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KODAI MATH. J.
36 (2013), 174–178

CHERN CLASSES AND THE ROST COHOMOLOGICAL INVARIANT

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1. Introduction

Let G be a simple simply connected Lie group G and G_k the corresponding split linear algebraic group over a field $k \subset \mathbf{C}$. Let p be a prime number. The cohomological invariant $Inv^*(G_k; \mathbf{Z}/p)$ is the ring of natural maps $H^1(F; G_k) \to H^*(F; \mathbf{Z}/p)$ for finitely generated field F over k. When the complex Lie group G has p-torsion in $H^*(G)$, Rost constructed a nonzero invariant $R(G_k) \in Inv^3(G_k; \mathbf{Z}/p)$ ([Ga-Me-Se]).

In this paper, we give a short proof of the existence of the Rost invariant for $k = \mathbf{C}$, by using the motivic cohomology and Chern classes of complex representations of G.

2. Motivic cohomology

Recall that $H^1(k; G_k)$ is the first non abelian Galois cohomology set of G_k, which represents the set of G_k-torsors over k. The cohomological invariant is defined by

$$Inv^i(G_k, \mathbf{Z}/p) = Func(H^1(F; G_k) \to H^i(F; \mathbf{Z}/p))$$

where *Func* means the additive group of natural functions for each field F which is finitely generated over k. (For detailed definition or properties, see the books [Ga-Me-Se], [Ga].)

Let BG_k be the classifying space ([To]) of G_k. Totaro proved [Ga-Me-Se] the following theorem in a letter to Serre.

Theorem 2.1 (Totaro). $Inv^*(G_k; \mathbf{Z}/p) \cong H^0(BG_k; H^*_{\mathbf{Z}/p})$.

Here $H^*(X; H^{*\prime}_{\mathbf{Z}/p})$ is the cohomology of the Zariski sheaf induced from the presheaf $H^*_{et}(V; \mathbf{Z}/p)$ for open subsets V of X. This sheaf cohomology is also

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2010 *Mathematics Subject Classification*. Primary 11E72, 12G05, 55P35; Secondary 55T25, 57T05.

Key words and phrases. Rost invariant, motivic cohomology, Chern classes.
Received April 17, 2012.

isomorphic to the E_2-term

$$E_2^{*,*'} \cong H^*(BG_k; H^{*'}_{\mathbf{Z}/p}) \Rightarrow H^*(BG_k; \mathbf{Z}/p)$$

of the coniveau spectral sequence by Bloch-Ogus [Bl-Og].

Next we recall the motivic cohomology. Let X be a smooth (quasi projective) variety over a field $k \subset \mathbf{C}$. Let $H^{*,*'}(X; \mathbf{Z}/p)$ be the mod(p) motivic cohomology defined by Voevodsky and Suslin ([Vo1,2]).

Recently M. Rost and V. Voevodsky ([Vo4], [Su-Jo], [Ro]) proved the Bloch-Kato conjecture. The Bloch-Kato conjecture implies the Beilinson-Lichtenbaum conjecture. Hence, there holds

$$H^{m,n}(X; \mathbf{Z}/p) \cong H^m_{et}(X; \mu_p^{\otimes n}) \quad \text{for all } m \le n.$$

In this paper, we assume that k contains a primitive p-th root of unity. Then there exists an isomorphism $H^m_{et}(X; \mu_p^{\otimes n}) \cong H^m_{et}(X; \mathbf{Z}/p)$. Let τ be a generator of $H^{0,1}(Spec(k); \mathbf{Z}/p) \cong \mathbf{Z}/p$, so that

$$colim_i\, \tau^i H^{*,*'}(X; \mathbf{Z}/p) \cong H^*_{et}(X; \mathbf{Z}/p).$$

The Beilinson-Lichtenbaum conjecture also implies the exact sequences of cohomology theories below

Theorem 2.2 ([Or-Vi-Vo], [Vo4]). *There is the long exact sequence*

$$\begin{aligned} &\to H^{m,n-1}(X; \mathbf{Z}/p) \xrightarrow{\times\tau} H^{m,n}(X; \mathbf{Z}/p) \\ &\to H^{m-n}(X; H^n_{\mathbf{Z}/p}) \to H^{m+1,n-1}(X; \mathbf{Z}/p) \xrightarrow{\times\tau} \cdots \end{aligned}$$

In particular, we have

Corollary 2.3. *The cohomology $H^{m-n}(X; H^n_{\mathbf{Z}/p})$ is additively isomorphic to*

$$H^{m,n}(X; \mathbf{Z}/p)/(\tau) \oplus Ker(\tau)|H^{m+1,n-1}(X; \mathbf{Z}/p)$$

where $H^{m,n}(X; \mathbf{Z}/p)/(\tau) = H^{m,n}(X; \mathbf{Z}/p)/(\tau H^{m,n-1}(X; \mathbf{Z}/p))$.

Corollary 2.4. *The map $\times\tau : H^{m,m-1}(X; \mathbf{Z}/p) \to H^{m,m}(X; \mathbf{Z}/p)$ is injective.*

3. Lie groups

In this section, we assume that $k = \mathbf{C}$ the field of complex numbers. Suppose that G is a simple simply connected Lie group having p-torsion in $H^*(G)$, namely ([Mi-To], [Ka])

$$(G, p) = \begin{cases} G_2, F_4, E_6, E_7, E_8, Spin_n \ (n \ge 7) & \text{for } p = 2 \\ F_4, E_6, E_7, E_8 & \text{for } p = 3, \\ E_8 & \text{for } p = 5. \end{cases}$$

It is well known ([Mi-To], [Ka]) that G is 2-connected and there is an element $x_3(G) \in H^3(G; \mathbf{Z}/p) \cong \mathbf{Z}/p$ with $Q_1 x_3(G) \neq 0$ for the Milnor operation Q_1. Note that for each inclusion $i : G \subset G'$ for above groups, we know $i^*(x_3(G')) = x_3(G)$. Consider the classifying space BG and its cohomology. Denote by $x_4(G)$ the transgression of $x_3(G)$ in $H^4(BG; \mathbf{Z}/p)$, namely, $x_4(G)$ generates $H^4(BG; \mathbf{Z}/p) \cong \mathbf{Z}/p$ and $Q_1(x_4(G)) \neq 0$. We write by $\bar{x}_4(G) \in H^*(BG : \mathbf{Z}_{(p)})$ the integral lift of $x_4(G)$.

LEMMA 3.1. *The element $p\bar{x}_4(G) \in H^4(BG)_{(p)}$ is represented by the Chern class $c_2(\xi)$ of a complex representation $\xi : G \to U(M)$ for some $M > 0$.*

Proof. We only need to prove for $G = Spin_n$, $p = 2$ and $G = E_8$ for odd primes. Because when $p = 2$, there is an inclusion $i : G \subset Spin_N$ for some N so that $i^*(\bar{x}_4(Spin_N)) = \bar{x}_4(G)$. For odd prime cases, there is an inlusion $i : G \subset E_8$, such that $i^*(\bar{x}_4(E_8)) = \bar{x}_4(G)$.

The complex representation ring is known ([Ad]) for $N = 2n + 1$

$$R(Spin_N) \cong \mathbf{Z}[\lambda_1, \dots, \lambda_{n-1}, \Delta_{\mathbf{C}}],$$

where λ_i is the i-th elementary symmetric function in variables $z_1^2 + z_1^{-2}, \dots, z_n^2 + z_n^{-2}$ in $R(T) \cong \mathbf{Z}[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]$ for the maximal torus T in $Spin_N$. Let T^1 be the first factor of T and $\eta : T^1 \subset Spin_N$. Then it is proved (page 1052 in [Sc-Ya]) that

$$\eta^* c_2(\lambda_1) = 4u^2, \quad \eta^* \bar{x}_4(Spin_N) = 2u^2$$

where u is the generator of $H^2(BT^1; \mathbf{Z}) = \mathbf{Z}$. This implies $2\bar{x}_4(Spin_N) = c_2(\lambda_1)$.

Let $\alpha : E_8 \to SO(248)$ be the adjoint representation of E_8. By the construction of the exceptional Lie group E_8 in [Ad], there exists a homomorphism $\beta : \mathrm{Spin}(16) \to E_8$ such that the induced representation of $\alpha \circ \beta$ is the direct sum of $\lambda_{16}^2 : \mathrm{Spin}(16) \to SO(120)$ and $\Delta_{16}^+ : \mathrm{Spin}(16) \to SO(128)$. Let T^8 be the maximal torus of Spin(16). Let T^1 be the first factor of T^8 and $\eta : T^1 \to \mathrm{Spin}(16)$ the inclusion of T^1 into Spin(16). Then it is proved (Proposition 1.2, page 372 in [Ka-Ya]) that the total Chern class of the complexification of $\alpha \circ \beta \circ \eta$ is

$$1 - 120u^2 + \cdots \in \mathbf{Z}[u] \cong H^*(BT^1; \mathbf{Z}).$$

Since $120 = 2^3 \cdot 3 \cdot 5$, the Chern class $c_2(\alpha)$ represents $\gamma p\bar{x}_4(E_8)$ for $p = 3, 5$ in $H^4(BE_8; \mathbf{Z}_{(p)})$, where γ is a unit in $\mathbf{Z}_{(p)}$. (Note $\gamma\bar{x}_4(E_8) \neq c_2(\alpha)$ since $Q_1(x_4(E_8)) \neq 0$.) □

Let $t_{\mathbf{C}} : H^{*,*'}(X; \mathbf{Z}/p) \to H^*(X(\mathbf{C}); \mathbf{Z}/p)$ be the realization map ([Vo1]) for a variety X over $k \subset \mathbf{C}$. Voevodsky defines the Milnor operation Q_i ([Vo1,3]) also in the mod p motivic cohomology

$$Q_i : H^{*,*'}(X; \mathbf{Z}/p) \to H^{*+2p^i-1, *'+p^i-1}(X; \mathbf{Z}/p)$$

which are compatible with the usual (topological) cohomology operations by the realization map $t_{\mathbf{C}}$. For a smooth variety X, the operation

$$Q_i : H^{2*,*}(X;\mathbf{Z}/p) = CH^*(X)/p \to H^{2*+2p^i-1,*+p^i-1}(X;\mathbf{Z}/p) = 0$$

is zero since $2(*+p^i-1)-(2*+2p^i-1) = -1 < 0$.

Theorem 3.2. *Suppose that G is a simple simply connected Lie groups having p-torsion in $H^*(G)$. Let $k = \mathbf{C}$. Then there is a nonzero element $R(G_k) \in Inv^3(G_k;\mathbf{Z}/p)$.*

Proof. From Corollary 2.3, we see

$$Ker(\tau) \mid H^{4,2}(BG_k;\mathbf{Z}/p) \subset H^0(BG_k;H^3_{\mathbf{Z}/p}) \cong Inv^3(G_k;\mathbf{Z}/p).$$

Hence we only need to see the existence of a nonzero element $c \in H^{4,2}(BG_k;\mathbf{Z}/p)$ with $\tau c = 0$.

Since $Q_1(x_4(G)) \neq 0$, there is no element x in $H^{4,2}(BG_k;\mathbf{Z}/p)$ such that $t_{\mathbf{C}}(x) = x_4(G)$, while there exists in $H^{4,4}(BG_k;\mathbf{Z}/p)$ from the Beilinson-Lichtenbaum conjecture.

On the other hand, $c_2(\xi) \in CH^2(BG_k)$, in fact Chow rings have Chern classes. Since $t_{\mathbf{C}}(c_2(\xi)) = p\bar{x}_4(G)$, we see that $c_2(\xi)$ is an additive generator of $H^{4,2}(BG_k)_{(p)}$, so is nonzero in $H^{4,2}(BG_k;\mathbf{Z}/p)$.

Consider the element

$$\tau^2(c_2(\xi)) = px = 0 \in H^{4,4}(BG_k;\mathbf{Z}/p) \cong H^4(BG;\mathbf{Z}/p) \cong \mathbf{Z}/p.$$

From Corollary 2.4, the map $\times\tau : H^{4,3}(BG_k;\mathbf{Z}/p) \to H^{4,4}(BG_k;\mathbf{Z}/p)$ is injective. Hence $\tau c_2(\xi) = 0$ in $H^{4,3}(BG_k;\mathbf{Z}/p)$. □

From the construction of the representation ξ in Lemma 3.1, $c_2(\xi)$ is natural for the natural inclusion $G \subset G'$ between simple Lie groups. Hence the invariant $R(G_k)$ is natural for such embedding $G_k \subset G'_k$.

References

[Ad] J. F. Adams, Lectures on exceptional Lie groups, Univ. Chicago Press, Chicago, IL, 1996.

[Bl-Og] S. Bloch and A. Ogus, Gersten's conjecture and the homology of schemes, Ann. Scient. Éc. Norm. Sup. **7** (1974), 181–202.

[Ga] S. Garibaldi, Cohomological invariants: exceptional groups and spin groups (with an appendix by D. Hoffmann), Memor. Amer. Math. Soc. **200**, no. 937, 2009.

[Ga-Me-Se] S. Garibaldi, A. Merkurjev and J. P. Serre, Cohomological invariants in Galois cohomology, University lect. series **28**, AMS, 2003.

[Ka-Ya] M. Kameko and N. Yagita, Chern subrings, Proc. of Amer. Math. Soc. **138** (2010), 307–324.

[Ka] R. Kane, The homology of Hopf spaces, Noth-Holland Mathematical Library, 1988.

[Mi-To] M. Mimura and H. Toda, Topology of Lie groups, Translations of mathematical monographs **91**, AMS Providence, RI., 1991.

[Or-Vi-Vo] D. Orlov, A. Vishik and V. Voevodsky, An exact sequence for $K_*^M/2$ with applications to quadratic forms, Ann. of Math. **165** (2007), 1–13.

[Ro] M. Rost, On the basic correspondence of a splitting variety, preprint, 2006.

[Sc-Ya] B. Schuster and N. Yagita, Transfers of Chern classes in BP-cohomology and Chow rings, Trans. Amer. Math. Soc. **353** (2001), 1039–1054.

[Su-Jo] A. Suslin and S. Joukhovitski, Norm variety, J. Pure and Appl. Algebra **206** (2006), 245–276.

[To] B. Totaro, The Chow ring of classifying spaces, Algebraic K-theory Univ. of Washington, Seattle, 1997, Proc. of symposia in pure math. **67** (1999), 248–281.

[Vo1] V. Voevodsky, The Milnor conjecture, www.math.uiuc.edu/K-theory/0170, 1996.

[Vo2] V. Voevodsky, Motivic cohomology with $\mathbf{Z}/2$ coefficient, Publ. Math. IHES **98** (2003), 59–104.

[Vo3] V. Voevodsky, Reduced power operations in motivic cohomology, Publ. Math. IHES **98** (2003), 1–57.

[Vo4] V. Voevodsky, On motivic cohomology with $\mathbf{Z}/l$-coefficients, Ann. of Math. **174** (2011), 401–438.

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