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INTERSECTION THEORY ON MIXED CURVES

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Abstract

We consider two mixed curves $C, C' \subset \mathbf{C}^2$ which are defined by mixed functions of two variables $\mathbf{z} = (z_1, z_2)$. We have shown in [4], that they have canonical orientations. If C and C' are smooth and intersect transversely at P, the intersection number $I_{top}(C, C'; P)$ is topologically defined. We will generalize this definition to the case when the intersection is not necessarily transversal or either C or C' may be singular at P using the defining mixed polynomials.

1. Introduction

First we recall the complex analytic situation. Consider complex polynomials $f(\mathbf{z})$ and $g(\mathbf{z})$ of two variables $\mathbf{z} = (z_1, z_2)$ and consider complex analytic curves defined by $C : f(\mathbf{z}) = 0$ and $C' : g(\mathbf{z}) = 0$. Suppose that P is an isolated intersection point of $C \cap C'$. Then the local algebraic intersection number $I(f, g; P)$ is defined by the dimension of the quotient module $\dim \mathcal{O}_P/(f, g)$ where $\mathcal{O}_P$ is the local ring of the holomorphic functions at P and (f, g) is the ideal generated by f and g. Thus $I(f, g; P)$ is a strictly positive integer and it is equal to 1 if and only if C and C' are non-singular at P and transversal to each other. On the other hand, the complex curves C, C' have canonical orientations which come from their complex structures (see for example, [1]) and the local algebraic intersection number is equal to the local topological intersection number if the intersection is transverse. Moreover this is also true for a non-transverse intersection in the sense that under a slight perturbation, an intersection P of algebraic intersection number ν splits into ν transverse intersections. In particular, the topological local intersection number can be defined by the algebraic local intersection number.

The purpose of this note is to define the local intersection multiplicity for two mixed curves using the defining polynomials and study the analogous properties. The problem in this case is that the local intersection number is not necessarily positive. This makes the algebraic calculation more difficult. Let $C : f(\mathbf{z}, \bar{\mathbf{z}}) = 0$ and $C' : g(\mathbf{z}, \bar{\mathbf{z}}) = 0$ be mixed curves which have at worst an isolated mixed singularity at $P \in C \cap C'$. We will define the intersection

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multiplicity $I_{top}(C, C'; P)$ using a certain mapping degree which is described by the defining polynomials f, g (Definition 5, §2 and Theorem 2). This definition coincides with the usual one for complex analytic curves.

In §4, we consider the roots of a mixed polynomial $h(u, \bar{u})$ of one variable u as a special case. We introduce the notion of *multiplicity with sign* $\mathrm{m_s}(h, \alpha)$ for a root α of $h(u, \bar{u}) = 0$ and we give a formula for the description of $\mathrm{m_s}(h, \alpha)$ for an admissible mixed polynomial $h(u, \bar{u})$ (Theorem 20).

2. Mixed curves

2.1. A mixed singular point. Let $f(\mathbf{z}, \bar{\mathbf{z}})$, $\mathbf{z} = (z_1, z_2) \in \mathbf{C}^2$, be a mixed polynomial. See [6, 5, 4] for further details about a mixed polynomial. Using real coordinates (x_1, y_1, x_2, y_2) with $z_j = x_j + iy_j$, $j = 1, 2$, f can be understood as a sum of two polynomials with real coefficients:

$$f(\mathbf{z}, \bar{\mathbf{z}}) = f_{\mathbf{R}}(x_1, y_1, x_2, y_2) + i f_I(x_1, y_1, x_2, y_2).$$

where $f_{\mathbf{R}}$, f_I are the real part and the imaginary part of f respectively. Recall that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a polynomial of x_1, y_1, x_2, y_2 by the substitution

$$x_j = \frac{z_j + \bar{z}_j}{2}, \quad y_j = \frac{z_j - \bar{z}_j}{2i}, \quad j = 1, 2.$$

We say that $C : f(\mathbf{z}, \bar{\mathbf{z}}) = 0$ is *mixed non-singular at* $P \in C$ if the Jacobian matrix of $(f_{\mathbf{R}}, f_I)$ has rank two at P ([3, 5]). We recall that $\mathbf{C}^2$ has a canonical orientation given from the complex structure. We identify $\mathbf{C}^2$ with $\mathbf{R}^4$ by $(z_1, z_2) \leftrightarrow (x_1, y_1, x_2, y_2)$ and thus a positive frame of $\mathbf{R}^4$ is given by $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}\right)$. If P is a mixed non-singular point, C is locally a real two dimensional manifold. The normal bundle $N_{C,P}$ of $C \subset \mathbf{C}^2$ at P has the canonical orientation so that the orientation is compatible with the complex valued function f, namely $df_P : N_{C,P} \to T_0\mathbf{C}$ is an orientation preserving isomorphism. Thus the orientation of C at P is defined as follows. A frame $(\mathbf{v}_1, \mathbf{v}_2) \subset T_P C$, $\mathbf{v}_1 = (v_{11}, v_{12}, v_{13}, v_{14})$, $\mathbf{v}_2 = (v_{21}, v_{22}, v_{23}, v_{24})$, is positive if and only if the frame

$$M := \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ \mathrm{grad}\, f_{\mathbf{R}} \\ \mathrm{grad}\, f_I \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} & v_{13} & v_{14} \\ v_{21} & v_{22} & v_{23} & v_{24} \\ \frac{\partial f_{\mathbf{R}}}{\partial x_1} & \frac{\partial f_{\mathbf{R}}}{\partial y_1} & \frac{\partial f_{\mathbf{R}}}{\partial x_2} & \frac{\partial f_{\mathbf{R}}}{\partial y_2} \\ \frac{\partial f_I}{\partial x_1} & \frac{\partial f_I}{\partial y_1} & \frac{\partial f_I}{\partial x_2} & \frac{\partial f_I}{\partial y_2} \end{pmatrix}$$

is a positive frame of $\mathbf{C}^2 = \mathbf{R}^4$. The gradient vector $\mathrm{grad}\, h(x_1, y_1, x_2, y_2)$ of a real valued function h is defined by

$$\mathrm{grad}\, h(x_1, y_1, x_2, y_2) = \left(\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial y_1}, \frac{\partial h}{\partial x_2}, \frac{\partial h}{\partial y_2}\right).$$

2.2. Mixed homogenization and the closure in $\mathbf{P}^2$. Assume that $f(\mathbf{z},\bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ is a mixed polynomial of two variables $\mathbf{z} = (z_1, z_2)$. Put $C = f^{-1}(0) \subset \mathbf{C}^2$. We assume that C is non-empty and that C has only a finite number of mixed singular points. We consider the affine space $\mathbf{C}^2$ with coordinates $\mathbf{z}$ as the affine chart $Z_0 \neq 0$ of the projective space $\mathbf{P}^2$ with homogeneous coordinates (Z_0, Z_1, Z_2). The coordinates are related by $z_1 = Z_1/Z_0$, $z_2 = Z_2/Z_0$. Let d^+ and d^- be the degree of $f(\mathbf{z},\bar{\mathbf{z}})$ in $\mathbf{z}$ and $\bar{\mathbf{z}}$ respectively. That is,

$$d^+ = \max\{|\nu| \,|\, c_{\nu\mu} \neq 0\}, \quad d^- = \max\{|\mu| \,|\, c_{\nu\mu} \neq 0\}$$

where $|\nu| = \nu_1 + \nu_2$ for a multi-integer $\nu = (\nu_1, \nu_2)$. We associate with f a strongly polar homogeneous mixed polynomial $F(\mathbf{Z},\bar{\mathbf{Z}})$ as follows, where $\mathbf{Z} = (Z_0, Z_1, Z_2)$ and $\bar{\mathbf{Z}} = (\bar{Z}_0, \bar{Z}_1, \bar{Z}_2)$ by $F(\mathbf{Z},\bar{\mathbf{Z}}) := Z_0^{d_+}\bar{Z}_0^{d^-} f\left(\frac{Z_1}{Z_0}, \frac{Z_2}{Z_0}, \frac{\bar{Z}_1}{\bar{Z}_0}, \frac{\bar{Z}_2}{\bar{Z}_0}\right)$ and we call F *the mixed homogenization of* f. Here a mixed polynomial $g(\mathbf{Z},\bar{\mathbf{Z}})$ is called strictly polar homogeneous polynomial of radial degree d_r and polar degree d_p if it is a linear combination of monomials $\mathbf{Z}^\nu\bar{\mathbf{Z}}^\mu$ with $|\nu| + |\mu| = d_r$, $|\nu| - |\mu| = d_p$. We define $\bar{C} \subset \mathbf{P}^2$ by the topological closure of $C \subset \mathbf{C}^2 \subset \mathbf{P}^2$ and we define a mixed projective curve $\tilde{C} := \{((Z_0 : Z_1 : Z_2) \in \mathbf{P}^2 \,|\, F(\mathbf{Z},\bar{\mathbf{Z}}) = 0\}$. It is easy to see that the closure $\bar{C}$ of C in $\mathbf{P}^2$ is a subset of $\tilde{C}$ but in general, $\bar{C}$ might be a proper subset of $\tilde{C}$. See Remark 3.3.1. F is a strongly polar homogeneous polynomial of radial degree $d_r = d^+ + d^-$ and the polar degree $d_p = d^+ - d^-$ respectively and $F|_{Z_0 \neq 0} = f$.

Remark 1. In [4], we have assumed that the polar degree is non-zero for the definition of strongly polar homogeneous polynomials, but in this paper, we consider also the case $d^+ = d^-$. In this case, $F(\mathbf{Z},\bar{\mathbf{Z}}) : \mathbf{C}^3 \backslash F^{-1}(0) \to \mathbf{C}^*$ does not give a global fibration but $F^{-1}(0)$ is $\mathbf{C}^*$-action stable where $\mathbf{C}^*$-action is the usual one:

$$\mathbf{C}^* \times \mathbf{C}^3 \to \mathbf{C}^3, \quad (\rho, (Z_0, Z_1, Z_2)) \mapsto (\rho Z_0, \rho Z_1, \rho Z_2).$$

In particular, the zero set $F = 0$ is well-defined in $\mathbf{P}^2$.

3. Intersection numbers

3.1. Local intersection number I (Smooth and transversal intersection case). In this section, we denote vectors in $\mathbf{R}^4$ by **column vectors** for brevity's sake. Assume that $C : f = 0$ and $C' : g = 0$ are two mixed curves and assume that $P \in C \cap C'$ and C, C' are mixed non-singular at P and the intersection is transverse at P. Let $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{v}_1$, $\mathbf{v}_2$ be positive frames of T_PC and T_PC'. Then the local (topological) intersection number $I_{top}(C, C'; P)$ is defined by the sign of the determinant $\det(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2)$ (See for example [2]). Namely

$$I_{top}(C, C'; P) = \begin{cases} 1, & \det(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2) > 0, \\ -1, & \det(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2) < 0. \end{cases}$$

For any frames $\mathbf{w}_1, \dots, \mathbf{w}_4$ of $\mathbf{R}^4$, we define

$$\mathrm{Sign}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4) := \begin{cases} 1, & \text{if } \det(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4) > 0 \\ -1, & \text{if } \det(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4) < 0. \end{cases}$$

By the definition of the orientation of C and C',

$$\mathrm{Sign}(\mathbf{u}_1, \mathbf{u}_2, {}^t\mathrm{grad}\, f_{\mathbf{R}}(P), {}^t\mathrm{grad}\, f_I(P)) = 1,$$
$$\mathrm{Sign}(\mathbf{v}_1, \mathbf{v}_2, {}^t\mathrm{grad}\, g_{\mathbf{R}}(P), {}^t\mathrm{grad}\, g_I(P)) = 1.$$

Now our first result is the following.

THEOREM 2. *The intersection number $I_{top}(C, C'; P)$ is given by*

$$\mathrm{Sign}({}^t\mathrm{grad}\, f_{\mathbf{R}}(P), {}^t\mathrm{grad}\, f_I(P), {}^t\mathrm{grad}\, g_{\mathbf{R}}(P), {}^t\mathrm{grad}\, g_I(P)).$$

Recall that the tangent space T_PC is generated by the vectors orthogonal to the two dimensional subspace $\langle {}^t\mathrm{grad}\, f_{\mathbf{R}}(P), {}^t\mathrm{grad}\, f_I(P)\rangle_{\mathbf{R}}$. Thus two dimensional planes $\langle \mathbf{u}_1, \mathbf{u}_2\rangle_{\mathbf{R}}$ and $\langle {}^t\mathrm{grad}\, f_{\mathbf{R}}(P), {}^t\mathrm{grad}\, f_I(P)\rangle_{\mathbf{R}}$ are orthogonal. Here $\langle \mathbf{w}_1, \mathbf{w}_2\rangle_{\mathbf{R}}$ is the two dimensional plane spanned by $\mathbf{w}_1$, $\mathbf{w}_2$.

3.1.1. *Gram-Schmidt orthonormalization.* First we consider a simple assertion. Let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ be column vectors in $\mathbf{R}^4$ and let P, Q be 2×2 matrices. Then

ASSERTION 3.

$$\det((\mathbf{a}_1, \mathbf{a}_2)P, (\mathbf{a}_3, \mathbf{a}_4)Q) = \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)\, \det(P)\, \det(Q)$$

Proof. The assertion follows from the simple equality in 4×4 matrices:

$$((\mathbf{a}_1, \mathbf{a}_2)P, (\mathbf{a}_3, \mathbf{a}_4)Q) = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4)\begin{pmatrix} P & O \\ O & Q \end{pmatrix}. \qquad \square$$

Now we consider Gram-Schmidt orthonormalization of $(\mathbf{u}_1, \mathbf{u}_2)$ and $({}^t\mathrm{grad}\, f_{\mathbf{R}}, {}^t\mathrm{grad}\, f_I)$. They are orthonormal frames $(\mathbf{u}_1', \mathbf{u}_2')$ and $({}^t\mathrm{grad}\, f_{\mathbf{R}}(P)'$, ${}^t\mathrm{grad}\, f_I(P)')$ such that they satisfy the equalities:

$$(\mathbf{u}_1', \mathbf{u}_2') = (\mathbf{u}_1, \mathbf{u}_2)Q_1, \quad \text{and}$$
$$({}^t\mathrm{grad}\, f_{\mathbf{R}}(P)', {}^t\mathrm{grad}\, f_I(P)') = ({}^t\mathrm{grad}\, f_{\mathbf{R}}(P), {}^t\mathrm{grad}\, f_I(P))Q_2$$

where Q_1, Q_2 are upper triangular 2×2 matrices with positive entries in their diagonals. Similarly we consider the orthonormalization

$$(\mathbf{v}_1', \mathbf{v}_2') = (\mathbf{v}_1, \mathbf{v}_2)R_1$$
$$({}^t\mathrm{grad}\, g_{\mathbf{R}}(P)', {}^t\mathrm{grad}\, g_I(P)') = ({}^t\mathrm{grad}\, g_{\mathbf{R}}(P), {}^t\mathrm{grad}\, g_I(P))R_2,$$

where R_1, R_2 are upper triangular matrices with positive entries in their diagonals. Using Assertion 3, we get

$$\begin{aligned}\mathrm{Sign}(\mathbf{u}_1', \mathbf{u}_2', \mathbf{v}_1', \mathbf{v}_2') &= \mathrm{Sign}((\mathbf{u}_1, \mathbf{u}_2)Q_1, (\mathbf{v}_1, \mathbf{v}_2)R_1) \\ &= \mathrm{Sign}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2),\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Sign}({}^t\mathrm{grad}\, f_{\mathbf{R}}(P)', {}^t\mathrm{grad}\, f_I(P)', {}^t\mathrm{grad}\, g_{\mathbf{R}}(P)', {}^t\mathrm{grad}\, g_I(P)') \\ &\quad = \mathrm{Sign}(({}^t\mathrm{grad}\, f_{\mathbf{R}}(P), {}^t\mathrm{grad}\, f_I(P))Q_2, ({}^t\mathrm{grad}\, g_{\mathbf{R}}(P), {}^t\mathrm{grad}\, g_I(P))R_2) \\ &\quad = \mathrm{Sign}({}^t\mathrm{grad}\, f_{\mathbf{R}}(P), {}^t\mathrm{grad}\, f_I(P), {}^t\mathrm{grad}\, g_{\mathbf{R}}(P), {}^t\mathrm{grad}\, g_I(P)).\end{aligned}$$

Thus the calculation of the intersection number can be done using these orthonormal frames

$$(\mathbf{u}_1', \mathbf{u}_2', {}^t\mathrm{grad}\, f_{\mathbf{R}}(P)', {}^t\mathrm{grad}\, f_I(P)'), \quad (\mathbf{u}_1', \mathbf{v}_2', {}^t\mathrm{grad}\, g_{\mathbf{R}}(P)', {}^t\mathrm{grad}\, g_I(P)').$$

Thus the proof of Theorem 2 is reduced to the following.

LEMMA 4. *Assume that* $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)$ *and* $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ *be positive orthonormal frames of* $\mathbf{R}^4$. *Then*

$$\det(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2) = \det(\mathbf{u}_3, \mathbf{u}_4, \mathbf{v}_3, \mathbf{v}_4).$$

Proof. Assume that

$$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)A, \tag{1}$$

with $A \in SO(4; \mathbf{R})$. Write A by 2×2 matrices as

$$A = \begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix}$$

The equality (1) can be rewritten as

$$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4)\,{}^tA \tag{2}$$

where

$${}^tA = \begin{pmatrix} {}^tA_1 & {}^tB_1 \\ {}^tA_2 & {}^tB_2 \end{pmatrix}.$$

First we consider the equality from (1):

$$\begin{aligned}\det(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2) &= \det((\mathbf{v}_1, \mathbf{v}_2)A_1 + (\mathbf{v}_3, \mathbf{v}_4)B_1, \mathbf{v}_1, \mathbf{v}_2) \\ &= \det(\mathbf{v}_1, \mathbf{v}_2, (\mathbf{v}_3, \mathbf{v}_4)B_1) \\ &= \det B_1.\end{aligned}$$

On the other hand, we have also from (2):

$$\det(\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2) = \det(\mathbf{u}_1, \mathbf{u}_2, (\mathbf{u}_3, \mathbf{u}_4)\,{}^tA_2) \tag{3}$$

$$= \det {}^tA_2 = \det A_2 \tag{4}$$

Thus $\det A_2 = \det B_1$. Similarly we get

$$\begin{aligned}\det(\mathbf{u}_3, \mathbf{u}_4, \mathbf{v}_3, \mathbf{v}_4) &= \det((\mathbf{v}_1, \mathbf{v}_2)A_2, \mathbf{v}_3, \mathbf{v}_4) = \det A_2 \\ &= \det(\mathbf{u}_3, \mathbf{u}_4, (\mathbf{v}_1, \mathbf{v}_2)\,{}^tB_1) = \det B_1.\end{aligned}$$

Thus the assertion follows from these equalities. $\square$

3.2. Local intersection number II (General case). Assume that $C : f(\mathbf{z}, \bar{\mathbf{z}}) = 0$ and $C' : g(\mathbf{z}, \bar{\mathbf{z}}) = 0$ are mixed curves as above and let P be an isolated intersection point of $C \cap C'$. We assume also that both C and C' have at worst an isolated mixed singularity at P.

Definition 5. Let $\varphi = (f_{\mathbf{R}}, f_I, g_{\mathbf{R}}, g_I) : \mathbf{R}^4 \to \mathbf{R}^4$. We define the local intersection number $I_{top}(C, C'; P)$ by the local mapping degree of the normalized mapping ψ of φ:

$$\psi := \varphi/\|\varphi\| : S^3_\varepsilon(P) \to S^3.$$

Here $S^3_\varepsilon(P) := \{\mathbf{x} \in \mathbf{R}^4 \mid \|\mathbf{x} - P\| = \varepsilon\}$ and ε is a sufficiently small positive number so that P is the only intersection of C and C' in $B_\varepsilon(P)$ where $B_\varepsilon(P)$ is the disk of radius ε centered at P.

Suppose that P is a transverse intersection of C and C' and assume that C and C' are mixed smooth at P. Take a small positive number ε so that

$$\|\varphi^{(1)}(\mathbf{z})\| \geq 2\|\varphi - \varphi^{(1)}(\mathbf{z})\|, \quad \|\mathbf{z} - P\| = \varepsilon$$

where $\varphi^{(1)}$ is the linear term of φ at P. Then we consider the homotopy $\varphi_t = (1-t)\varphi + t\varphi^{(1)}$, $0 \leq t \leq 1$. Then the normalized mapping ψ is homotopic to that of $\varphi^{(1)}$ on $S^3_\varepsilon(P)$. The latter is nothing but the normalization of (${}^t\mathrm{grad}\, f_{\mathbf{R}}$, ${}^t\mathrm{grad}\, f_I$, ${}^t\mathrm{grad}\, g_{\mathbf{R}}$, ${}^t\mathrm{grad}\, g_I$). Thus

Proposition 6. *This definition coincides with the topological local intersection number if the intersection is transverse and two curves C, C' are mixed nonsingular at P.*

3.2.1. *Stability of the intersection number under a bifurcation.* Consider two mixed algebraic curves $C : f = 0$ and $C' : g = 0$ and assume that $P \in C \cap C'$ is an isolated point of $C \cap C'$ (but probably not a transversal intersection). Let f_t, g_t, $|t| \leq \rho$ be two continuous families of mixed polynomials such that $f_0 = f$, $g_0 = g$. We take a fixed $\varepsilon > 0$ so that $C \cap C' \cap B^4_\varepsilon(P) = \{P\}$ with $B^4_\varepsilon(P) = \{\mathbf{z} \in \mathbf{C}^2 \mid \|\mathbf{z} - P\| \leq \varepsilon\}$ and put $C_t = \{\mathbf{z} \in \mathbf{C}^2 \mid f_t(\mathbf{z}, \bar{\mathbf{z}}) = 0\}$ and $C'_s =$

$\{\mathbf{z} \in \mathbf{C}^2 \mid g_s(\mathbf{z}, \bar{\mathbf{z}}) = 0\}$. Take a sufficiently small $\gamma > 0$ so that

$$\{\mathbf{z} \in S_\varepsilon \mid f_\alpha(\mathbf{z}, \bar{\mathbf{z}}) = g_\beta(\mathbf{z}, \bar{\mathbf{z}}) = 0\} = \emptyset, \quad |\alpha|, |\beta| \le \gamma \le \rho.$$

Take δ, δ' with $|\delta|, |\delta'| \le \gamma$ and assume that $C_\delta \cap C'_{\delta'} \cap B^4_\varepsilon(P) = \{P_1, \dots, P_\nu\}$ and at each point P_j, two curves C_δ and $C'_{\delta'}$ are smooth and they intersect transversely. Then we claim:

Theorem 7. *Suppose that $P \in C \cap C'$ is bifurcated into ν transverse intersections in the near fibers $C_\delta \cap C'_{\delta'}$ as above. Let a and b the number of positive and negative intersection points among $\{P_1, \dots, P_\nu\}$ $(a + b = \nu)$. Then $I_{top}(C, C'; P) = a - b$.*

Proof. The assertion follows from the following standard topological argument. First, we consider the map of the pair $\varphi_{t,s} = (f_t, g_s)$ and its normalized one:

$$\psi_{t,s} : S^3_\varepsilon \to S^3, \quad \psi_{t,s}(\mathbf{z}, \bar{\mathbf{z}}) = \varphi_{t,s}(\mathbf{z}, \bar{\mathbf{z}}) / \|\varphi_{t,s}(\mathbf{z}, \bar{\mathbf{z}})\|.$$

The mapping degree of $\psi_{t,s}$ is independent of t and s for any $|t| \le \gamma$, $|s| \le \gamma$.

Secondly, take a sufficiently small positive number $0 < r \ll \varepsilon$ so that the disks $B^4_r(P_j)$, $j = 1, \dots, \nu$ are mutually disjoint and do not intersect with $S^3_\varepsilon(P)$. Then $\psi_{\delta,\delta'}$ is extended to a mapping

$$\psi_{\delta,\delta'} : X := B^4_\varepsilon(P) \Big\backslash \bigcup_{j=1}^{\nu} \operatorname{Int} B^4_r(P_j) \to S^3$$

where $\operatorname{Int} B^4_r(P_j) = B^4_r(P_j) \backslash S^3_r(P_j)$. Thus the fundamental class $[S_\varepsilon(P)]$ is equal to the sum of fundamental classes $\sum_{i=1}^{\nu} [S_r(P_j)]$ in $H_3(X)$, the mapping degree of $\psi_{\delta,\delta'} : S^3_\varepsilon(P) \to S^3$ is the sum of the local mapping degrees of $\psi_{\delta,\delta'} : S^3_r(P_j) \to S^3$. □

Remark 8. Note that a, b in the above theorem depends on the bifurcation but $a - b$ is independent of the chosen bifurcation. Note also that a, b can be 0 which implies $C_\delta \cap C'_{\delta'} \cap B^4_\varepsilon(P) = \emptyset$. See Example 10.

3.3. Global intersection number. We consider the global intersection number. Let $C : F(\mathbf{X}, \overline{\mathbf{X}}) = 0$ and $C' : G(\mathbf{X}, \overline{\mathbf{X}}) = 0$ be mixed projective curves in $\mathbf{P}^2$ defined by strongly polar homogeneous polynomials F and G of polar degree d and d' respectively. We assume also that the mixed singularities of C and C' are at worst isolated singularities. Then by Theorem 11, [4], they have respective fundamental cycles $[C]$ and $[C']$. Here $\mathbf{X} = (X_0, X_1, X_2)$ are homogeneous coordinates of $\mathbf{P}^2$. Assume that $C \cap C' \cap \{X_0 = 0\} = \emptyset$. We consider the affine space $\mathbf{C}^2$ with coordinates $z_1 = X_1/X_0$ and $z_2 = X_2/X_0$ respectively and put

$$f(\mathbf{z}, \bar{\mathbf{z}}) := F(1, z_1, z_2, 1, \bar{z}_1, \bar{z}_2), \quad g(\mathbf{z}, \bar{\mathbf{z}}) := G(1, z_1, z_2, 1, \bar{z}_1, \bar{z}_2)$$

respectively. Let $C \cap C' = \{P_1, \dots, P_\mu\}$. Then by Theorem 11, [4], the fundamental classes $[C]$, $[C']$ of C, C' exist and they satisfy, in $H_2(\mathbf{P}^2)$

$$[C] = d[\mathbf{P}^1], \quad [C'] = d'[\mathbf{P}^1]$$

where $[\mathbf{P}^1]$ is the homology class corresponding to the fundamental class of the complex line $\mathbf{P}^1 \subset \mathbf{P}^2$. Thus we have the equality $[C] \cdot [C'] = dd'$. Now we have the equality:

THEOREM 9.

$$\sum_{j=1}^{\mu} I_{top}(C, C'; P_j) = dd'.$$

Example 10. Consider the special case:

$$C : z_1 = 0, \quad C' : g(\mathbf{z}, \bar{\mathbf{z}}) = 2z_1 + z_1\bar{z}_1 + z_2\bar{z}_2 = 0.$$

Then $\tilde{C}$ is the projective line $z_1 = 0$ of degree 1 and $\tilde{C}'$ is the mixed curve of polar degree 0 which is defined by $G(\mathbf{Z}, \bar{\mathbf{Z}}) = 2\bar{Z}_0 Z_1 + Z_1\bar{Z}_1 + Z_2\bar{Z}_2 = 0$. Actually C' is a 2 dimensional sphere

$$C' : y_1 = 0, \quad (x_1 + 1)^2 + x_2^2 + y_2^2 = 1$$

and it has a mixed singular point $(-1, 0)$. We see that $\tilde{C} \cap \tilde{C}' = \{(1:0:0)\}$ and $\tilde{C} \cdot \tilde{C}' = 0$. This implies $I(C, C'; (0,0)) = 0$. In fact, consider the bifurcation $C_t = \{z_1 - t = 0\}$. It is easy to see that $C_t \cap C' = \emptyset$ if $t > 0$. For $t < 0$ small, the intersection $C_t \cap C'$ is a circle.

3.3.1. *Remark.* 1. **Twisted line.** The singular locus of a mixed curve can be non-isolated, even if we assume that it does not have any real codimension 1 components.

Consider the curve $f(\mathbf{z}, \bar{\mathbf{z}}) = z_1 - \bar{z}_2$. Then C is a smooth real two-plane and $\tilde{C}$ is defined by $F = Z_1\bar{Z}_0 - Z_0\bar{Z}_2 = 0$. We call C (and $\tilde{C}$) *a twisted line*. Let $\bar{C} \subset \mathbf{P}^2$ be the topological closure. The complex line at infinity L_∞ is defined by $Z_0 = 0$. To see more details about the structure of these mixed curves, we consider the coordinate chart $U_2 = \{Z_2 \neq 0\}$ with complex coordinates $(u_0, u_1) = (Z_0/Z_2, Z_1/Z_2)$. Then $\tilde{C} \cap U_2$ is defined by

$$f_2(u_0, u_1) = u_1\bar{u}_0 - u_0 = 0. \tag{5}$$

We observe that

(a) $\tilde{C} = L_\infty \cup \bar{C}$ and $S := L_\infty \cap \bar{C}$ is a circle defined by $L_\infty \cap \{|Z_1/Z_2| = 1\}$. This follows from (5), as $|u_1| = 1$.

(b) The singular locus of $\tilde{C}$ is equal to S. Using the coordinates (u_0, u_1) on U_2, S is defined by $|u_1| = 1$ on $L_\infty = \{u_0 = 0\}$. As a 1-cycle, we orient it counterclockwise. Inside the circle S (*i.e.*, $|u_1| < 1$), the orientation is the same with the disk $\Delta := \{u_1 \in \mathbf{C} \mid |u_1| < 1\}$. Outside $\{|u_1| > 1\}$ of S, the orientation is opposite to the complex structure with coordinates u_1. The singular locus can be computed by the Jacobian matrix of $(f_{2\mathbf{R}}, f_{2I})$ or by Proposition 1, [3]

(c) Let $U_0 = \{Z_0 \neq 0\}$. In this coordinate, $p : C \to \mathbf{C}$, $p(z_1, z_2) = z_2$ is an orientation preserving diffeomorphism. The circle $S_R := \{|z_2| = R\}$ converges to $-2S$ when $R \to \infty$.

Proof. To see this, consider the large circle S_R parametrized by $z_1 = Re^{-i\theta}$, $z_2 = Re^{i\theta}$, $0 \leq \theta \leq 2\pi$. In the chart U_2, this corresponds to

$$u_0(\theta) = \frac{1}{R}e^{-i\theta}, \quad u_1(\theta) = z_1/z_2 = e^{-2i\theta}. \qquad \square$$

In [4], we have observed that there exists a fundamental class $[D] \in H_2(D)$ for any mixed projective curve D with at most isolated mixed singularities. Our curve $\tilde{C}$ has non-isolated singularities along S. However we claim that

Claim 11. *$\tilde{C}$ has a fundamental class.*

To see this, triangulate $\tilde{C}$ so that S is a union of 1-simplices. Then the sum ω of all two simplices with positive orientation in L_∞ satisfies $\partial\omega = 2S$ by the observation (b). The sum σ of 2 simplices in $\bar{C}$ satisfies $\partial\sigma = -2S$ as we have observed in (c). Thus $\omega + \sigma$ is a cycle and it gives the fundamental class.

2. It is possible that a projective mixed curve D with at most isolated singularities may have some 0-dimensional components. The fundamental class $[D] \in H_2(D)$ is the sum of 2 simplices with positive orientation under a triangulation where singular points are vertices.

Problem 12. Assume that a projective mixed curve C has at most 1 dimensional singular locus. Does C have always a fundamental class as above?

3.3.2. *Remark on complex analytic cases.* Assume that C and C' are complex analytic curves. Assume first that $P = (\alpha, \beta) \in C \cap C'$ is a transverse intersection where C, C' are non-singular. Let J be the complex Jacobian matrix at P

$$J = \det \begin{pmatrix} \dfrac{\partial f}{\partial z_1}(\alpha, \beta) & \dfrac{\partial f}{\partial z_2}(\alpha, \beta) \\ \dfrac{\partial g}{\partial z_1}(\alpha, \beta) & \dfrac{\partial g}{\partial z_2}(\alpha, \beta) \end{pmatrix}$$

Then using the Cauchy-Riemann equality, we can easily show that

$$\det \frac{\partial(f_{\mathbf{R}}, f_I, g_{\mathbf{R}}, g_I)}{\partial(x_1, y_1, x_2, y_2)}(\alpha, \beta) = |J|^2 > 0.$$

This implies that the local intersection number is 1 if the intersection is transversal at a regular point P. For a generic case, we have

$$I_{top}(C, C'; P) = \dim_{\mathbf{C}} \mathcal{O}_P/(f, g) = I(C, C'; P) \in \mathbf{N}$$

where $I(C, C'; P)$ is the algebraic local intersection multiplicity and (f, g) is the ideal generated by f, g.

4. Multiplicity with sign

In this section, we consider the special case that $C : \hat{f}(\mathbf{z}, \bar{\mathbf{z}}) = 0$ is a mixed curve and C' is a complex line in $\mathbf{C}^2$. So $\mathbf{z} = (z_1, z_2) \in \mathbf{C}^2$ and we assume that $g := z_2$ and $\hat{f}|_{z_2=0}$ is a mixed polynomial of one complex variable, z_1. Put $f := \hat{f}|_{z_2=0}$. Suppose that $\alpha \in \mathbf{C}$ is an isolated mixed root of $f(z_1, \bar{z}_1) = 0$, *i.e.*, $f(\alpha, \bar{\alpha}) = 0$ and $f(z_1, \bar{z}_1) \neq 0$ for any sufficiently near $z_1 \neq \alpha$. For a positive number $\varepsilon > 0$, we put

$$S^1_\varepsilon(\alpha) := \{z_1 \in \mathbf{C} \mid |z_1 - \alpha| = \varepsilon\}.$$

We define *the multiplicity with sign* of the root $z_1 = \alpha$ by the mapping degree of the normalized function

$$f/|f| : S^1_\varepsilon(\alpha) \to S^1, \quad \mathbf{z} \mapsto f(z_1, \bar{z}_1)/|f(z_1, \bar{z}_1)|.$$

for a sufficiently small ε and we denote the multiplicity with sign by $\mathrm{m_s(f, \alpha)}$. The mapping degree $\mathrm{m_s(f, \alpha)}$ is also called *the rotation number.* We claim

Lemma 13. *Let f, $\hat{f}$ be as above. Let $g(\mathbf{z}, \bar{\mathbf{z}}) = z_2$. Let $C = \{\hat{f}(\mathbf{z}, \bar{\mathbf{z}}) = 0\}$ and $C' = \{z_2 = 0\}$. Let $\alpha \in \mathbf{C}$ be a root of f and let $\hat{\alpha} = (\alpha, 0)$. Then $\hat{\alpha} \in V(\hat{f}, g)$ and $I_{top}(C, C'; \hat{\alpha}) = \mathrm{m_s(f, \alpha)}$.*

Proof. We use the notations:

$$D_\varepsilon(\alpha) := \{z \mid |z - \alpha| \leq \varepsilon\}, \quad S^1_\varepsilon(\alpha) = \partial D_\varepsilon(\alpha),$$
$$D_\varepsilon := D_\varepsilon(0), \quad S^1_\varepsilon := S^1_\varepsilon(0).$$

Put $f_t(\mathbf{z}, \bar{\mathbf{z}}) = f(z_1, \bar{z}_1) + t(\hat{f}(\mathbf{z}, \bar{\mathbf{z}}) - f(z_1, \bar{z}_1))$. Note that $f_1 = \hat{f}$, $f_0 = f$. Take a positive number ε_1 small enough so that 0 is the unique root of $f(z_1, \bar{z}_1) = 0$ in $D_{\varepsilon_1}(\alpha)$. Then take $0 < \varepsilon_2 \ll \varepsilon_1$ so that f_t is non-zero on $S^1_{\varepsilon_1}(\alpha) \times D_{\varepsilon_2}$, that is, $f_t(\mathbf{z}, \bar{\mathbf{z}}) \neq 0$ if $|z_1 - \alpha| = \varepsilon_1$, $|z_2| \leq \varepsilon_2$. For the calculation of the mapping degree of the normalization ψ of $(\hat{f}_{\mathbf{R}}, \hat{f}_I, g_{\mathbf{R}}, g_I)$, we can use the boundary of $\partial(D_{\varepsilon_1}(\alpha) \times D_{\varepsilon_2})$ instead of the sphere $S^3_\varepsilon(\hat{\alpha})$. We use the Mayer-Vietoris exact sequence of $\partial(D_{\varepsilon_1}(\alpha) \times D_{\varepsilon_2})$ associated with the decomposition $\{D_{\varepsilon_1}(\alpha) \times S^1_{\varepsilon_2}, S^1_{\varepsilon_1}(\alpha) \times D_{\varepsilon_2}\}$. Then we have the following commutative diagram where the horizontal arrows are isomorphisms.

$$\begin{array}{ccc} H_3(\partial(D_{\varepsilon_1}(\alpha)\times D_{\varepsilon_2})) & \xrightarrow{\delta} & H_2(S^1_{\varepsilon_1}(\alpha)\times S^1_{\varepsilon_2}) \\ \Big\downarrow \psi_* & & \Big\downarrow \psi'_{*1} \\ H_3(\partial(D_{\varepsilon}(\alpha)\times D_{\varepsilon_2})) & \xrightarrow{\delta} & H_2(S^1_{\varepsilon_1}(\alpha)\times S^1_{\varepsilon_2}) \end{array}$$

The right vertical map ψ'_{*1} is induced by $\hat{f} = f_1$ and f_1 is homotopic to $f_0 = f$. Therefore ψ'_{*1} coincides with $(\varepsilon_1 f/|f|)_* \times \mathrm{id}$. The homotopy is given by the normalization of $(f_t(\mathbf{z},\bar{\mathbf{z}}), g(\mathbf{z},\bar{\mathbf{z}}))$. Here the normalization ψ_* of $f_t(\mathbf{z},\bar{\mathbf{z}})$ is defined by $\psi_*(\mathbf{z},\bar{\mathbf{z}}) = \hat{\alpha} + f_t(\mathbf{z},\bar{\mathbf{z}})\lambda$ where λ is the unique positive number so that the right hand side is in $\partial(D_\varepsilon(\alpha)\times D_{\varepsilon_2})$. Thus we get

$$\begin{aligned} I_{top}(C, C'; \hat{\alpha}) &= \text{mapping degree of } \psi_* \\ &= \text{mapping degree of } (\varepsilon_1 f/|f|)_* = \mathrm{m_s}(\mathrm{f},\alpha). \qquad \square \end{aligned}$$

We define *the total multiplicity with sign* by the sum of $\mathrm{m_s}(\mathrm{f},\alpha)$ for all $\alpha \in V(f)$ where $V(f) = \{\alpha \in \mathbf{C} \mid f(\alpha,\bar{\alpha}) = 0\}$ and denote it by $\mathrm{m_{s,tot}}(\mathrm{f}) = \sum_{\alpha\in \mathrm{V(f)}} \mathrm{m_s}(\mathrm{f},\alpha)$. Note that $\mathrm{m_s}(\mathrm{f},\alpha)$ and $\mathrm{m_{s,tot}}(\mathrm{f})$ is not necessarily positive and it can be any integer.

4.1. A criterion for the positivity. Let us study some details for a simple root $\alpha \in V(f)$. First $f(z_1,\bar{z}_1)$ can be written as a polynomial of w_1, $\overline{w}_1$ with $w_1 = z_1 - \alpha$ by the substitution $f_\alpha(w_1,\overline{w}_1) := f(w_1+\alpha, \overline{w}_1+\bar{\alpha})$. Put $a := \dfrac{\partial f}{\partial z_1}(\alpha,\bar{\alpha})$ and $b := \dfrac{\partial f}{\partial \bar{z}_1}(\alpha,\bar{\alpha})$. This implies that $L(w_1,\overline{w}_1) = aw_1 + b\overline{w}_1$ is the linear term of $f_\alpha(w_1,\overline{w}_1)$. Put

$$a = a_1 + a_2 i, \quad b = b_1 + b_2 i, \quad \alpha = \alpha_1 + \alpha_2 i, \quad a_1, a_2, b_1, b_2, \alpha_1, \alpha_2 \in \mathbf{R}.$$

Then the expansions of the real polynomials $f_{\mathbf{R}}$, f_I in two real variables $(x_\alpha, y_\alpha) := (x - \alpha_1, y - \alpha_2)$ are given as follows:

$$\begin{aligned} f_{\mathbf{R}}(x_\alpha, y_\alpha) &= \Re f(w_1+\alpha, \overline{w}_1+\bar{\alpha}) \\ &= (a_1+b_1)x_\alpha + (-a_2+b_2)y_\alpha + (\text{higher terms}) \\ f_I(x_\alpha, y_\alpha) &= \Im f(w_1+\alpha, \overline{w}_1+\bar{\alpha}) \\ &= (a_2+b_2)x_\alpha + (a_1-b_1)y_\alpha + (\text{higher terms}). \end{aligned}$$

Thus we observe that

$$\begin{aligned} \det\left(\frac{\partial(f_{\mathbf{R}}, f_I)}{\partial(x,y)}(\alpha_1,\alpha_2)\right) &= \left|\begin{pmatrix} a_1+b_1 & -a_2+b_2 \\ a_2+b_2 & a_1-b_1 \end{pmatrix}\right| \\ &= (a_1^2+a_2^2) - (b_1^2+b_2^2) = |a|^2 - |b|^2. \end{aligned}$$

DEFINITION 14. We say that α is a *positive simple root* if α is a mixed-regular point for f and $\mathrm{m_s}(\mathrm{f},\alpha) > 0$ which is equivalent to

$$\det\left(\frac{\partial(f_{\mathbf{R}}, f_I)}{\partial(x,y)}(\alpha_1,\alpha_2)\right) > 0$$

Similarly α is a *negative simple root* if α is a mixed-regular point for f and $\mathrm{m_s}(\mathrm{f},\alpha) < 0$. This is equivalent to

$$\det\left(\frac{\partial(f_{\mathbf{R}}, f_I)}{\partial(x,y)}(\alpha_1,\alpha_2)\right) < 0.$$

Thus we have the criterion:

PROPOSITION 15. (1) *Assume that α is a mixed regular root of f. Then α is a positive* (*resp. negative*) *simple root if and only if $|a| > |b|$. That is,*

$$\mathrm{m_s}(\mathrm{f},\alpha) = 1 \Leftrightarrow \left|\frac{\partial \mathrm{f}}{\partial \mathrm{z}_1}(\alpha,\bar{\alpha})\right| > \left|\frac{\partial \mathrm{f}}{\partial \bar{\mathrm{z}}_1}(\alpha,\bar{\alpha})\right|$$

$$\mathrm{m_s}(\mathrm{f},\alpha) = -1 \Leftrightarrow \left|\frac{\partial \mathrm{f}}{\partial \mathrm{z}_1}(\alpha,\bar{\alpha})\right| < \left|\frac{\partial \mathrm{f}}{\partial \bar{\mathrm{z}}_1}(\alpha,\bar{\alpha})\right|$$

(2) *If* $\left|\frac{\partial f}{\partial z_1}(\alpha,\bar{\alpha})\right| = \left|\frac{\partial f}{\partial \bar{z}_1}(\alpha,\bar{\alpha})\right|$, *$\alpha$ is a mixed singularity of f.*

4.2. Bifurcation. Suppose that 0 is an isolated root of a mixed polynomial $f(u,\bar{u})$. Consider a bifurcation family $f_t(u,\bar{u}) = 0$ and let $\{P_1(t),\dots, P_\nu(t)\}$ be the roots of $f_t(u,\bar{u}) = 0$ which are bifurcating from $u = 0$. Then we have

PROPOSITION 16. $\sum_{i=1}^{\nu} \mathrm{m_s}(\mathrm{f_t},\mathrm{P_i}(\mathrm{t})) = \mathrm{m_s}(\mathrm{f},0)$. *In particular, if the roots $P_i(t)$ are simple,* $\mathrm{m_s}(\mathrm{f},0)$ *is equal to the difference of the number of positive roots and the negative roots.*

The proof is similar with that of Theorem 4. Note that ν depends on the chosen bifurcation.

Example 17. 1. Let $f(u,\bar{u}) = u^2\bar{u}$. It is easy to see that $u = 0$ is a non-simple singularity and $\mathrm{m_s}(\mathrm{f},0) = 1$. (For a complex polynomial singularity, $\mathrm{m_s}(\mathrm{f},0) = 1$ implies that 0 is a simple root.) Consider two bifurcation families:

$$f_t(u,\bar{u}) = (u^2 - t)\bar{u}, \quad g_s(u,\bar{u}) = u(u\bar{u} + s) \quad \textit{for } t,s \geq 0.$$

Note that $f_t = 0$ has two positive roots $u = \pm\sqrt{t}$ and a negative root $u = 0$. $g_s = 0$ has only one positive root $u = 0$ for $s > 0$.

Assertion 18. *Let* $f(u,\bar u)=u^n+u+\bar u$ *for any* $n\ge 2$. *Then* $u=0$ *is a mixed singular root of* f *and*

$$\mathrm{m_s}(\mathrm{f},0)=\begin{cases}1 & n: even\\ -1 & n\equiv 3 \mod 4\\ 1 & n\equiv 1 \mod 4\end{cases}$$

For the proof see the Appendix (§4.4.4).

4.3. Admissible mixed polynomial and the main theorem. We consider a mixed polynomial $f(u,\bar u)=\sum_{\nu,\mu}c_{\nu,u}u^\nu\bar u^\mu$ of one variable u. The *maximal degree* of f is defined by $\bar d=\max\{\nu+\mu\,|\,c_{\nu,\mu\neq 0}\}$. We denote $\bar d=\bar d(f)$. Similarly we define the *minimal degree* of f at the origin by $\underline{d}:=\min\{\nu+\mu\,|\,c_{\nu,\mu}\neq 0\}$ and we denote $\underline{d}=\underline{d}(f)$. Note that the minimal degree is a local invariant but the maximal degree is a global invariant. That is, $\bar d(f)$ is invariant under the change of coordinate $v=c(u-a)$, $a\in\mathbf{C}$, $c\in\mathbf{C}^*$ and $\underline{d}(f)$ is invariant under a local change of coordinates $u\mapsto u'=cu$, $c\in\mathbf{C}^*$.

For a positive integer ℓ, we put

$$f_\ell(u,\bar u):=\sum_{\nu+\mu=\ell}c_{\nu,\mu}u^\nu\bar u^\mu.$$

Then we can write

$$\begin{aligned}f(u,\bar u)&=f_{\bar d}(u,\bar u)+f_{\bar d-1}(u,\bar u)+\cdots+f_{\underline{d}+1}(u,\bar u)+f_{\underline{d}}(u,\bar u)\\&=f_{\bar d}(u,\bar u)+k(u,\bar u),\\&=f_{\underline{d}}(u,\bar u)+j(u,\bar u)\end{aligned}$$

with $\bar d(k)<\bar d$ and $\underline{d}(j)>\underline{d}$. Note that we have a unique factorization of $f_{\bar d}$ and $f_{\underline{d}}$ as follows.

$$f_{\bar d}(u,\bar u)=cu^p\bar u^q\prod_{j=1}^{s}(u+\gamma_j\bar u)^{\nu_j},\quad p+q+\sum_{j=1}^{s}\nu_j=\bar d,\quad c\in\mathbf{C}^* \tag{6}$$

$$f_{\underline{d}}(u,\bar u)=c'u^a\bar u^b\prod_{j=1}^{s'}(u+\delta_j\bar u)^{\mu_j},\quad a+b+\sum_{j=1}^{s'}\mu_j=\underline{d},\quad c'\in\mathbf{C}^* \tag{7}$$

where $\gamma_1,\dots,\gamma_s$ (respectively $\delta_1,\dots,\delta_{s'}$) are mutually distinct non-zero complex numbers. We say that f is *admissible at infinity* (respectively *admissible at the origin*) if $|\gamma_j|\neq 1$ for $j=1,\dots,s$ (resp. $|\delta_j|\neq 1$, $j=1,\dots,s'$). For non-zero complex number ξ, we put

$$\varepsilon(\xi)=\begin{cases}1 & |\xi|<1\\ 0 & |\xi|=1\\ -1 & |\xi|>1\end{cases}$$

and we consider the following integers:

$$\beta(f) := p - q + \sum_{j=1}^{s} \varepsilon(\gamma_j)v_j, \quad \rho(f,0) := a - b + \sum_{j=1}^{s'} \varepsilon(\delta_j)\mu_j.$$

Remark 19. Assume that f is not admissible at infinity and assume $|\gamma_1| = 1$ for example. Put $\gamma_1 = \exp(i\theta_1)$. Then $f_{\bar{d}}$ vanishes at $u = R\exp(i\theta_1/2)$ for any $R > 0$. Thus the behavior of f on the big circle $|u| = R$ is not controlled by the highest term $f_{\bar{d}}$. The same reason can be applied for $f_{\underline{d}}$ for a small circle $|u| = r$ with $r \ll 1$, if there exists some j such that $|\delta_j| = 1$.

Our main result is the following.

THEOREM 20. (1) *Assume that* $f(u,\bar{u})$ *is an admissible mixed polynomial at infinity. Then* $\mathrm{m_{s,tot}(f)} = \beta(\mathrm{f})$.
(2) *Assume that* $f(u,\bar{u})$ *is an admissible mixed polynomial at the origin. Then* $\mathrm{m_s(f,0)} = \rho(\mathrm{f},0)$.

Proof. Put $\bar{d} = \bar{d}(f)$ and assume that $f_{\bar{d}}$ is factored as in (6). In the case $s = 0$, the proof is the same with that of Theorem 11, [4]. In the general case, we first assume that

$$|\gamma_1| \le \cdots \le |\gamma_\ell| < 1 < |\gamma_{\ell+1}| \le \cdots \le |\gamma_s|.$$

Let R be a positive number. First we observe that for any $u \in S_R^1$,

$$\begin{aligned} |f_{\bar{d}}(u,\bar{u})| &= |c|R^{\bar{d}} \prod_{j=1}^{\ell} |1 + \gamma_j \bar{u}/u|^{v_j} \prod_{j=\ell+1}^{s} |u/\bar{u} + \gamma_j|^{v_j} \\ &\ge |c|R^{\bar{d}} \prod_{j=1}^{\ell} (1 - |\gamma_j|)^{v_j} \prod_{j=\ell+1}^{s} (|\gamma_j| - 1)^{v_j} \\ &\ge MR^{\bar{d}} \end{aligned}$$

for some positive constant $M > 0$. We can choose a sufficiently large $R > 0$ so that

$$|f_{\bar{d}}(u,\bar{u})| > 2|k(u,\bar{u})|, \quad \forall u,\ |u| \ge R.$$

The rest of the argument is exactly the same as the proof of Theorem 11, [4]. Let $V(f) = \{\alpha_1, \ldots, \alpha_m\}$ and take a small positive number ε so that $D_\varepsilon(\alpha_j) \cap V(f) = \{\alpha_j\}$ where $D_\varepsilon(a) := \{u \mid |u - a| \le \varepsilon\}$. First, as $f/|f| : S_R^1 \to S^1$ is extended to $D_R(O) \setminus \bigcup_{i=1}^{m} D_\varepsilon(\alpha_j)$, we have

$$\text{mapping degree } (f/|f| : S_R^1 \to S^1) = \sum_{j=1}^{m} \mathrm{m_s(f,\alpha_j)}.$$

To compute the mapping degree $f/|f| : S_R^1 \to S^1$, we consider the family of polynomials $f(u,\bar u,t) := f_{\bar d}(u,\bar u) + (1-t)k(u,\bar u)$. This family is non-vanishing on S_R^1. Note that $f(u,\bar u,0) = f(u,\bar u)$ and $f(u,\bar u,1) = f_{\bar d}(u,\bar u)$. As $f/|f| \simeq f_{\bar d}/|f_{\bar d}|$ on S_R^1, we have

$$\sum_{j=1}^m \mathrm{m_s}(\mathrm{f},\alpha_\mathrm{j}) = \text{mapping degree of } f/|f| : S_R^1 \to S^1$$
$$= \text{mapping degree of } f_{\bar d}/|f_{\bar d}|.$$

Now we will show that the mapping degree of $f_{\bar d}/|f_{\bar d}|$ is equal to the integer $\beta(f)$. For this purpose, we write $f_{\bar d}$ as

$$f_{\bar d}(u,\bar u) = u^{\hat p}\bar u^{\hat q}\prod_{j=1}^{\ell}\left(1+\gamma_j\frac{\bar u}{u}\right)^{v_j}\prod_{k=\ell+1}^{s}\left(\frac{u}{\bar u}+\gamma_j\right)^{v_k}$$

$$\text{where } \hat p = p + \sum_{j=1}^{\ell} v_j, \quad \hat q = q + \sum_{j=\ell+1}^{s} v_j.$$

Note that

$$\beta(f) = \hat p - \hat q = p - q + \sum_{j=1}^{\ell} v_j - \sum_{j=\ell+1}^{s} v_j$$

in the above notation. We observe that

$$1+\gamma_j\frac{\bar u}{u} \in D_{|\gamma_j|}(1), \quad 1 \le j \le \ell,\ u \in S_R^1$$
$$\frac{u}{\bar u}+\gamma_k \in D_1(\gamma_k), \quad \ell+1 \le k \le s,\ u \in S_R^1$$

where $D_\varepsilon(\eta) = \{\zeta \in \mathbf{C} \mid |\zeta-\eta| < \varepsilon\}$. It is easy to observe that

$$0 \notin D_{|\gamma_j|}(1) \quad (j \le \ell), \qquad 0 \notin D_1(\gamma_k) \quad (k \ge \ell+1).$$

Consider the family of polynomials

$$f_{\bar d}(u,\bar u,t) := u^{\hat p}\bar u^{\hat q}\prod_{j=1}^{s}\left(1+t\gamma_j\frac{\bar u}{u}\right)^{v_j}\prod_{k=s+1}^{s}\left(t\frac{u}{\bar u}+\gamma_j\right)^{v_k}, \quad 0 \le t \le 1.$$

Note that $f_{\bar d}(u,\bar u,1) = f_{\bar d}(u,\bar u)$ and $f_{\bar d}(u,\bar u,0) = u^{\hat p}\bar u^{\hat q}$. As $f_{\bar d}(u,\bar u,t)$, $0 \le t \le 1$ give a homotopy on S_R^1, the assertion follows from the fact that the mapping degree of $u^{\hat p}\bar u^{\hat q}$ is $\beta(f)$. This proves the first assertion (1).

The second assertion (2) is proved by the same argument:

– Take a sufficiently small $r > 0$ so that

$$|f_{\underline d}(u,\bar u)| \ge 2|j(u,\bar u)|, \quad \forall u,\ |u| \le r$$

where $f = f_{\underline d} + j$.

– Observe that the homotopy $f(u, \bar{u}, t) = f_{\underline{d}}(u, \bar{u}) + tj(u, \bar{u})$, $0 \le t \le 1$ is non-vanishing on the circle $S^1_{\underline{r}}$.
– The normalization $f_{\underline{d}}/|f_{\underline{d}}|$ of $f_{\underline{d}}(u, \bar{u}, 0)$ is homotopic to that of $u^{\rho(f,0)}$. □

4.4. Compactification

4.4.1. *Generic line at infinity and a generic affine chart.* Let $F(\mathbf{z})$ be a strongly polar homogeneous polynomial of two variables $\mathbf{z} = (z_0, z_1)$ of radial and polar degree d and q. We can write $2r = d - q$ for some integer $r \ge 0$. Then F is a linear combination of monomials $\mathbf{z}^\nu \bar{\mathbf{z}}^\mu$ such that $|\nu| + |\mu| = d$ and $|\nu| - |\mu| = q$. Assume that $z_0 = 0$ is generic so that this line does not contain any root of $F = 0$. This implies that F has a monomial $z_1^\alpha \bar{z}_1^\beta$ where $\alpha = \dfrac{d+q}{2}$, $\beta = \dfrac{d-q}{2}$. Then in the affine coordinate $z_0 \ne 0$ with the coordinate $u = z_1/z_0$, the mixed polynomial $f(u, \bar{u}) = F(1, u, 1, \bar{u})$ can be written as

$$f(u, \bar{u}) = cu^\alpha \bar{u}^\beta + \sum_{a,b} c_{a,b} u^a \bar{u}^b, \quad c \ne 0,\ 0 \le a < \alpha,\ 0 \le b < \beta.$$

Note also that $F(\mathbf{z}, \bar{\mathbf{z}}) = z_0^\alpha \bar{z}_0^\beta f(z_1/z_0, \bar{z}_1/\bar{z}_0)$ and

$$f_d(u, \bar{u}) = cu^{q+r} \bar{u}^r, \quad f = f_d + (\text{lower terms}),\ u = z_1/z_0.$$

In this case, we have that $\mathrm{m_{s,tot}(f) = q}$ in Theorem 11, [4]. Thus Theorem 11 [4] is a special case of Theorem 20.

4.4.2. *Polar homogeneous compactification.* We consider now the opposite situation. Suppose that we are given a mixed polynomial $f(u, \bar{u}) = \sum_{\nu,\mu} c_{\nu,\mu} u^\nu \bar{u}^\mu$. Let $\bar{d} = \overline{\deg}\, f$ and put

$$d_+ = \max\{\nu \mid c_{\nu,\mu} \ne 0\}, \quad d_- = \max\{\mu \mid c_{\nu,\mu} \ne 0\}.$$

Define

$$F(z_0, z_1, \bar{z}_0, \bar{z}_1) := z_0^{d_+} \bar{z}_0^{d_-} f(z_1/z_0, \bar{z}_1/\bar{z}_0).$$

$F(z_0, z_1, \bar{z}_0, \bar{z}_1)$ is the mixed homogenization defined in §1. Put $d_h = d_+ + d_-$ and $q_h = d_+ - d_-$. By the definition, we have the following assertion.

Proposition 21. *Assume that $f_{\bar{d}}(u, \bar{u})$ be factorized as* (2) *and let $F(z_0, z_1, \bar{z}_0, \bar{z}_1)$ be as above. F is a strongly polar homogeneous polynomial of radial degree d_h and polar degree q_h and we have the inequality $d_h \ge \bar{d} = \overline{\deg}\, f$.*

(1) *The equality $d_h = \bar{d}$ holds if and only if*

$$p = d_+, \quad q = d_-, \quad s = 0.$$

(2) *Assume that* $d_h > \bar{d}$. *Then* $(0:1) \in V(F)$. *Namely each monomial in* $F(z_0, z_1, \bar{z}_0, \bar{z}_1)$ *contains either* z_0 *or* $\bar{z}_0$.

4.4.3. *Example.* Consider the polynomial:

$$f(u, \bar{u}) = u^2\bar{u}(u - 2\bar{u}) + 1.$$

$V(k)$ consists of 4 points

$$u = \pm\sqrt[4]{1/3}i,\ \pm 1.$$

The multiplicities with sign of the first two roots $\{\pm\sqrt[4]{1/3}i\}$ are 1 and the latter two roots $\{\pm 1\}$ are -1. This implies that $\mathrm{m_{s,tot}(f)} = 0$ as Theorem 20 asserts.

The mixed homogenization $f(\mathbf{z}, \bar{\mathbf{z}})$ is given by

$$F(\mathbf{z}, \bar{\mathbf{z}}) = z_1^2\bar{z}_1(z_1\bar{z}_0 - 2\bar{z}_1 z_0) + z_0^3\bar{z}_0^2.$$

We see that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a strongly polar homogeneous polynomial of radial and polar degrees 5 and 1 respectively. We observe that $(0:1)$ is on $V(f)$ and it has multiplicity with sign 1. Now take the generic affine coordinate chart $U_1 := \{z_1 \neq 0\}$ with the coordinate $v = z_0/z_1$. Then the affine equation of $V(f) \cap U_1$ is given as

$$f'(v) = \bar{v} - 2v + v^3\bar{v}^2$$

and $V(f) \cap U_1$ consists of 5 points. Note that $\mathrm{m_s(f', 0)} = 1$.

4.4.4. *Appendix: Proof of Assertion* 18. Recall $f(u, \bar{u}) = u^n + u + \bar{u}$. The proof follows the following observations.

1. $\mathrm{m_{s,tot}(f)} = \mathrm{n}$ by Theorem 20.
2. For any $\alpha \in V(f)\backslash\{0\}$, α is a simple mixed root with $\mathrm{m_s(f, \alpha)} = 1$.
3. The number, say β, of non-zero mixed roots of f is given as follows:

$$\beta = \begin{cases} n-1 & n \text{ even} \\ n+1 & n \equiv 3 \mod 4 \\ n-1 & n \equiv 1 \mod 4 \end{cases}$$

Let us show the observation 2. So assume that $\alpha \in V(f)$ and $\alpha \neq 0$. Take the coordinate $v := u - \alpha$. Then

$$\begin{aligned} f(v + \alpha) &= \alpha^n + \alpha + \bar{\alpha} + n\alpha^{n-1}v + v + \bar{v} + (\text{higher terms in } v) \\ &= (n\alpha^{n-1} + 1)v + \bar{v} + (\text{higher terms in } v) \\ &= \left(-(n-1) - n\frac{\bar{\alpha}}{\alpha}\right)v + \bar{v} + (\text{higher terms in } v) \end{aligned}$$

Now we conclude the assertion by Proposition 9 as

$$\left|(n-1) - n\frac{\bar{\alpha}}{\alpha}\right| \geq n - (n-1) = 1$$

and by the equality takes place if and only if $\bar{\alpha} = -\alpha$, that is α is purely imaginary. This does not happen by the following calculation.

Now we show the observation 3. As the calculation is easy, we only show the result. Assume $f(u) = 0$ with $u \neq 0$. Put $u = r\exp(ia)$, $0 \leq a < 2\pi$ in the polar coordinates. Then we have

$$r^n \sin(na) = 0, \quad r^n \cos(na) + 2r\cos(a) = 0.$$

Thus the first equality says that

$$na = j\pi, \quad j = 0, \dots, 2n-1$$

The second equality has a positive solution for r if and only if $\cos(na)\cos(a) < 0$. This implies that α is not a pure imaginary complex number. Assume $n = 4k$ for example. Then the solution exists for the following.

$$\frac{a}{\pi} = \{1, 3, \dots, 2k-1, 2k+2, 2k+4, \dots, 6k-2, 6k+1, \dots, 8k-1\}$$
$$\beta = 4k-1, \quad \mathrm{m_s(f,0) = 4k} - \beta = 1$$

For the case $n = 4k+2$,

$$\frac{a}{\pi} = \{1, 3, \dots, 2k-1, 2k+2, 2k+4, \dots, 6k+2, 6k+5, \dots, 8k+3\}$$
$$\beta = 4k+1, \quad \mathrm{m_s(f,0) = 4k + 2} - \beta = 1$$

For the case $n = 4k-1$, we have

$$\frac{a}{\pi} = \{1, 3, \dots, 2k-1, 2k, 2k+2, \dots, 6k-2, 6k-1, \dots, 8k-3\}$$
$$\beta = 4k, \quad \mathrm{m_s(f,0) = 4k - 1} - \beta = -1$$

For $n = 4k+1$, we have

$$\frac{a}{\pi} = \{1, 3, \dots, 2k-1, 2k+2, 2k+4, \dots, 6k, 6k+3, \dots, 8k+1\}$$
$$\beta = 4k, \quad \mathrm{m_s(f,0) = 4k + 1} - \beta = 1$$

4.5. Figure. Let us consider the case $n = 2$, $f(u) = u^2 + u + \bar{u}$. Note that $f(u)$ has two mixed singular points, O and $P = (-2, 0)$.

The following figures shows the trace of $f(u(\theta), \bar{u}(\theta))$, $u(\theta) = r\exp(i\theta)$, $0 \leq \theta \leq 2\pi$ for $r = 3/2, 2, 3$ respectively.

Case $r = 3/2$. The next figure (Figure 1) shows that $\mathrm{m_s}(\mathrm{f}, 0) = 1$.

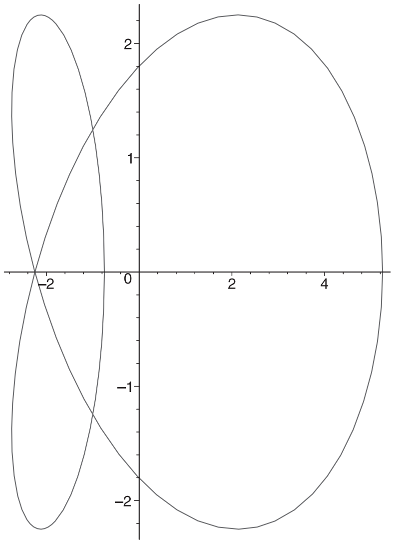

Figure 1. $n = 2,\ r = 3/2$

Case $r = 2$. This figure (Figure 2) corresponds to the critical case that $|u| = 2$ passes through the mixed singular point $(-2, 0)$.

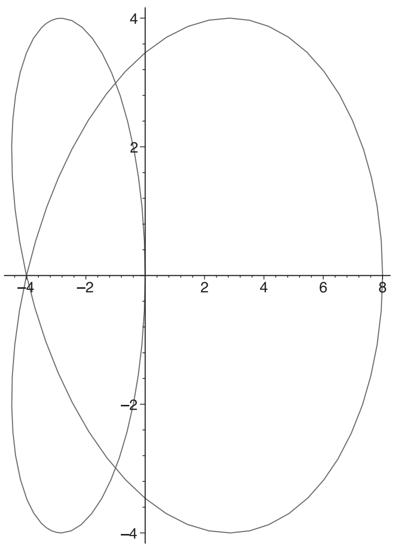

Figure 2. $n = 2,\ r = 2$

Case $r = 3$. The disk $|u| \le 3$ contains a mixed singular point $(-2, 0)$ and $\mathrm{m}_{\mathrm{s,tot}}(\mathrm{f}) = 2$.

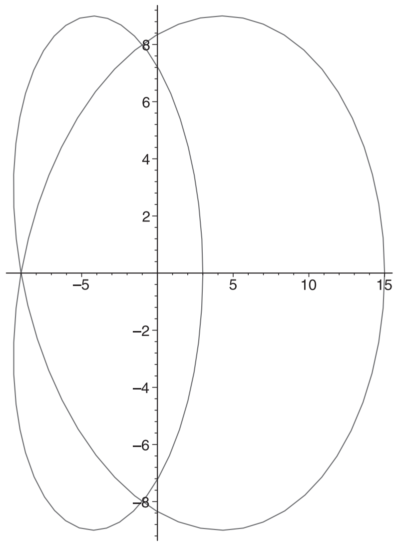

Figure 3. $n = 2,\ r = 3$

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