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THE UNIQUENESS PROBLEM FOR MEROMORPHIC MAPPINGS WITH TRUNCATED MULTIPLICITIES

Feng Lü

Abstract

The purpose of this work is twofold. The first is to solve a uniqueness problem of meromorphic mappings posed by T. Cao and H. Yi in [1]. The second is to generalize several previous uniqueness theorems of meromorphic mappings "partially" sharing a few moving targets, which were given by Z. Chen and M. Ru [2], Z. Chen and Q. Yan [3], D. Thai and S. Quang [13].

1. The uniqueness problem for hyperplanes

In 1926, R. Nevanlinna [10] showed that if two meromorphic functions have the same inverse images for five distinct values, then these two functions must be identical. In 1975, the Nevanlinna's result was generalized to the case of meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ by H. Fujimoto [6]. In fact, he obtained that for two linearly non-degenerate meromorphic mappings f and g of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$, if they have the same inverse images counted with multiplicities for $3n+2$ hyperplanes in general position in $\mathbf{P}^n(\mathbf{C})$, then $f=g$. Over the last few decades, there have been a lot of results related this problem. (see H. Fujimoto [7], S. Ji [9], M. Ru [11], Z. Chen and Q. Yan [15])

Let f be a linearly non-degenerate meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. For each hyperplane H we denote by $v_{(f,H)}$ the map of $\mathbf{C}^m$ into N_0 such that $v_{(f,H)}(a)$ ($a \in \mathbf{C}^m$) is the intersection multiplicity of the image of f and H at $f(a)$. Take q hyperplanes $H_1, \dots, H_q$ in $\mathbf{P}^n(\mathbf{C})$ in general position and a positive integer l_0.

Consider the family $\mathscr{F}_{\le m}(\{H_j\}_{j=1}^q, f, l_0)$ of all linearly non-degenerate meromorphic mappings $g : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ satisfying the conditions:

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(a) $\min\{v_{(g,H_j),\le m}, l_0\} = \min\{v_{(f,H_j),\le m}, l_0\}$ for all $j \in \{1,\dots,q\}$,
(b) $\dim(f^{-1}(H_i) \cap f^{-1}(H_j)) \le m-2$, for all $1 \le i < j \le q$, and
(c) $f(z) = g(z)$ on $\bigcup_{j=1}^{q}\{z : 0 < v_{(f,H_j)} \le m\}$.
In particular, if $m = \infty$, we omit it for brevity.
In 1983, L. Smiley [12] showed that

Theorem A. *If* $q \ge 3n+2$, *then* $g_1 = g_2$ *for any* $g_1, g_2 \in \mathscr{F}(\{H_j\}_{j=1}^{q}, f, 1)$.

In [8], P. Hu obtained anther uniqueness result of meromorphic functions on $\mathbf{C}^m$ with the idea of truncated multiplicities.

Theorem B. *Let* f *and* g *be two meromorphic functions in* $\mathbf{C}^m$, *let* $a_j \in \mathbf{P}^1(\mathbf{C})$ $(j = 1,\dots,q)$ *be* q *distinct elements, and let* $m_1 \ge m_2 \ge \cdots \ge m_q$ *be* q *positive integers or* ∞. *If* $v^1_{(f,H_j),\le m_j} = v^1_{(g,H_j),\le m_j}$ $(j = 1,\dots,q)$ *and* $\sum_{i=3}^{q}\frac{m_i}{m_i+1} > 2$, *then* $f = g$.

In order to deduce the more smaller number q, T. Cao and H. Yi [1] deduced the following result. They generalized Theorem B from meromorphic functions to meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$.

Theorem C. *Let* f *and* g *be two linearly non-degenerate meromorphic mappings of* $\mathbf{C}^m$ *into* $\mathbf{P}^n(\mathbf{C})$, *let* H_j $(1 \le j \le q)$ *be* $q(\ge 2n)$ *hyperplanes in general position such that* $\dim f^{-1}(H_i \cap H_j) \le n-2$ *for* $i \ne j$, *and let* $m_1 \ge m_2 \ge \cdots \ge m_q \ge n$ *be* q *integers or* ∞. *Assume that*
(a) $v^1_{(f,H_j),\le m_j} = v^1_{(g,H_j),\le m_j}$ $(j = 1,\dots,q)$ *and*
(b) $f(z) = g(z)$ *on* $\bigcup_{j=1}^{q}\{z \in \mathbf{C}^m : 0 < v_{(f,H_j)}(z) \le m_j\}$. *If* $\sum_{i=3}^{q}\frac{m_i}{m_i+1} > A_0$, *where*

$$A_0 = \frac{(n-1)q+n+1}{n} - \frac{4(n-1)}{q+2n-2} + \frac{1}{m_1+1} + \frac{1}{m_2+1},$$

then $f = g$.

In the paper [1], the authors pointed out that if $n = 1$ in Theorem C, then the condition $\sum_{i=3}^{q}\frac{m_i}{m_i+1} > A_0$ reduces to that $\sum_{i=3}^{q}\frac{m_i}{m_i+1} > 2 + \frac{1}{m_1+1} + \frac{1}{m_2+1}$.
But it dot not coincide with the related condition in Theorem B. So, they asked whether one can deduce a better result. In this section, we solve the problem and obtain the following theorem.

Theorem 1.1. *With the same assumptions* (a), (b) *as in Theorem* C, *if*

$$\sum_{i=3}^{q}\frac{m_i}{m_i+1} > A_1 = \frac{(n-1)q+n+1}{n} - \frac{2(n-1)}{m_2+1} - \frac{4(n-1)}{q+2n-2}\left(1 - \frac{n}{m_2+1}\right),$$

then $f = g$.

Remark 1. Noting that $q \geq 2n$, we have

$$\frac{4(n-1)}{q+2n-2}\frac{n}{m_2+1} \leq \frac{4(n-1)}{q}\frac{n}{m_2+1} \leq \frac{4(n-1)}{2n}\frac{n}{m_2+1} \leq \frac{2(n-1)}{m_2+1}.$$

Then, compare Theorem C to Theorem 1.1, we derive

$$\begin{aligned} A_0 - A_1 &= \frac{1}{m_1+1} + \frac{1}{m_2+1} + \frac{2(n-1)}{m_2+1} - \frac{4(n-1)}{q+2n-2}\frac{n}{m_2+1} \\ &\geq \frac{1}{m_1+1} + \frac{1}{m_2+1}. \end{aligned}$$

Obviously, the number A_1 is smaller than A_0. So, we improve Theorem C.

Remark 2. When $n = 1$, we see that the condition $\sum_{i=3}^{q} \frac{m_i}{m_i+1} > A_1$ reduces to $\sum_{i=3}^{q} \frac{m_i}{m_i+1} > 2$. Thus, we solve the above problem posed by T. Cao and H. Yi in [1].

By Theorem 1.1, we obtain the following corollaries which are improvements of the related corollaries in [1].

Corollary 1.2. *If* $q \geq 2n+3$, *then* $\mathscr{F}_{\leq m}(\{H_j\}_{j=1}^{q}, f, 1) = 1$, *where*

$$m > \frac{(n-1)q^2 - (n-3)q + 2n^2 - 2}{(q+n-1)(q-2n-2)}.$$

Corollary 1.3. *If* $q = 3n+2$, *then* $\mathscr{F}_{\leq m}(\{H_j\}_{j=1}^{q}, f, 1) = 1$, *where*

$$m > \frac{9n^2 + 2n - 1}{4n+1}.$$

Corollary 1.4. *If* $q = 2n+3$, *then* $\mathscr{F}_{\leq m}(\{H_j\}_{j=1}^{q}, f, 1) = 1$, *where*

$$m > \frac{4n^3 + 8n^2 - 2}{3n+2}.$$

Corollary 1.5. *If* $q = 3n+1$ *and* $n \geq 2$, *then* $\mathscr{F}_{\leq m}(\{H_j\}_{j=1}^{q}, f, 1) = 1$, *where*

$$m > \frac{9n^2 - 4n + 3}{4(n-1)}.$$

Corollary 1.6. *If* $q = 3n$ *and* $n \geq 3$, *then* $\mathscr{F}_{\leq m}(\{H_j\}_{j=1}^{q}, f, 1) = 1$, *where*

$$m > \frac{9n^3 - 10n^2 + 9n - 2}{(4n-1)(n-2)}.$$

Corollary 1.7. *If* $q = 3n - 1$ *and* $n \geq 4$, *then* $\mathscr{F}_{\leq m}(\{H_j\}_{j=1}^q, f, 1) = 1$, *where*

$$m > \frac{9n^3 - 16n^2 + 17n - 6}{(4n-2)(n-3)}.$$

2. The uniqueness problem for moving targets

Recently, motivated by the establishment of the second main theorem of value distribution theory for moving targets, the study of the uniqueness problem of meromorphic mappings from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ intersecting a finite set of moving targets has started. At the same time, many outstanding results were derived. (See Z. Chen and Q. Yan [2, 3], Z. Tu [14].)

In this section, we still focus on the uniqueness problem for moving targets. In order to state our results, we recall the following.

Let $a_1, \dots, a_q$ $(q \geq n+1)$ be q meromorphic mappings from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ with reduced representations $a_j = (a_{j0} : \cdots : a_{jn})$ $(j = 1, \dots, q)$. We say that $a_1, \dots, a_q$ are located in general position if $\det(a_{j_k l}) \neq 0$ for any $1 \leq j_0 < j_1 < \cdots < j_n \leq q$.

Let f be a linearly non-degenerate meromorphic mapping from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. We say that a_j is "small" (with respects of f) if $T_{a_j}(r) = o(T_f(r))$ as $r \to \infty$.

Let M_m be the field of all meromorphic functions on $\mathbf{C}^m$. Denote by $R(\{a_j\}_{j=1}^q) \subset M_m$ the smallest subfield which contains $\mathbf{C}$ and all $\dfrac{a_{jk}}{a_{jl}}$ with $a_{jl} \neq 0$. Let f be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ with reduced representation $f = (f_0 : \cdots : f_n)$. We say that f is linearly non-degenerate over $R(\{a_j\}_{j=1}^q)$ if $f_0, \dots, f_n$ are linearly independent over $R(\{a_j\}_{j=1}^q)$.

Suppose that f be a meromorphic mapping from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and d be a positive integer. Let $\{a_j\}_{j=1}^q$ be "small" (with respect to f) meromorphic mappings from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ in general position such that

$$\dim\{z \in \mathbf{C}^m : (f, a_i)(z) = (f, a_j)(z) = 0\} \leq m - 2 \quad (1 \leq i \neq j \leq q).$$

Assume that f is linearly non-degenerate over $R(\{a_j\}_{j=1}^q)$. Consider the family $\mathscr{F}(\{a_j\}_{j=1}^q, f, d)$ of all linearly non-degenerate over $R(\{a_j\}_{j=1}^q)$ meromorphic mappings $g : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ satisfying the conditions:

(I) $\min\{v_{(g,a_j)}(z), d\} = \min\{v_{(f,a_j)}(z), d\}$ for all $j \in \{1, \dots, q\}$;

(II) $f(z) = g(z)$ on $\bigcup_{j=1}^q \{z : (f, a_j)(z) = 0\}$.

In [2], Z. Chen and M. Ru studied the uniqueness problem of holomorphic curves and proved the following.

Theorem D. *If* $q \geq 2n^2 + 4n$, *then* $\#\mathscr{F}(\{a_j\}_{j=1}^q, f, 2) \leq 2$.

In 2007, D. Thai and S. Quang [13] improved Theorem D and obtain the following theorem.

THEOREM E. *If $q \geq 2n^2 + 4n$ and $n \geq 2$, then $\#\mathscr{F}(\{a_j\}_{j=1}^q, f, 1) = 1$.*

In 2006, Z. Chen and Q. Yan [3] considered the uniqueness problem of meromorphic mappings in another direction. In fact, they weakened the assumption of sharing moving targets to "partially" sharing moving targets. Here, we say that two meromorphic mappings f, g partially share a moving target a if $\overline{E}(a, f) \subseteq \overline{E}(a, g)$, where $\overline{E}(a, h) = \{z \in \mathbf{C}^m : (h, a)(z) = 0\}$ for a meromorphic mapping h from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. Their result can be described as follows.

THEOREM F. *Let f and g be two meromorphic mappings, let $\{a_j\}_{j=1}^q$ be q "small" (with respect to f) meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ in general position such that $(f, a_j) \not\equiv 0$ and $(g, a_j) \not\equiv 0$ $(1 \leq j \leq q)$, and let f, g be linearly non-degenerate over $R(\{a_j\}_{j=1}^q)$. Assume that*
(1) $\overline{E}(f, a_j) \subseteq \overline{E}(g, a_j)$ $1 \leq j \leq q$;
(2) $\dim \overline{E}(f, a_i) \cap \overline{E}(f, a_j) \leq m - 2$ *for* $1 \leq i \neq j \leq q$;
(3) $f(z) = g(z)$ *on* $\bigcup_{j=1}^q \{z \in \mathbf{C}^m : (f, a_j)(z) = 0\}$. *If* $q = 2n^2 + 4n + 1$ *and*

$$\liminf_{r \to \infty} \sum_{j=1}^q N_{(f, a_j)}^1(r) \Big/ \sum_{j=1}^q N_{(g, a_j)}^1(r) > \frac{n(n+2)}{n(n+2)+1},$$

then $f = g$.

Nowadays, to seek the smaller number q in the above theorems becomes an interesting and meaningful job. In the section, the aim is to replace the number q by a smaller one in Theorem E and F. In fact, we obtain the following two results.

THEOREM 2.1. *If $q \geq 2n^2 + 2n + 3$, then $\#\mathscr{F}(\{a_j\}_{j=1}^q, f, 1) = 1$.*

THEOREM 2.2. *Assume that the conditions are stated as in Theorem F. If $q \geq 2n^2 + 2n + 3$, then $f = g$.*

Remark 3. If $n \geq 2$, our results are improvements of Theorem E and F, respectively.

3. Preliminaries and some lemmas

Set $\|z\| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2}$ for $z = (z_1, \ldots, z_m)$ and define

$$B(r) = \{z \in \mathbf{C}^m : \|z\| < r\}, \quad S(r) = \{z \in \mathbf{C}^m : \|z\| = r\} \quad (0 < r < \infty),$$

and

$$\upsilon_{m-1}(z) = (dd^c \|z\|^2)^{m-1}, \quad \sigma_m(z) = d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1}$$

on $\mathbf{C}^m \backslash \{0\}$.

Let f be a non-constant meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$. We take the holomorphic functions $f_0, \dots, f_n$ on $\mathbf{C}^m$ such that $\mathscr{I}_f = \{z \in \mathbf{C}^m : f_0(z) = \cdots f_n(z) = 0\}$ is of dimension at most $m-2$, and $f = \{f_0, \dots, f_n\}$ is called a reduced representation of f. The characteristic function of f is defined as

$$T_f(r) = \int_{S(r)} \log\|f\|\sigma_m - \int_{S(1)} \log\|f\|\sigma_m.$$

Note that $T_f(r)$ is independent of the choice of the reduced representation of f.

For a divisor v on $\mathbf{C}^m$ and positive integers k, p (or $k, p = \infty$), we define some divisors as follows.

$$v^p(z) = \min\{p, v(z)\},$$

$$v^p_{\le k}(z) = \begin{cases} 0, & \text{if } v(z) > k, \\ v^p(z), & \text{if } v(z) \le k, \end{cases}$$

$$v^p_{>k}(z) = \begin{cases} v^p(z), & \text{if } v(z) > k, \\ 0, & \text{if } v(z) \le k. \end{cases}$$

Define $n(t)$ by

$$n(t) = \begin{cases} \int_{|v| \cap B(t)} v(z) \upsilon_{m-1}, & \text{if } m \ge 2, \\ \sum_{|z| \le t} v(z), & \text{if } m = 1. \end{cases}$$

Similarly, we define $n^p(t)$, $n^p_{\le k}(t)$, $n^p_{>k}(t)$. Define the counting function of v as

$$N(r, v) = \int_1^r \frac{n(t)}{t^{2n-1}}\, dt \quad (1 < r < \infty).$$

Similarly, we define $N(r, v^p)$, $N(r, v^p_{\le k})$, $N(r, v^p_{>k})$ and denote them by $N^p(r, v)$, $N^p_{\le k}(r, v)$, $N^p_{>k}(r, v)$, respectively.

Let $\phi : \mathbf{C}^m \to \mathbf{P}^1(\mathbf{C})$ be a meromorphic function. Define

$$N_\phi(r) = N(r, v_\phi), \quad N^p_\phi(r) = N^p(r, v_\phi),$$

$$N^p_{\phi, \le k}(r) = N^p_{\le k}(r, v_\phi), \quad N^p_{\phi, >k}(r) = N^p_{>k}(r, v_\phi).$$

In order to prove our results, we need the second main theorem for meromorphic mappings.

Lemma 3.1 [13]. *Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerate meromorphic mapping and $H_1, \dots, H_q$ be q hyperplanes in general position in $\mathbf{P}^n(\mathbf{C})$. Then*

$$\| \ (q - n - 1) T_f(r) \le \sum_{j=1}^q N^n_{(f, H_j)}(r) + o(T_f(r)).$$

As usual, by the notation "$\| \ P$" we mean the assertion P holds for all $r \in [0, \infty)$ excluding a Borel subset E of the interval $[0, \infty)$ with $\int_E dr < \infty$.

The following lemma is a modification of the second main theorem, which is essential to the proof of Theorem 1.1.

LEMMA 3.2. *Let $f: \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly non-degenerate meromorphic mapping and $H_1, \dots, H_q$ be q hyperplanes in general position in $\mathbf{P}^n(\mathbf{C})$. Then*

$$\Big\| \quad \left[q - n - 1 - \sum_{i=3}^{q} \frac{n}{m_i + 1} - \frac{2n}{m_2 + 1}\right] T_f(r) \\ \le \sum_{j=1}^{q} \left(1 - \frac{n}{m_2 + 1}\right) N^n_{(f, H_j), \le m_j}(r) + o(T_f(r)),$$

where $m_1 \ge m_2 \ge, \cdots, \ge m_q \ge n$ are integers.

Proof. With Lemma 3.1, we have

$$(3.1) \quad \| \ (q - n - 1) T_f(r) \\ \le \sum_{j=1}^{q} N^n_{(f, H_j)}(r) + o(T_f(r)) \\ = \sum_{j=1}^{q} [N^n_{(f, H_j), \le m_j}(r) + N^n_{(f, H_j), > m_j}(r)] + o(T_f(r)) \\ \le \sum_{j=1}^{q} \left[N^n_{(f, H_j), \le m_j}(r) + \frac{n}{m_j + 1} N_{(f, H_j), > m_j}(r)\right] + o(T_f(r)) \\ = \sum_{j=1}^{q} N^n_{(f, H_j), \le m_j}(r) + \frac{n}{m_j + 1} [N_{(f, H_j)}(r) - N_{(f, H_j), \le m_j}(r)] + o(T_f(r)) \\ \le \sum_{j=1}^{q} N^n_{(f, H_j), \le m_j}(r) + \frac{n}{m_j + 1} [N_{(f, H_j)}(r) - N^n_{(f, H_j), \le m_j}(r)] + o(T_f(r)) \\ = \sum_{j=1}^{q} \left[1 - \frac{n}{m_j + 1}\right] N^n_{(f, H_j), \le m_j}(r) + \frac{n}{m_j + 1} N_{(f, H_j)}(r) + o(T_f(r)) \\ \le \sum_{j=1}^{q} \left[1 - \frac{n}{m_j + 1}\right] N^n_{(f, H_j), \le m_j}(r) + \frac{n}{m_j + 1} T_f(r) + o(T_f(r)) \\ \le \left[1 - \frac{n}{m_1 + 1}\right] N^n_{(f, H_1), \le m_1}(r) + \sum_{j=2}^{q} \left[1 - \frac{n}{m_2 + 1}\right] N^n_{(f, H_j), \le m_j}(r) \\ + \sum_{j=1}^{q} \frac{n}{m_j + 1} T_f(r) + o(T_f(r))$$

$$\le \left[\frac{n}{m_2+1} - \frac{n}{m_1+1}\right] N^n_{(f,H_1),\le m_1}(r) + \sum_{j=1}^{q}\left[1 - \frac{n}{m_2+1}\right] N^n_{(f,H_j),\le m_j}(r)$$

$$+ \sum_{j=1}^{q} \frac{n}{m_j+1} T_f(r) + o(T_f(r))$$

$$\le \left[\frac{n}{m_2+1} - \frac{n}{m_1+1}\right] T_f(r) + \sum_{j=1}^{q}\left[1 - \frac{n}{m_2+1}\right] N^n_{(f,H_j),\le m_j}(r)$$

$$+ \sum_{j=1}^{q} \frac{n}{m_j+1} T_f(r) + o(T_f(r)),$$

which implies that the conclusion of Lemma 3.2 holds.

Hence, we finish the proof of this lemma.

LEMMA 3.3 [13]. *Let f be a meromorphic mapping, and $\{a_j\}_{j=1}^q$ be q $(\ge 2n+1)$ meromorphic mappings from $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ in general position such that f is linearly non-degenerate over $R(\{a_j\}_{j=1}^q)$. Then*

$$\Big\| \quad \frac{q}{n+2} T_f(r) \le \sum_{j=1}^{q} N^n_{(f,a_j)}(r) + o(T_f(r)) + O\left(\max_{1\le j\le q} T_{a_j}(r)\right).$$

4. The proof of Theorem 1.1

On the contrary, suppose that $f \ne g$. In the following, we use the methods of Z. Chen and Q. Yan [4], T. Cao and H. Yi [1], G. Dethloff and T. Tan [5] to handle the problem.

We first introduce an equivalence relation on $L := \{1, \dots, q\}$ as follows $i \sim j$ if and only if $\dfrac{(f,H_i)}{(f,H_j)} - \dfrac{(g,H_i)}{(g,H_j)} = 0$. Set $\{L_1, \dots, L_s\} = L/\sim$. Since $f \ne g$ and $\{H_j\}_{j=1}^q$ are in general position, we have that $\# L_k \le n$ for all $k \in \{1, \dots, s\}$. Without loss of generality, we assume that $L_k := \{i_{k-1}+1, \dots, i_k\}$ $(k \in \{1, \dots, s\})$ where $0 = i_0 < \cdots < i_s = q$.

Define the map $\sigma : \{1, \dots, q\} \to \{1, \dots, q\}$ by

$$\sigma(i) = \begin{cases} i+n, & \text{if } i+n \le q, \\ i+n-q, & \text{if } i+n > q. \end{cases}$$

It is easy to see that σ is bijective and $|\sigma(i) - i| \ge n$ (note that $q \ge 2n$). This implies that i and $\sigma(i)$ belong two distinct sets of $\{L_1, \dots, L_s\}$ and

$$\frac{(f,H_i)}{(f,H_{\sigma(i)})} - \frac{(g,H_i)}{(g,H_{\sigma(i)})} \ne 0. \tag{4.1}$$

Let $P_i = (f, H_i)(g, H_{\sigma(i)}) - (g, H_i)(f, H_{\sigma(i)})$. Obviously, $P_i \neq 0$. With the Jensen formula, we obtain

$$
\begin{aligned}
(4.2) \qquad N_{P_i}(r) &= \int_{S(r)} \log|P_i|\sigma_m + O(1) \\
&\leq \int_{S(r)} \log(|(f, H_i)|^2 + |(f, H_{\sigma(i)})|^2)^{1/2}\sigma_m \\
&\quad + \int_{S(r)} \log(|(g, H_i)|^2 + |(g, H_{\sigma(i)})|^2)^{1/2}\sigma_m + O(1) \\
&\leq T_f(r) + T_g(r) + O(1) = T(r) + O(1),
\end{aligned}
$$

where $T(r) = T_f(r) + T_g(r)$.

Let $k \in \{i, \sigma(i)\}$. Since $v^1_{(f,H_k),\leq m_k} = v^1_{(g,H_k),\leq m_k}$, we have that a zero point z_0 of (f, H_k) with multiplicity $\leq m_k$ is a zero point of (g, H_k) with multiplicity $\leq m_k$. Then z_0 is a zero point of P_i with multiplicity $\geq \min\{v_{(f,H_k)}(z_0), v_{(g,H_k)}(z_0)\}$ (outside an analytic set of codimension ≥ 2).

We also have

$$
\begin{aligned}
(4.3) \qquad &\min\{v_{(f,H_k)}(z_0), v_{(g,H_k)}(z_0)\} \\
&\quad \geq v^n_{(f,H_k),\leq m_k}(z_0) + v^n_{(g,H_k),\leq m_k}(z_0) - nv^1_{(f,H_k),\leq m_k}(z_0).
\end{aligned}
$$

For any $j \in \{1, \dots, q\} \setminus \{i, \sigma(i)\}$, by $f(z) = g(z)$ on $\bigcup_{j=1}^q \{z \in \mathbf{C}^m : 0 < v_{(f,H_j)}(z) \leq m_j\}$, we have that a zero point z_0 of (f, H_j) with multiplicity $\leq m_j$ is a zero point of P_i (outside an analytic set of codimension ≥ 2).

From the above discussions, we deduce

$$
\begin{aligned}
(4.4) \quad & N^n_{(f,H_i),\leq m_i}(r) + N^n_{(g,H_i),\leq m_i}(r) - nN^1_{(f,H_i),\leq m_i}(r) + N^n_{(f,H_{\sigma(i)}),\leq m_{\sigma(i)}}(r) \\
&\quad + N^n_{(g,H_{\sigma(i)}),\leq m_{\sigma(i)}}(r) - nN^1_{(f,H_{\sigma(i)}),\leq m_{\sigma(i)}}(r) + \sum_{j \neq i, \sigma(i)}^{q} N^1_{(f,H_j),\leq m_j}(r) \\
&\leq N_{P_i}(r) \leq T(r).
\end{aligned}
$$

By taking the sum of both sides of (4.4) over $(1 \leq i \leq q)$, we have

$$
\begin{aligned}
(4.5) \qquad & 2\sum_{i=1}^{q} [N^n_{(f,H_i),\leq m_i}(r) + N^n_{(g,H_i),\leq m_i}(r)] \\
&\quad + (q - 2n - 2)\sum_{i=1}^{q} N^1_{(f,H_i),\leq m_i}(r) \leq qT(r).
\end{aligned}
$$

Similarly, we deduce that

$$2\sum_{i=1}^{q}[N^{n}_{(f,H_i),\le m_i}(r)+N^{n}_{(g,H_i),\le m_i}(r)] + (q-2n-2)\sum_{i=1}^{q}N^{1}_{(g,H_i),\le m_i}(r)\le qT(r). \tag{4.6}$$

Combining (4.5), (4.6) and $N^{1}_{(f,H_i),\le m_i}(r)\ge \frac{1}{n}N^{n}_{(f,H_i),\le m_i}(r)$ yields that

$$\frac{q+2n-2}{n}\sum_{i=1}^{q}[N^{n}_{(g,H_i),\le m_i}(r)+N^{n}_{(f,H_i),\le m_i}(r)]\le 2qT(r). \tag{4.7}$$

Rewriting (4.7) as

$$\left[1-\frac{n}{m_2+1}\right]\sum_{i=1}^{q}[N^{n}_{(g,H_i),\le m_i}(r)+N^{n}_{(f,H_i),\le m_i}(r)] \le \frac{2qn}{q+2n-2}\left[1-\frac{n}{m_2+1}\right]T(r). \tag{4.8}$$

With Lemma 3.2 and (4.8), we get

$$\left[q-n-1-\sum_{i=3}^{q}\frac{n}{m_i+1}-\frac{2n}{m_2+1}\right]T(r) \le \frac{2qn}{q+2n-2}\left(1-\frac{n}{m_2+1}\right)T(r)+o(T(r)). \tag{4.9}$$

Furthermore, we obtain

$$\begin{aligned}
&\left[q-n-1-\sum_{i=3}^{q}\frac{n}{m_i+1}-\frac{2n}{m_2+1}\right]T(r) \\
&\left[n(q-2)-\sum_{i=3}^{q}\frac{n}{m_i+1}-n(q-2)+q-n-1-\frac{2n}{m_2+1}\right]T(r) \\
&=\left[n\sum_{i=3}^{q}\frac{m_i}{m_i+1}-(n-1)q+n-1-\frac{2n}{m_2+1}\right]T(r) \\
&\le \frac{2qn}{q+2n-2}\left(1-\frac{n}{m_2+1}\right)T(r)+o(T(r)) \\
&=\left[2n-\frac{2n^2}{m_2+1}-\frac{4n(n-1)}{q+2n-2}\left(1-\frac{n}{m_2+1}\right)\right]T(r)+o(T(r)),
\end{aligned} \tag{4.10}$$

which implies that

$$\sum_{i=3}^{q}\frac{m_i}{m_i+1} \le \frac{(n-1)q+n+1}{n} - \frac{2(n-1)}{m_2+1} - \frac{4(n-1)}{q+2n-2}\left(1-\frac{n}{m_2+1}\right).$$

It contradicts with assumption.

Hence, we complete the proof of Theorem 1.1.

5. The proof of Theorem 2.1

On the contrary, suppose that $f, g \in \mathscr{F}(\{a_j\}_{j=1}^q, f, 1)$ such that $f \not\equiv g$.

Similarly as above, we first introduce an equivalence relation on $L := \{1, \dots, q\}$ as follows $i \sim j$ if and only if $\dfrac{(f, a_i)}{(f, a_j)} - \dfrac{(g, a_i)}{(g, a_j)} \equiv 0$. Set $\{L_1, \dots, L_s\} = L/\sim$. Since $f \not\equiv g$ and $\{a_j\}_{j=1}^q$ are in general position, we have that $\# L_k \le n$ for all $k \in \{1, \dots, s\}$. Without loss of generality, we assume that $L_k := \{i_{k-1}+1, \dots, i_k\}$ $(k \in \{1, \dots, s\})$ where $0 = i_0 < \cdots < i_s = q$.

Define the map $\sigma : \{1, \dots, q\} \to \{1, \dots, q\}$ by

$$\sigma(i) = \begin{cases} i+n, & \text{if } i+n \le q, \\ i+n-q, & \text{if } i+n > q. \end{cases}$$

It is easy to see that σ is bijective and $|\sigma(i) - i| \ge n$ (note that $q \ge 2n^2 + 2n + 2 \ge 2n$). This implies that i and $\sigma(i)$ belong two distinct sets of $\{L_1, \dots, L_s\}$ and

$$\frac{(f, a_i)}{(f, a_{\sigma(i)})} - \frac{(g, a_i)}{(g, a_{\sigma(i)})} \not\equiv 0.$$

Let

$$P_i = \frac{(f, a_i)}{(f, a_{\sigma(i)})} - \frac{(g, a_i)}{(g, a_{\sigma(i)})}. \tag{5.1}$$

Obviously, $P_i \not\equiv 0$.

Since $\min\{v_{(f,a_i)}(z), 1\} = \min\{v_{(g,a_i)}(z), 1\}$, we obtain from (5.1) that

$$\begin{aligned} v_{P_i}(z_0) &\ge \min\{v_{(f,a_i)}(z_0), v_{(g,a_i)}(z_0)\} \\ &\ge v^n_{(f,a_i)}(z_0) + v^n_{(g,a_i)}(z_0) - n v^1_{(g,a_i)}(z_0). \end{aligned} \tag{5.2}$$

For any $j \in \{1, \dots, q\} \setminus \{i, \sigma(i)\}$, by the assumption (II) and (5.1), we have that a zero point z_0 of (g, a_j) is a zero point of P_i (outside an analytic set of codimension ≥ 2).

From the above discussions, we deduce

$$N^n_{(f,a_i)}(r) + N^n_{(g,a_i)}(r) - nN^1_{(g,a_i)}(r) + \sum_{j\neq i,\sigma(i)}^{q} N^1_{(g,a_j)}(r) \tag{5.3}$$
$$\leq N_{P_i}(r) \leq T_{P_i}(r) \leq m(r,P_i) + N_{1/P_i}(r)$$
$$\leq T(r) - N_{(f,a_{\sigma(i)})}(r) - N_{(g,a_{\sigma(i)})}(r) + N_{1/P_i}(r) + O(1),$$

where $T(r) = T_f(r) + T_g(r)$. Obviously,

$$v_{1/P_i}(z) - v_{(f,a_{\sigma(i)})}(z) - v_{(g,a_{\sigma(i)})}(z) \tag{5.4}$$
$$\leq \max\{v_{(f,a_{\sigma(i)})}(z), v_{(g,a_{\sigma(i)})}(z)\} - v_{(f,a_{\sigma(i)})}(z) - v_{(g,a_{\sigma(i)})}(z)$$
$$= -\min\{v_{(f,a_{\sigma(i)})}(z), v_{(g,a_{\sigma(i)})}(z)\}$$
$$\leq -v_{(f,a_{\sigma(i)})}(z) - v_{(g,a_{\sigma(i)})}(z) + nv^1_{(g,a_{\sigma(i)})}(z).$$

Combing (5.3) and (5.4) yields that

$$N^n_{(f,a_i)}(r) + N^n_{(g,a_i)}(r) - nN^1_{(g,a_i)}(r) + N^n_{(f,a_{\sigma(i)})}(r) \tag{5.5}$$
$$+ N^n_{(g,a_{\sigma(i)})}(r) - nN^1_{(g,a_{\sigma(i)})}(r) + \sum_{j\neq i,\sigma(i)}^{q} N^1_{(g,a_j)}(r)$$
$$\leq T(r) + O(1).$$

By taking the sum of both sides of (5.5) over $(1 \leq i \leq q)$, we have

$$\Big\| \ 2\sum_{i=1}^{q}[N^n_{(f,a_i)}(r) + N^n_{(g,a_i)}(r)] + (q-2n-2)\sum_{i=1}^{q} N^1_{(g,a_i)}(r) \tag{5.6}$$
$$\leq qT(r) + O(1) \leq (n+2)\sum_{i=1}^{q}[N^n_{(f,a_i)}(r) + N^n_{(g,a_i)}(r)] + o(T(r)),$$

which implies that

$$\Big\| \ (q-2n-2)\sum_{i=1}^{q} N^1_{(g,a_i)}(r) \tag{5.7}$$
$$\leq n\sum_{i=1}^{q}[N^n_{(f,a_i)}(r) + N^n_{(g,a_i)}(r)] + o(T(r))$$
$$\leq n^2\sum_{i=1}^{q}[N^1_{(f,a_i)}(r) + N^1_{(g,a_i)}(r)] + o(T(r))$$
$$\leq 2n^2\sum_{i=1}^{q} N^1_{(g,a_i)}(r) + o(T(r)).$$

Noting that $q \geq 2n^2 + 2n + 3$ and (5.7), we deduce that $\sum_{i=1}^{q} N^1_{(g,a_i)}(r) = \sum_{i=1}^{q} N^1_{(f,a_i)}(r) = o(T(r))$. By Lemma 3.3, we derive a contradiction.

Thus, we finish the proof of Theorem 2.1.

6. The proof of Theorem 2.2

In the following, we will prove Theorem 2.2 as the way in Theorem 2.1. Obviously, the inequality (5.2) still holds.

For any $j \in \{1, \dots, q\} \backslash \{i, \sigma(i)\}$, by the assumption $\overline{E}(f, a_j) \subseteq \overline{E}(g, a_j)$ and (5.1), we have that a zero point z_0 of (f, a_j) is a zero point of P_i (outside an analytic set of codimension ≥ 2).

From the above discussions, we derive that

$$
\begin{aligned}
(6.1) \qquad & N^n_{(f,a_i)}(r) + N^n_{(g,a_i)}(r) - nN^1_{(g,a_i)}(r) + \sum_{j \neq i, \sigma(i)}^{q} N^1_{(f,a_j)}(r) \\
& \leq N_{P_i}(r) \leq T_{P_i}(r) \leq m(r, P_i) + N_{1/P_i}(r) \\
& \leq T(r) - N_{(f,a_{\sigma(i)})}(r) - N_{(g,a_{\sigma(i)})}(r) + N_{1/P_i}(r) + O(1),
\end{aligned}
$$

where $T(r) = T_f(r) + T_g(r)$. Obviously,

$$
\begin{aligned}
(6.2) \qquad & v_{1/P_i}(z) - v_{(f,a_{\sigma(i)})}(z) - v_{(g,a_{\sigma(i)})}(z) \\
& \leq \max\{v_{(f,a_{\sigma(i)})}(z), v_{(g,a_{\sigma(i)})}(z)\} - v_{(f,a_{\sigma(i)})}(z) - v_{(g,a_{\sigma(i)})}(z) \\
& = -\min\{v_{(f,a_{\sigma(i)})}(z), v_{(g,a_{\sigma(i)})}(z)\} \\
& \leq -v_{(f,a_{\sigma(i)})}(z) - v_{(g,a_{\sigma(i)})}(z) + nv^1_{(g,a_{\sigma(i)})}(z).
\end{aligned}
$$

Combing (6.1) and (6.2) yields that

$$
\begin{aligned}
(6.3) \qquad & N^n_{(f,a_i)}(r) + N^n_{(g,a_i)}(r) - nN^1_{(g,a_i)}(r) + N^n_{(f,a_{\sigma(i)})}(r) \\
& \quad + N^n_{(g,a_{\sigma(i)})}(r) - nN^1_{(g,a_{\sigma(i)})}(r) + \sum_{j \neq i, \sigma(i)}^{q} N^1_{(f,a_j)}(r) \\
& \leq T(r) + O(1).
\end{aligned}
$$

By taking the sum of both sides of (6.3) over $(1 \leq i \leq q)$, we have

$$
\begin{aligned}
(6.4) \qquad \Big\| \; & 2\sum_{i=1}^{q} [N^n_{(f,a_i)}(r) + N^n_{(g,a_i)}(r)] - 2n\sum_{i=1}^{q} N^1_{(g,a_i)}(r) + (q-2)\sum_{i=1}^{q} N^1_{(f,a_i)}(r) \\
& \leq qT(r) + O(1) \leq (n+2)\sum_{i=1}^{q} [N^n_{(f,a_i)}(r) + N^n_{(g,a_i)}(r)] + o(T(r)),
\end{aligned}
$$

which implies that

$$\text{(6.5)} \qquad \left\| \; -2n\sum_{i=1}^{q} N^{1}_{(g,a_i)}(r) + (q-2)\sum_{i=1}^{q} N^{1}_{(f,a_i)}(r) \right.$$
$$\leq n\sum_{i=1}^{q}[N^{n}_{(f,a_i)}(r) + N^{n}_{(g,a_i)}(r)] + o(T(r))$$
$$\leq n^2\sum_{i=1}^{q}[N^{1}_{(f,a_i)}(r) + N^{1}_{(g,a_i)}(r)] + o(T(r)).$$

It follows from (6.5) that

$$\text{(6.6)} \qquad \left\| \; (q-n^2-2)\sum_{i=1}^{q} N^{1}_{(f,a_i)}(r) \leq (n^2+2n)\sum_{i=1}^{q} N^{1}_{(g,a_i)}(r) + o(T(r)), \right.$$

which indicates that

$$\liminf_{r\to\infty} \sum_{i=1}^{q} N^{1}_{(f,a_i)}(r) \Big/ \sum_{i=1}^{q} N^{1}_{(g,a_i)}(r) \leq \frac{n(n+2)}{q-n^2-2}.$$

For $q \geq 2n^2+2n+3$, we obtain that

$$\liminf_{r\to\infty} \sum_{i=1}^{q} N^{1}_{(f,a_i)}(r) \Big/ \sum_{i=1}^{q} N^{1}_{(g,a_i)}(r) \leq \frac{n(n+2)}{n^2+2n+1},$$

which contradicts with the assumption.

Thus, we finish the proof of Theorem 2.2.

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References

[1] T. B. Cao and H. X. Yi, Uniqueness theorems for meromorphic mappings sharing hyperplanes in the general position, Sci. Sin. Math. **41** (2011), 135–144 (in Chinese).

[2] Z. H. Chen and M. Ru, A uniqueness theorem of moving targets with truncated multiplicities, Houston. J. Math. **32** (2006), 589–601.

[3] Z. H. Chen and Q. M. Yan, Uniqueness theorem of meromorphic mappings with moving targets, China Ann. Math. **27B** (2006), 393–400.

[4] Z. H. Chen and Q. M. Yan, Uniqueness theorem of meromorphic mappings into $\mathbf{P}^N(\mathbf{C})$ sharing $2N+3$ hyperplanes regardless of multiplicities, Int. J. Math. **20** (2009), 717–726.

[5] G. Dethloff and T. V. Tan, Uniqueness theorems for meromorphic mappings with few hyperplanes, Bull. Sci. Math. **133** (2009), 501–514.

[6] H. Fujimoto, The uniqueness problem of meromorphic maps into the complex projective space, Nagoya Math. J. **58** (1975), 1–23.

[7] H. Fujimoto, Uniqueness problem with truncated multiplicities in value distribution theory, Nagoya Math. J. **152** (1998), 131–152.

[8] P. C. Hu, P. Li P and C. C. Yang, Unicity of meromorphic mappings, Kluwer, Alphen aan den Rijn, 2003.

[9] S. Y. Ji, Uniqueness problem without multiplicities in value distribution theory, Pacific J. Math. **135** (1988), 323–348.

[10] R. Nevanlinna, Einige Eideutigkeitssätze in der Theorie der meromor-phen Funktionen, Acta. Math. **48** (1926), 367–391.

[11] M. Ru, Nevanlinna theory and its relation to Diophantine Approximation, World Science Pub, Singapore, 2001.

[12] L. Smiley, Geometric conditions for unicity of holomorphic curves, Contemp. Math. **25** (1983), 149–154.

[13] D. D. Thai and S. D. Quang, Uniqueness problem with truncated multiplicities of meromorphic mappings in several complex variables for moving targets, Int. J. Math. **16** (2005), 903–939.

[14] Z. H. Tu, Uniqueness problem of meromorphic mappings in several complex variables for moving targets, Tohoku Math. J. **54** (2002), 567–579.

[15] Q. M. Yan and Z. H. Chen, A note on the uniqueness theorem of meromorphic mappings, Science in China, Ser. A. Mathematics **49** (2006), 360–365.

Feng Lü
College of Science
China University of Petroleum
Qingdao, Shandong, 266580
P.R. China
E-mail: lvfeng18@gmail.com