yields that

$$w_- \le u \le w_+ \quad \text{on } \Gamma_{\rho_\varepsilon} \times (0, \tau_\varepsilon]. \tag{6.12}$$

Observe that

$$w_- < \beta = u < w_+ \quad \text{on } \partial\Omega \times (0, \tau_\varepsilon], \tag{6.13}$$

$$w_- = u = w_+ = 0 \quad \text{on } \Omega_{\rho_\varepsilon} \times \{0\}. \tag{6.14}$$

Therefore, by combining these with (6.12) and (6.10), we get the conclusion (6.9) from the comparison principle. □

Let us next consider the solution u of problem (1.11). Namely, we consider the bounded solution $u = u(x,t)$ of the Cauchy problem:

$$\partial_t u = \Delta u^m \quad \text{in } \mathbf{R}^N \times (0,\infty) \qquad \text{and} \qquad u = \beta \mathscr{X}_{\Omega^c} \quad \text{on } \mathbf{R}^N \times \{0\},$$

where $\mathscr{X}_{\Omega^c}$ is the characteristic function of the set $\Omega^c = \mathbf{R}^N \backslash \Omega$. Let us set $\phi(s) = s^m$ for $s \ge 0$. We use a result from [MS3]: for every $\gamma > 0$, there exists a unique C^2 solution $f_\gamma = f_\gamma(\xi)$ of the problem:

$$(\phi'(f_\gamma) f_\gamma')' + \frac{1}{2}\xi f_\gamma' = 0 \quad \text{in } \mathbf{R}, \tag{6.15}$$

$$f_\gamma(-\infty) = \gamma, \quad f_\gamma(\infty) = 0, \tag{6.16}$$

$$f_\gamma' < 0 \quad \text{in } \mathbf{R}. \tag{6.17}$$

Moreover, [AP, Theorem 5 and its example 3, p. 388 and p. 390] also gives (6.4). Note that, if we put $h(s,t) = f_\gamma(t^{-1/2}s)$ for $s \in \mathbf{R}$ and $t > 0$, then h satisfies the one-dimensional problem:

$$\partial_t h = \partial_s^2 \phi(h) \quad \text{in } \mathbf{R} \times (0,\infty) \qquad \text{and} \qquad h = \gamma \mathscr{X}_{(-\infty,0]} \quad \text{on } \mathbf{R} \times \{0\}.$$

Let $0 < \varepsilon < \frac{1}{4}$. By the same proof as in [MS3, Proof of (3.35), pp. 251–252], we find a sufficiently small $0 < \eta_\varepsilon \ll \varepsilon$ and two C^2 functions $f_\pm = f_\pm(\xi)$ for $\xi \in \mathbf{R}$ satisfying:

$$f_\pm(\xi) = f_{\beta\pm\varepsilon}(\sqrt{1 \mp 2\eta_\varepsilon}\,\xi) \quad \text{if } \xi \geq \eta_\varepsilon, \tag{6.18}$$

$$f'_\pm < 0 \quad \text{in } \mathbf{R}, \tag{6.19}$$

$$f_-(-\infty) < \beta = f_\beta(-\infty) < f_+(-\infty) \quad \text{and} \quad f_- < f_\beta < f_+ \quad \text{in } \mathbf{R}, \tag{6.20}$$

$$(\phi'(f_\pm)f'_\pm)' + \frac{1}{2}\xi f'_\pm = h_\pm(\xi) f'_\pm \quad \text{in } \mathbf{R}, \tag{6.21}$$

and

$$f_\pm \to f_\beta \quad \text{as } \varepsilon \to 0^+ \text{ uniformly on } \mathbf{R}. \tag{6.22}$$

Moreover, by (6.4) we also have (6.7).

By setting

$$w_\pm(x,t) = f_\pm(t^{-1/2}d^*(x)) \quad \text{for } (x,t) \in \mathbf{R}^N \times (0,\infty), \tag{6.23}$$

we obtain

PROPOSITION 6.2. *Let u be the solution of problem* (1.11) *where $\partial\Omega$ is bounded and of class C^2 and $g \equiv \beta$ for some positive constant $\beta > 0$. For every small $\varepsilon > 0$ there exist $\rho_\varepsilon \in (0,\rho_0)$ and $\tau_\varepsilon > 0$ satisfying*

$$w_- \leq u \leq w_+ \quad \text{in } \Omega_{\rho_\varepsilon} \times (0,\tau_\varepsilon], \tag{6.24}$$

where $w_\pm$ are given by (6.23) *and $\Omega_{\rho_\varepsilon}$ is defined by* (1.13) *with $\rho = \rho_\varepsilon$.*

Proof. The proof is similar to that of Proposition 6.1. The ingredient (6.13) is replaced by the corresponding inequalities on $\{x \in \mathbf{R}^N : d^*(x) = -\rho_\varepsilon\} \times (0,\tau_\varepsilon]$. □

7. Proofs of Theorems 1.1 and 1.2

By virtue of section 4, we can assume that $\partial\Omega$ is bounded and of class C^2 and $f \equiv g \equiv \beta$ for some positive constant $\beta > 0$. We will use a geometric lemma from [MS1] adjusted to our situation.

Lemma 7.1 ([MS1, Lemma 2.1, p. 376]). *Let* $\kappa_j(y_0) < \dfrac{1}{R}$ *for every* $j = 1, \dots, N-1$. *Then we have*:

$$\lim_{s\to 0^+} s^{-(N-1)/2}\mathscr{H}^{N-1}(\Gamma_s \cap B_R(x_0)) = 2^{(N-1)/2}\omega_{N-1}\left\{\prod_{j=1}^{N-1}\left(\frac{1}{R}-\kappa_j(y_0)\right)\right\}^{-1/2},$$

where $\mathscr{H}^{N-1}$ *is the standard* $(N-1)$*-dimensional Hausdorff measure, and* ω_{N-1} *is the volume of the unit ball in* $\mathbf{R}^{N-1}$.

Let us first prove Theorem 1.1 for the solution u of problem (1.3)–(1.5) by using Proposition 5.1. Take a small $\varepsilon > 0$. Let $\alpha > \dfrac{(N+1)(2-p)}{2p}$. Then Proposition 5.1 yields that for every $t \in (0, \tau_\varepsilon]$

$$\int_{B_R(x_0)\cap\Omega_{\rho_\varepsilon}} (w_-(x,t))^\alpha\, dx \le \int_{B_R(x_0)\cap\Omega_{\rho_\varepsilon}} (u(x,t))^\alpha\, dx \le \int_{B_R(x_0)\cap\Omega_{\rho_\varepsilon}} (w_+(x,t))^\alpha\, dx. \tag{7.1}$$

For $(x,t) \in (\Omega\backslash\Omega_{\rho_\varepsilon}) \times (0,\infty)$, by (3.12) of Proposition 3.1, we have

$$t^{-(N+1)/2p}(u(x,t))^\alpha \le t^{-(N+1)/2p+\alpha/(2-p)}(v(x))^\alpha. \tag{7.2}$$

Therefore, since $\overline{B_R(x_0)}\backslash\Omega_{\rho_\varepsilon}$ is a compact set contained in Ω and $-\dfrac{N+1}{2p} + \dfrac{\alpha}{2-p} > 0$, we see that

$$t^{-(N+1)/2p}\int_{B_R(x_0)\backslash\Omega_{\rho_\varepsilon}} (u(x,t))^\alpha\, dx \to 0 \quad \text{as } t \to 0^+. \tag{7.3}$$

With the aid of the co-area formula, we have

$$\begin{aligned}&\int_{B_R(x_0)\cap\Omega_{\rho_\varepsilon}} (w_\pm(x,t))^\alpha\, dx\\ &\quad = t^{(N+1)/2p}\int_0^{\rho_\varepsilon t^{-1/2}} (\varphi_\pm(\xi))^\alpha \xi^{(N-1)/2}(t^{1/p}\xi)^{-(N-1)/2}\mathscr{H}^{N-1}(B_R(x_0)\cap\Gamma_{t^{1/p}\xi})\, d\xi.\end{aligned}$$

Thus, when $\kappa_j(y_0) < \dfrac{1}{R}$ for every $j = 1, \dots, N-1$, by Lebesgue's dominated convergence theorem and Lemma 7.1, we get

$$\begin{aligned}&\lim_{t\to 0^+} t^{-(N+1)/2p}\int_{B_R(x_0)\cap\Omega_{\rho_\varepsilon}} (w_\pm)^\alpha\, dx\\ &\quad = 2^{(N-1)/2}\,\omega_{N-1}\left\{\prod_{j=1}^{N-1}\left(\frac{1}{R}-\kappa_j(y_0)\right)\right\}^{-1/2}\int_0^\infty (\varphi_\pm(\xi))^\alpha \xi^{(N-1)/2}\, d\xi.\end{aligned}$$

Here (5.10) together with the inequality $-\dfrac{p\alpha}{2-p}+\dfrac{N-1}{2}<-1$ guarantees that the right-hand side of this formula is finite. Moreover, by Lebesgue's dominated convergence theorem and (5.6), we see that

$$\lim_{\varepsilon\to 0}\int_0^{\infty}(\varphi_{\pm}(\xi))^{\alpha}\xi^{(N-1)/2}\,d\xi=\int_0^{\infty}(\varphi(\xi))^{\alpha}\xi^{(N-1)/2}\,d\xi.$$

Therefore, since $\varepsilon>0$ is arbitrarily small, it follows from (7.1) and (7.3) that (1.7) holds true, where we set

$$c=2^{(N-1)/2}\omega_{N-1}\int_0^{\infty}(\varphi(\xi))^{\alpha}\xi^{(N-1)/2}\,d\xi.$$

It remains to consider the case where $\kappa_j(y_0)=\dfrac{1}{R}$ for some $j\in\{1,\dots,N-1\}$. Choose a sequence of balls $\{B_{R_k}(x_k)\}_{k=1}^{\infty}$ satisfying:

$R_k<R$, $y_0\in\partial B_{R_k}(x_k)$, and $B_{R_k}(x_k)\subset B_R(x_0)$ for every $k\geq 1$, and $\lim_{k\to\infty}R_k=R$.

Since $\kappa_j(y_0)\leq\dfrac{1}{R}<\dfrac{1}{R_k}$ for every $j=1,\dots,N-1$ and every $k\geq 1$, we can apply the previous case to each $B_{R_k}(x_k)$ to see that for every $k\geq 1$

$$\begin{aligned}\liminf_{t\to 0^+}t^{-(N+1)/2p}\int_{B_R(x_0)}(u(x,t))^{\alpha}\,dx&\geq\liminf_{t\to 0^+}t^{-(N+1)/2p}\int_{B_{R_k}(x_k)}(u(x,t))^{\alpha}\,dx\\&=c\left\{\prod_{j=1}^{N-1}\left(\frac{1}{R_k}-\kappa_j(y_0)\right)\right\}^{-1/2}.\end{aligned}$$

Hence, letting $k\to\infty$ yields that

$$\liminf_{t\to 0^+}t^{-(N+1)/2p}\int_{B_R(x_0)}(u(x,t))^{\alpha}\,dx=\infty,$$

which completes the proof for problem (1.3)–(1.5).

The proof of Theorem 1.1 for problem (1.6) runs similarly with the aid of Proposition 5.2. Also, the proof of Theorem 1.2 runs similarly with the aid of Propositions 6.1 and 6.2. Of course, for problems (1.8)–(1.10) and (1.11), we use Proposition 3.2 and the assumption that $\alpha>\dfrac{(N+1)(1-m)}{4}$ instead of Proposition 3.1 and the assumption that $\alpha>\dfrac{(N+1)(2-p)}{2p}$.

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