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KODAI MATH. J.
36 (2013), 154–166

CONTACT METRIC STRUCTURES ON S^3

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Abstract

In this paper, we construct a new family of contact metric structures on the unit sphere $\mathbf{S}^3$. Especially, the above family has the property that $\nabla_\xi \tau = 2\alpha\tau\phi$.

1. Introduction

Chern and Hamilton ([6]) introduced the torsion $\tau = \mathscr{L}_\xi g$, where $\mathscr{L}_\xi$ is the Lie derivative of g with respect to the characteristic vector field ξ, in their study of compact contact three-manifolds. G. Calvaruso and D. Perrone ([4]) proved that a 3-dimensional contact metric manifold is locally homogeneous if and only if it is ball homogeneous and, moreover, satisfies the condition

$$\nabla_\xi \tau = 2\alpha\tau\phi, \tag{1.1}$$

where α is a constant. Here, the composition $\tau\phi(X, Y)$ has to be interpreted as $\tau(\phi X, Y)$. Especially, the condition (1.1) with $\alpha = 0$ is equivalent to the condition that at a given point, the sectional curvature of all planes perpendicular to the contact subbundle, are equal ([14]). These manifolds are said to be $3 - \tau$ manifolds ([7]).

On the other hand, D. E. Blair ([1, pp. 133–137]) constructed examples of conformally flat contact metric three-manifolds which do not have constant sectional curvature. G. Calvaruso ([3]) pointed out that Blair' s examples satisfy the condition (1.1) with α smooth function which is constant along the geodesic foliation generated by ξ. The same author proved that a conformally flat contact metric 3-manifold satisfying condition (1.1) with $\alpha = const. \neq 2$, has constant sectional curvature 0 or 1. More generally, the authors in ([8]) investigated conformally flat contact metric 3-manifolds satisfying the condition (1.1), where α is a smooth function constant along the flow of ξ. A contact metric manifold satisfying the condition (1.1), where α is an arbitrary smooth function, is called $3 - \tau - \alpha$ manifold ([8]).

In contact metric geometry, there are few examples of compact contact metric manifolds. These examples include the odd dimensional spheres, the

1991 *Mathematics Subject Classification*. Primary 53C15; Secondary 53C25.
Key words and phrases. Contact metric manifolds, 3-τ-α-manifolds.
Received May 1, 2012; revised August 28, 2012.

unit tangent sphere bundle over a Riemannian manifold and some of the 3-dimensional unimodular Lie groups, endowed with a left-invariant metric, listed in Table I of [2]. On the contrary, the previously mentioned Blair's examples are non compact conformally flat and $3-\tau-\alpha$ manifolds. To this direction, the authors in [8], proved that the solid torus $S^1 \times D^2$ is a conformally flat, semi-K contact, $3-\tau-\alpha$ manifold. In this paper, we prove that the unit sphere $\mathbf{S}^3$, equipped with a specific family of contact metric structures (η, ξ, ϕ, g_f), is a $3-\tau-\alpha$ manifold (see Theorem 3.1). More precisely, this family of contact metric structures on $\mathbf{S}^3$ depends on a differentiable function f of $\mathbf{S}^3$ which satisfies a particular partial differential equation. Furthermore, we give special solutions of this differential equation (see Examples 3.1 and 3.2). Also, we clarify that this family of structures on $\mathbf{S}^3$ isn't invariant for D-homothetic deformations (see Remark 3.5). Additionally, we thoroughly investigate some curvature properties of the contact metric manifold $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ (see Theorem 3.2 and Theorem 3.3). Finally, we point out that in the case which the function f is non-constant, $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ is not a generalized (κ, μ)-contact metric manifold (see Remark 3.3).

2. Contact metric manifolds

We start with some fundamental notions about contact Riemannian geometry. We refer to [1] for further details.

A differentiable $(2n+1)$-dimensional manifold is called *contact manifold* if it admits a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It is well known that a contact manifold admits an almost contact metric structure (η, ξ, ϕ, g), i.e. a global vector field ξ, which is called the *characteristic vector field* or the *Reeb vector field*, a tensor field ϕ of type $(1,1)$ and a Riemannian metric g (*associated metric*) such that

$$\eta(\xi) = 1, \quad \phi^2 = -Id + \eta \otimes \xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.1}$$

for all vector fields X, Y on M. Moreover, the structure (η, ξ, ϕ, g) can be chosen so that

$$d\eta(X, Y) = g(X, \phi Y), \tag{2.2}$$

for all vector fields X, Y on M. The manifold M together with the structure tensors (η, ξ, ϕ, g) is called a *contact metric manifold* (c.m.m., in short) and is denoted by $[M, (\eta, \xi, \phi, g)]$. We denote by ∇ the Levi-Civita connection, and by R the corresponding Riemann curvature tensor field given by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$$

for all vector fields X, Y on M. Moreover, we denote by S the Ricci tensor and by r the scalar curvature.

We define on M the operators l, h and τ by

$$lX = R(X, \xi)\xi, \quad hX = \frac{1}{2}(\mathscr{L}_\xi \phi)X, \quad \tau(X, Y) = (\mathscr{L}_\xi g)(X, Y)$$

where $\mathscr{L}_\xi$ is the Lie differentiation in the direction of ξ. The tensors h and τ are symmetric and satisfy ([9]):

$$h\xi = 0, \quad \operatorname{tr} h = \operatorname{tr} h\phi = 0, \quad h\phi = -\phi h, \tag{2.3}$$
$$\tau = 2g(\phi\cdot, h\cdot), \quad \nabla_\xi \tau = 2g(\phi\cdot, \nabla_\xi h\cdot)$$

for all vector fields X on M.

The tensor l is symmetric and satisfies ([1, p. 111]):

$$\phi l \phi - l = 2(\phi^2 + h^2), \quad \nabla_\xi h = \phi - \phi l - \phi h^2. \tag{2.4}$$

Combining relations (2.4), we get

$$l\phi - \phi l = 2\nabla_\xi h. \tag{2.5}$$

A contact metric manifold for ξ being a Killing vector field is called a *K-contact manifold*. It is well known that a contact metric manifold is K-contact if and only if $h = 0$. (or, equivalently, $\tau = 0$).

A contact structure on M gives rise to an almost complex structure on the product $M \times \mathbf{R}$. If this structure is integrable, then the contact metric manifold is said to be *Sasakian*. Equivalently, a contact metric manifold is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

for all vector fields X, Y on M.

Every Sasakian manifold is K-contact, but the converse is true only in the three dimensional case.

The sectional curvature $K(X, \xi)$ of a plain section spanned by ξ and a vector field X orthogonal to ξ is called *ξ-sectional curvature*. The sectional curvature $K(X, \phi X)$ of a plain section spanned by the vector field X (orthogonal to ξ) and ϕX is called *ϕ-sectional curvature*.

A c.m.m. $[M, (\eta, \xi, \phi, g)]$ is said to be η-Einstein if the Ricci tensor S is of the form

$$S = ag + b\eta \otimes \eta, \tag{2.6}$$

where a and b are smooth functions on M. Every K-contact metric 3-manifold is η-Einstein and the Ricci tensor is given by ([14])

$$S = \left(\frac{r}{2} - 1\right)g + \left(-\frac{r}{2} + 3\right)\eta \otimes \eta.$$

A connected c.m.m. $[M, (\eta, \xi, \phi, g)]$ of which the Riemann curvature tensor satisfies the relation

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \tag{2.7}$$

with κ, μ smooth functions on M and every vector fields X, Y on M, is called *generalized (κ, μ)-c.m.m.* ([11]). Especially. if the functions κ, μ are constants on M, the c.m.m. $[M,(\eta,\xi,\phi,g)]$ is called (κ,μ)-*c.m.m.* ([2]). Generalized (κ,μ)-c.m.m. not (κ,μ)-c.m.m. appear only in dimension 3 ([11]).

We call $3-\tau-\alpha$ manifold a 3-dimensional c.m.m. satisfying $\nabla_\xi\tau = 2\alpha\tau\phi$, where the composition $\tau\phi(X,Y)$ has to be interpreted as $\tau(\phi X, Y)$ and α is an arbitrary smooth function. We mention that every generalized (κ,μ)-c.m.m. is a $3-\tau-\alpha$ manifold with $\alpha = \frac{\mu}{2}$ ([11]).

Let $[M,(\eta,\xi,\phi,g)]$ be a 3-dimensional contact metric manifold. Let U be the open subset of M where $h \neq 0$ and V the interior of U^c. Then $U \cup V$ is an open and dense subset of M. For every $p \in U$ there exists an open neighborhood W of p and a vector field e defined on W such that $he = \lambda e$ and $h\phi e = -\lambda\phi e$, where λ is a non-vanishing smooth function of U. We call the local orthonormal frame field $\{e,\phi e,\xi\}$ a *h-basis*.

Combining relations (2.3) and (2.5), we have the following Proposition:

PROPOSITION 2.1. *Let $[M,(\eta,\xi,\phi,g)]$ be a 3-dimensional contact metric manifold. Then, $[M,(\eta,\xi,\phi,g)]$ is a $3-\tau-\alpha$ manifold if and only if $l\phi-\phi l = 4\alpha h\phi$.*

3. New contact metric structures on $\mathbf{S}^3$

We consider the unit sphere

$$\mathbf{S}^3 = \left\{ p = (x_1,x_2,x_3,x_4) \in \mathbf{R}^4 \;\middle|\; \sum_{i=1}^{4}(x_i)^2 = 1 \right\}$$

embedded in $\mathbf{R}^4$. The orthonormal vectors

$$e_1((x_1,x_2,x_3,x_4)) = (-x_2,x_1,-x_4,x_3)$$
$$e_2((x_1,x_2,x_3,x_4)) = (-x_3,x_4,x_1,-x_2),$$
$$e_3((x_1,x_2,x_3,x_4)) = (-x_4,-x_3,x_2,x_1)$$

are orthogonal to $x = (x_1,x_2,x_3,x_4) \in \mathbf{S}^3$ with respect to the Euclidean metric and linearly independent everywhere on $\mathbf{S}^3$. Hence, they define the tangent space $T_x\mathbf{S}^3$ ([13, page 259]). We easily get

$$[e_1,e_2] = 2e_3, \quad [e_2,e_3] = 2e_1, \quad [e_3,e_1] = 2e_2. \tag{3.1}$$

Let f be an arbitrary smooth function of $\mathbf{S}^3$ non-vanishing everywhere on $\mathbf{S}^3$ which is a solution of the following partial differential equation:

$$-x_2\frac{\partial f}{\partial x_1} + x_1\frac{\partial f}{\partial x_2} - x_4\frac{\partial f}{\partial x_3} + x_3\frac{\partial f}{\partial x_4} = 0, \tag{3.2}$$

or, equivalently, $e_1(f) = 0$. Let g_f, ϕ be the Riemannian metric and the tensor field of type $(1,1)$ given by

$$g_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{f^2} & 0 \\ 0 & 0 & f^2 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -f^2 \\ 0 & \dfrac{1}{f^2} & 0 \end{pmatrix},$$

with respect to the basis $\{e_1, e_2, e_3\}$. We denote by η the 1-form defined by $\eta(W) = g_f(W, e_1)$ for every $W \in \mathscr{X}(\mathbf{S}^3)$. Then η is a contact form since $\eta \wedge d\eta \neq 0$ everywhere on $\mathbf{S}^3$. Using the definition of η and the linearity of ϕ, $d\eta$ and g_f, we easily obtain that $\eta(e_1) = 1$, $d\eta(Z, W) = g(\phi Z, W)$ and $g_f(\phi Z, \phi W) = g_f(Z, W) - \eta(Z)\eta(W)$ for every vector fields Z, W on $\mathbf{S}^3$. Hence $[\mathbf{S}^3, (\eta, e_1, \phi, g_f)]$ is a c.m.m.

Since $g_f(e_1, e_1) = 1$, $g_f(e_2, e_2) = \dfrac{1}{f^2}$, $g_f(e_3, e_3) = f^2$ and $g_f(e_i, e_j) = 0$ for all $i \neq j$, we easily get that the set $\left\{ w_1 = e_1, w_2 = fe_2, w_3 = \dfrac{1}{f} e_3 \right\}$ is an orthonormal frame field globally defined on $\mathbf{S}^3$. Using relations (3.1) and (3.2), we easily obtain that their Lie brackets are given by:

$$[w_1, w_2] = 2f^2 w_3, \quad [w_1, w_3] = -\frac{2}{f^2} w_2, \tag{3.3}$$
$$[w_2, w_3] = 2w_1 - \frac{e_3(f)}{f^2} w_2 - e_2(f) w_3.$$

Let ∇ be the Levi-Civita connection corresponding to g_f. By using the Koszul's formula

$$\begin{aligned} 2g(\nabla_Y Z, W) = {} & Yg(Z, W) + Zg(W, Y) - Wg(Y, Z) - g(Y, [Z, W]) \\ & - g(Z, [Y, W]) + g(W, [Y, Z]), \end{aligned}$$

and (3.3), we calculate

$$\nabla_{w_1} w_1 = 0, \quad \nabla_{w_2} w_1 = \left(-1 - f^2 + \frac{1}{f^2}\right) w_3, \quad \nabla_{w_3} w_1 = \left(1 - f^2 + \frac{1}{f^2}\right) w_2,$$
$$\nabla_{w_1} w_2 = \left(f^2 + \frac{1}{f^2} - 1\right) w_3, \quad \nabla_{w_2} w_2 = \frac{e_3(f)}{f^2} w_3,$$
$$\nabla_{w_3} w_2 = \left(f^2 - 1 - \frac{1}{f^2}\right) w_1 + e_2(f) w_3, \tag{3.4}$$
$$\nabla_{w_1} w_3 = \left(1 - f^2 - \frac{1}{f^2}\right) w_2, \quad \nabla_{w_2} w_3 = \left(1 + f^2 - \frac{1}{f^2}\right) w_1 - \frac{e_3(f)}{f^2} w_2,$$
$$\nabla_{w_3} w_3 = -e_2(f) w_2.$$

From the definition of the tensor field h and relations (3.3), we get that $hw_1 = 0$ and

$$hw_2 = \frac{1}{2}(\mathcal{L}_{w_1}\phi)w_2 = \frac{1}{2}\{[w_1, w_3] - \phi[w_1, w_2]\} = \left(f^2 - \frac{1}{f^2}\right)w_2. \tag{3.5}$$

Similarly, we obtain that

$$hw_3 = \left(\frac{1}{f^2} - f^2\right)w_3. \tag{3.6}$$

As a consequence, $\{w_1, w_2, w_3\}$ is a globally defined h-basis. Furthermore, combining relations (3.2), (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned}
&(\nabla_{w_1}h)w_1 = 0,\\
&(\nabla_{w_1}h)w_2 = 2\left(f^2 - \frac{1}{f^2}\right)\left(f^2 + \frac{1}{f^2} - 1\right)w_3,\\
&(\nabla_{w_1}h)w_3 = 2\left(f^2 - \frac{1}{f^2}\right)\left(f^2 + \frac{1}{f^2} - 1\right)w_2.
\end{aligned} \tag{3.7}$$

In the sequel, we compute the tensor field $\tau\phi$ of $\mathbf{S}^3$ with respect to the contact metric structure (η, ξ, ϕ, g). We remind that the tensor field $\tau\phi$ is given by $\tau\phi(X, Y) = \tau(\phi X, Y)$, for all $X, Y \in \mathcal{X}(\mathbf{S}^3)$. Indeed, by using relations (2.3), (3.5) and (3.6), we have

$$\begin{aligned}
&\tau\phi(w_1, w_1) = 0, \quad \tau\phi(w_1, w_2) = 0, \quad \tau\phi(w_1, w_3) = 0,\\
&\tau\phi(w_2, w_1) = 0, \quad \tau\phi(w_2, w_2) = -2\left(f^2 - \frac{1}{f^2}\right), \quad \tau\phi(w_2, w_3) = 0,\\
&\tau\phi(w_3, w_1) = 0, \quad \tau\phi(w_3, w_2) = 0, \quad \tau\phi(w_3, w_3) = 2\left(f^2 - \frac{1}{f^2}\right).
\end{aligned} \tag{3.8}$$

Now, combining relations (2.3), (3.7) and (3.8), we have

$$\nabla_{w_1}\tau = 2\alpha\tau\phi, \tag{3.9}$$

where $\alpha = 1 - f^2 - \dfrac{1}{f^2}$. As a consequence, we yield the following Theorem:

THEOREM 3.1. *Let $\mathbf{S}^3$ be the 3-unit sphere and f be an arbitrary smooth function of $\mathbf{S}^3$, non-vanishing everywhere on $\mathbf{S}^3$ which is a solution of the following partial differential equation*:

$$-x_2\frac{\partial f}{\partial x_1} + x_1\frac{\partial f}{\partial x_2} - x_4\frac{\partial f}{\partial x_3} + x_3\frac{\partial f}{\partial x_4} = 0.$$

We consider the quadruple (η, ξ, ϕ, g_f) *given by*

$$g_f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{f^2} & 0 \\ 0 & 0 & f^2 \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -f^2 \\ 0 & \dfrac{1}{f^2} & 0 \end{pmatrix}, \quad \xi = e_1, \quad \eta(W) = g_f(W, e_1),$$

with respect to the basis $\{e_1, e_2, e_3\}$ *and for all* $W \in \mathscr{X}(\mathbf{S}^3)$. *Then* $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ *is a* $3 - \tau - \alpha$ *manifold with* $\alpha = 1 - f^2 - \dfrac{1}{f^2}$.

Remark 3.1. We choose $f \equiv 1$ on $\mathbf{S}^3$. In this case, relations (3.5) and (3.6) give $h = 0$ i.e. $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ is a K-contact metric manifold or, equivalently, a Sasakian manifold. Furthermore, if we choose $f \equiv -1$ on $\mathbf{S}^3$, then we get again a Sasakian structure on $\mathbf{S}^3$. We mention that these two Sasakian structures are the only Sasakian structures of Theorem 3.1.

In the sequel, we thoroughly investigate some curvature properties of $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$. More precisely, we have

THEOREM 3.2. *Let* $\mathbf{S}^3$ *be the* 3-*unit sphere and* f *be an arbitrary smooth function of* $\mathbf{S}^3$, *non-vanishing everywhere on* $\mathbf{S}^3$ *which is a solution of the partial differential equation* (3.2). *Then,* $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ *is a* (κ, μ)-*c.m.m. if and only if* $f = c = const. \neq 0$. *Especially, if* $f = \pm 1$, *then we get the standard Sasakian structure on* $\mathbf{S}^3$. *If* $f = c = const. \neq \pm 1, 0$, *then* $\kappa = 3 - c^4 - \dfrac{1}{c^4}$ *and* $\mu = 2\left(1 - c^2 - \dfrac{1}{c^2}\right)$.

Proof. We assume that $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ is a (κ, μ)-c.m.m. or, equivalently, relation (2.7) holds for the real constants κ and μ. Furthermore, the Jacobi operator l of a (κ, μ)-contact metric manifold is given by ([2])

$$l = -\kappa\phi^2 + \mu h. \tag{3.10}$$

Combining relations (3.5) and (3.10), we have

$$l(w_2) = \left[\kappa + \mu\left(f^2 - \frac{1}{f^2}\right)\right] w_2. \tag{3.11}$$

On the other hand, by using relations (3.2), (3.3) and (3.4), we straightforward calculate

$$\begin{aligned} l(w_2) &= R(w_2, w_1)w_1 = \nabla_{w_2}\nabla_{w_1}w_1 - \nabla_{w_1}\nabla_{w_2}w_1 - \nabla_{[w_2, w_1]}w_1 \\ &= \left[3 - 3f^4 + 2f^2 - \frac{2}{f^2} + \frac{1}{f^4}\right] w_2. \end{aligned} \tag{3.12}$$

Comparing relations (3.11) and (3.12), we easily conclude that $f = c = const. \neq 0$. Conversely, we assume that $f = c = const. \neq 0$. In the case which $f = \pm 1$, we get the standard Sasakian structure on $\mathbf{S}^3$ (see Remark 3.1). In the following, we deal with the case $f = c = const. \neq \pm 1, 0$. Setting now, $\kappa = 3 - c^4 - \frac{1}{c^4}$, $\mu = 2\left(1 - c^2 - \frac{1}{c^2}\right)$ and using the relations (3.3), (3.4), (3.5) and (3.6), we easily deduce that

$$\begin{aligned}
R(w_2, w_1)w_1 &= \left[3 - 3c^4 + 2c^2 - \frac{2}{c^2} + \frac{1}{c^4}\right] w_2 \\
&= \kappa(\eta(w_1)w_2 - \eta(w_2)w_1) + \mu(\eta(w_1)hw_2 - \eta(w_2)hw_1), \\
R(w_3, w_1)w_1 &= \left[c^4 - \frac{3}{c^4} + 3 + \frac{2}{c^2} - 2c^2\right] w_3, \\
&= \kappa(\eta(w_1)w_3 - \eta(w_3)w_1) + \mu(\eta(w_1)hw_3 - \eta(w_3)hw_1), \\
R(w_2, w_3)w_1 &= 0 \\
&= \kappa(\eta(w_3)w_2 - \eta(w_2)w_3) + \mu(\eta(w_3)hw_2 - \eta(w_2)hw_3).
\end{aligned}$$

By direct calculation, these relations yield:

$$R(Z, W)\xi = \kappa[\eta(W)Z - \eta(Z)W] + \mu[\eta(W)hZ - \eta(Z)hW],$$

for all vector fields Z, W on $\mathbf{S}^3$. Hence, it has been shown that $[\mathbf{S}^3, (\eta, \xi, \phi, g_c)]$ is a (κ, μ)-c.m.m. □

Remark 3.2. The (κ, μ)-structures on $\mathbf{S}^3$ (which is diffeomorphic with the Lie group $SU(2)$), described in Theorem 3.2, coincide with the ones given in the main Theorem of [2]. Moreover, we explicitly exhibit the structure tensors (η, ξ, ϕ, g_c) on these structures.

Remark 3.3. Let f be a non-constant smooth function of $\mathbf{S}^3$ non-vanishing everywhere on $\mathbf{S}^3$ which additionally satisfies the partial differential equation (3.2). Then, $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ is a $3 - \tau - \alpha$ c.m.m. which is not a generalized (κ, μ)-c.m.m. Indeed, if it were a generalized (κ, μ)-c.m.m., then using relation (13) of [12] with $\lambda = f^2 - \frac{1}{f^2}$ (in the case which $\lambda > 0$), we would get

$$[w_2, w_3] = -\frac{\left(1 + \frac{1}{f^4}\right)e_3(f)}{f^2 - \frac{1}{f^2}} w_2 + \frac{\left(f^2 + \frac{1}{f^2}\right)e_2(f)}{f^2 - \frac{1}{f^2}} w_3 + 2w_1. \tag{3.13}$$

Comparing relations (3.3) and (3.13), we obtain that $e_2(f) = 0$. Since $e_1(f) = 0$, using relations (3.3), we have that $e_3(f) = 0$ i.e. f is a constant which is a contradiction. We mention that every generalized (κ, μ)-contact metric manifold is a $3-\tau-\alpha$ manifold ([10], [11]). Analogously, we work in the case which $\lambda < 0$.

Theorem 3.3. *Let* $\mathbf{S}^3$ *be the* 3*-unit sphere and* f *be an arbitrary smooth function of* $\mathbf{S}^3$, *non-vanishing everywhere on* $\mathbf{S}^3$ *which is a solution of the partial differential equation* (3.2). *Then, the following conditions are equivalent*:

(i) $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ *has constant* ξ*-sectional curvature.*

(ii) $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ *is* η*-Einstein.*

(iii) $f = \pm 1$ *and* $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ *is a Sasakian manifold.*

Proof. (i) $\mapsto$ (iii) We assume that $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ has constant ξ-sectional curvature equals to $\bar{c}$. This implies that

$$g_f(R(\xi, w_2)w_2, \xi) = \bar{c} \quad \text{and} \quad g_f(R(\xi, w_3)w_3, \xi) = \bar{c} \tag{3.14}$$

We will prove that $\lambda = f^2 - \frac{1}{f^2} \equiv 0$ on $\mathbf{S}^3$. On the contrary, we assume that there exists a point $p \in \mathbf{S}^3$ such that $\lambda(p) \neq 0$. Hence, we have either $\lambda(p) > 0$ or $\lambda(p) < 0$. We deal with the case $\lambda(p) > 0$. Since the function λ is continuous, there exists an open neighborhood W of p such that $\lambda(q) > 0$ for all $q \in W$. Combining relations (2-15) of [9] (see also [5]), (3.4) and (3.14), we get

$$-2\left(f^2 + \frac{1}{f^2} - 1\right)\lambda - \lambda^2 + 1 = \bar{c}, \tag{3.15}$$

$$2\left(f^2 + \frac{1}{f^2} - 1\right)\lambda - \lambda^2 + 1 = \bar{c}, \tag{3.16}$$

on W. Subtracting the relations (3.15) and (3.16), we get $\left(f^2 + \frac{1}{f^2} - 1\right) \cdot \left(f^2 - \frac{1}{f^2}\right) = 0$. Since $f^2(z) + \frac{1}{f^2(z)} - 1 \neq 0$ for every $z \in \mathbf{S}^3$, we deduce that $f^2 - \frac{1}{f^2} = 0$ on W, which is a contradiction. As a consequence, $\lambda = f^2 - \frac{1}{f^2} \equiv 0$ on $\mathbf{S}^3$ or, equivalently, $f = \pm 1$. Applying Theorem 3.2, $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ is a Sasakian manifold. Similarly, we deal with the case $\lambda(p) < 0$.

(ii) $\mapsto$ (iii) We assume that $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ is η-Einstein. We will prove that $\lambda = f^2 - \frac{1}{f^2} \equiv 0$ on $\mathbf{S}^3$. On the contrary, we assume that there exists a point $p \in \mathbf{S}^3$ such that $\lambda(p) \neq 0$. Hence, we have either $\lambda(p) > 0$ or $\lambda(p) < 0$. We deal with the case $\lambda(p) > 0$. Since the function λ is continuous, there exists an open neighborhood W of p such that $\lambda(q) > 0$ for all $q \in W$. By using the h-basis $\{w_1, w_2, w_3\}$ and combining relations (2-18) of [9] (see also [5]), (2.6) and (3.4), we obtain

$$\frac{r}{2} - 1 + \lambda^2 - 2\left(f^2 + \frac{1}{f^2} - 1\right)\lambda = \frac{r}{2} - 1 + \lambda^2 + 2\left(f^2 + \frac{1}{f^2} - 1\right)\lambda,$$

on W. Equivalently, $\left(f^2+\frac{1}{f^2}-1\right)\left(f^2-\frac{1}{f^2}\right)=0$. Since $f^2(z)+\frac{1}{f^2(z)}-1 \neq 0$ for every $z \in \mathbf{S}^3$, we deduce that $f^2-\frac{1}{f^2}=0$ on W, which is a contradiction. As a consequence, $\lambda = f^2-\frac{1}{f^2} \equiv 0$ on $\mathbf{S}^3$ or, equivalently, $f=\pm 1$. Applying Theorem 3.2, $[\mathbf{S}^3,(\eta,\xi,\phi,g_f)]$ is a Sasakian manifold. Similarly, we deal with the case $\lambda(p)<0$.

(iii) $\mapsto$ (i), (ii) We assume that $f=\pm 1$. By using relations (3.4), we deduce that $[\mathbf{S}^3,(\eta,\xi,\phi,g_{\pm 1})]$ is a space of constant sectional curvature equals 1. Hence, $\bar{c}=1$ and $S=2g$. □

Remark 3.4. By using relations (3.4), the ϕ-sectional curvature of $[\mathbf{S}^3,(\eta,\xi,\phi,g_f)]$ is given by

$$\begin{aligned} K(w_2,\phi w_2) &= g_f(R(w_2,w_3)w_3,w_2) \\ &= -fe_2(e_2(f))+\frac{1}{f}e_3\left(\frac{e_3(f)}{f^2}\right)-\left(\frac{e_3(f)}{f^2}\right)^2-(e_2(f))^2 \\ &\quad -\left[\left(1+f^2-\frac{1}{f^2}\right)\left(1-f^2+\frac{1}{f^2}\right)+2\left(1-f^2-\frac{1}{f^2}\right)\right]. \end{aligned}$$

Let $[M,(\eta,\xi,\phi,g)]$ be a contact metric 3-manifold. A ***D*-homothetic transformation** ([2], [11]) is the transformation:

$$\bar{\eta}=t\eta, \quad \bar{\xi}=\frac{1}{t}\xi, \quad \bar{\phi}=\phi, \quad \bar{g}=tg+t(t-1)\eta\otimes\eta \tag{3.17}$$

at the structure tensors where t is a positive constant. It is well known [11] that $[M,(\bar{\eta},\bar{\xi},\bar{\phi},\bar{g})]$ is also a contact metric manifold. Moreover, the curvature tensor R and the tensor h transform in the following manner ([2], [11]):

$$\bar{h}=\frac{1}{t}h \tag{3.18}$$

and

$$\begin{aligned} t\bar{R}(X,Y)\bar{\xi} &= R(X,Y)\xi+(t-1)^2[\eta(Y)X-\eta(X)Y] \\ &\quad -(t-1)[(\nabla_X\phi)Y-(\nabla_Y\phi)X+\eta(X)(Y+hY)-\eta(Y)(X+hX)]. \end{aligned}$$

Moreover, it is well known [1, p. 94] that every 3-dimensional contact metric manifold is a contact strongly pseudo-convex integrable CR manifold, or, equivalently, satisfies the condition

$$(\nabla_X\phi)Y=g(X+hX,Y)\xi-\eta(Y)(X+hX).$$

Using the above relations we finally obtain that

$$\bar{R}(X,Y)\bar{\xi} = \frac{1}{t}R(X,Y)\xi + \frac{t^2-1}{t}[\eta(Y)X - \eta(X)Y] + \frac{2(t-1)}{t}[\eta(Y)hX - \eta(X)hY].$$

Hence, by using relations (2.1), we get

$$\bar{l}(X) = \frac{1}{t^2}l(X) - \frac{t^2-1}{t^2}\phi^2 X + \frac{2(t-1)}{t^2}hX. \tag{3.19}$$

PROPOSITION 3.1. *Let $\mathbf{S}^3$ be the 3-unit sphere and f be an arbitrary smooth function of $\mathbf{S}^3$, nonvanishing everywhere on $\mathbf{S}^3$, which is a solution of the partial differential equation* (3.2). *For any positive parameter t, the corresponding D-homothetic transformation of (η, ξ, ϕ, g_f) yields a $3-\tau-\alpha$ contact metric structure on $\mathbf{S}^3$.*

Proof. Let $\mathbf{S}^3$ be the 3-unit sphere and f be an arbitrary smooth function of $\mathbf{S}^3$, non-vanishing everywhere on $\mathbf{S}^3$ which is a solution of the partial differential equation (3.2). Applying Theorem 3.1, we have that $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ is a $3-\tau-\alpha$ manifold with $\alpha = 1 - f^2 - \frac{1}{f^2}$. Furthermore, applying Proposition 2.1 we deduce that

$$l\phi - \phi l = 4\alpha h\phi. \tag{3.20}$$

Applying a D-homothetic transformation on $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$, we obtain a new contact metric structure on $\mathbf{S}^3$ which is denoted by $[\mathbf{S}^3, (\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g}_f)]$. Combining (3.17), (3.18), (3.19), (3.20) and the fact that h anticommutes with ϕ, we get

$$\bar{l}\bar{\phi} - \bar{\phi}\bar{l} = \bar{l}\phi - \phi\bar{l} = 4\frac{\alpha+t-1}{t}\bar{h}\bar{\phi}.$$

Therefore, by Proposition 2.1, we easily conclude that $[\mathbf{S}^3, (\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g}_f)]$ is a $3-\bar{\tau}-\bar{\alpha}$ manifold with $\bar{\alpha} = \frac{\alpha+t-1}{t}$. □

Remark 3.5. The family of contact metric structures on $\mathbf{S}^3$ described in Theorem 3.1 isn't invariant for D-homothetic transformations because the Reeb vector field $\bar{\xi}$ isn't the same with the initial Reeb vector field ξ. We remind that $\bar{\xi} = \frac{1}{t}\xi$. On the contrary, $[\mathbf{S}^3, (\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g}_f)]$ remains a $3-\tau-\alpha$ manifold.

Example 3.1. We consider the smooth function f on $\mathbf{S}^3$ given by:

$$f(x_1, x_2, x_3, x_4) = \begin{cases} e^{-1/(x_1^2+x_2^2-x_3^2-x_4^2)}, & x_1^2+x_2^2 > x_3^2+x_4^2, \\ 1, & x_1^2+x_2^2 \leqq x_3^2+x_4^2 \end{cases}$$

Obviously, the function f satisfies the differential equation (3.2). Hence, applying Theorem 3.1, we get that $[\mathbf{S}^3, (\eta, \xi, \phi, g_f)]$ is a $3-\tau-\alpha$ manifold. Furthermore, the set $C = \{(x_1, x_2, x_3, x_4) \in \mathbf{S}^3 : x_1^2 + x_2^2 < x_3^2 + x_4^2\}$ equipped with the structures tensors (η, ξ, ϕ, g_f) is a Sasakian manifold and the set $\mathbf{S}^3 - \bar{C} = \{(x_1, x_2, x_3, x_4) \in \mathbf{S}^3 : x_1^2 + x_2^2 > x_3^2 + x_4^2\}$ equipped with the structures tensors (η, ξ, ϕ, g_f) is not a Sasakian manifold.

Example 3.2. Let p be a positive real constant. We consider the differentiable function f_p on $\mathbf{S}^3$ given by:

$$f_p(x_1, x_2, x_3, x_4) = \begin{cases} e^{-1/(x_1^2+x_2^2-x_3^2-x_4^2)} + p, & x_1^2 + x_2^2 > x_3^2 + x_4^2, \\ 1 + p, & x_1^2 + x_2^2 \leqq x_3^2 + x_4^2 \end{cases}$$

Obviously, the function f_p satisfies the partial differential equation (3.2). Hence, applying Theorem 3.1, we get that $[\mathbf{S}^3, (\eta, \xi, \phi, g_{f_p})]$ is a $3-\tau-\alpha$ manifold. Furthermore, by using Example 3.2, the quadruple $[C, (\eta, \xi, \phi, g_f)]$ (mentioned in Example 3.3) is a non-Sasakian (κ, μ)-c.m.m. with $\kappa = 3 - (1+p)^4 - \dfrac{1}{(1+p)^4}$ and $\mu = 2\left(1 - (1+p)^2 - \dfrac{1}{(1+p)^2}\right)$ and the quadruple $[\mathbf{S}^3 - \bar{C}, (\eta, \xi, \phi, g_f)]$ is not a (κ, μ)-c.m.m.

Remark 3.6. Examples 3.1 and 3.2 are never $3-\tau$ since their function α cannot vanish. Additionally, applying Theorem 3.3, the ξ-sectional curvature of these Examples is never constant.

Remark 3.7. Let f be a differentiable function of $\mathbf{S}^3$ which satisfies the partial differential equation (3.2). We suppose that the closed set $A = \{p \in \mathbf{S}^3 : f(p) = 0\}$ is non-empty. Using the notations of Theorem 3.1, the quadruple $[A^c, (\eta, \xi, \phi, g_f)]$ is a $3-\tau-\alpha$ c.m.m. However, this structure (η, ξ, ϕ, g_f) cannot be extended to an open set W of $\mathbf{S}^3$ such that $A^c \subset W$ and $W \cap \partial A \neq \emptyset$. On the contrary, we assume that the quadruple $[A^c, (\eta, \xi, \phi, g_f)]$ is extended to the quadruple $[W, (\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})]$. Let $p \in W \cap \partial A$. Then, there exists a sequence $a_n \in A^c$ such that $\lim a_n = p$. Then, $g_{f|a_n}(e_2, e_2) = \dfrac{1}{f^2(a_n)}$. Using the fact that the $\bar{g}$ is a tensor field and the quadruple $(\bar{\eta}, \bar{\xi}, \bar{\phi}, \bar{g})$ is an extension of the quadruple (η, ξ, ϕ, g_f), we have $\bar{g}_p(e_2, e_2) = \lim \bar{g}_{a_n}(e_2, e_2) = \lim g_{f|a_n}(e_2, e_2) = \lim \dfrac{1}{f^2(a_n)} = +\infty$. As a consequence, the quadruple (η, ξ, ϕ, g_f) cannot be extended in the previous meaning.

Acknowledgment. The authors wish to thank the referee for useful comments on the manuscript.

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