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# A DISCRIMINANT CRITERION OF IRREDUCIBILITY

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## Abstract

In this paper we give a criterion of irreducibility for a complex power series in two variables, using the notion of jacobian Newton diagrams, defined with respect to any direction. Then we apply our method to study the branches of plane algebraic curves. For an affine plane curve with one point at infinity, we also obtain a criterion for an analytical irreducibility in terms of the Newton diagram of a discriminant, without using coordinates centered at the point at infinity.

## 1. Introduction

In [12] we give criteria of irreducibility for a complex power series in two variables, using the notion of jacobian Newton diagrams, defined with respect to a generic direction. In this paper we generalize these criteria to any direction and we use this new general criterion to study the branches of plane algebraic curves. The paper is organized as follows:

In 1.1 we recall the notion of the *Newton diagram.* Then in 1.2 we explain what is the *discriminant curve* of an analytic mapping $F : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$. If $D(u, v) = 0$ is the discriminant curve then the Newton diagram of $D$ will be called *jacobian Newton diagram* of $F$ and denoted $\mathcal{N}_J(F)$. At the end of the section we present formulas for computing equations of discriminants.

In Section 2 we consider $\mathcal{N}_J(l, f)$ where $l$ is a regular function and $f$ is a singular irreducible series. We shall call such diagrams *Merle type diagrams*. We recall Merle's result that equisingularity class of $f$ and the intersection multiplicity $(f, l)_0$ determine and are determined by $\mathcal{N}_J(l, f)$. In Theorem 2.3 we give necessary and sufficient conditions of arithmetical nature for a Newton diagram to be a Merle type diagram.

The main technical result of the paper is Theorem 3.1. It states that $f$ is an irreducible power series if and only if $\mathcal{N}_J(l, f)$ is a Merle type diagram. We apply our irreducibility criterion to power series taken from Kuo's paper [17].

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Then in Theorem 3.7 we give a criterion for local irreducibility of plane algebraic curves which requires only computing a usual discriminant of a polynomial in one variable.

Finally we study the singularity at infinity of a plane affine curve with one point at infinity for which the global counterpart of our main result holds. We obtain a criterion for an analytical irreducibility of a curve at the point at infinity in terms of the Newton diagram of a discriminant, without using coordinates centered at the point at infinity. To get this result we need Theorem 3.1 in its full generality (non-transverse case). Theorem 5.1 is important in the context of the Jacobian conjecture. Recall that Abhyankar proved in [1] that this conjecture is settled affirmatively in the case where there is only one branch at infinity.

### 1.1. Newton diagrams of plane analytic curves

In this section we recall the definition of a Newton diagram and introduce the needed notation. Write $\mathbf{R}_+ = \{x \in \mathbf{R} : x \geq 0\}$.

Let $f \in \mathbf{C}\{x, y\}$, $f(x, y) = \sum a_{i,j} x^i y^j$ be a non-zero convergent power series. Put $\operatorname{supp} f := \{(i, j) : a_{i,j} \neq 0\}$. Then by definition the *Newton diagram* $\Delta_f$ of $f$ is

$$\Delta_f = \text{Convex Hull } (\operatorname{supp} f + \mathbf{R}_+^2).$$

The basic property of Newton diagrams is that the Newton diagram of a product is the Minkowski sum of Newton diagrams. That is $\Delta_{fg} = \Delta_f + \Delta_g$ where $\Delta_f + \Delta_g = \{a + b : a \in \Delta_f, b \in \Delta_g\}$. In particular if $f$ and $g$ differ by an invertible factor $u \in \mathbf{C}\{x, y\}$, $u(0, 0) \neq 0$ then $\Delta_f = \Delta_g$. A plane analytic curve $f = 0$ is, for us, a principal ideal generated by $f$ in the ring of convergent power series of two variables. Thus the Newton diagram of a plane analytic curve is well defined because two arbitrary chosen generators of the principal ideal differ by an invertible factor.

Following Teissier [24] we introduce *elementary Newton diagrams*. For $m, n > 0$ we put $\left\{\frac{n}{m}\right\} = \Delta_{x^n + y^m}$. We put also $\left\{\frac{n}{\infty}\right\} = \Delta_{x^n}$ and $\left\{\frac{\infty}{m}\right\} = \Delta_{y^m}$.

One can check that every Newton diagram $\Delta \subsetneq \mathbf{R}_+^2$ has a unique representation $\Delta = \sum_{i=1}^r \left\{\frac{L_i}{M_i}\right\}$, where *inclinations* of successive elementary diagrams form an increasing sequence (by definition the inclination of $\left\{\frac{L}{M}\right\}$ is $L/M$ with the conventions that $L/\infty = 0$ and $\infty/M = +\infty$). We shall call this representation the *canonical form* of $\Delta$.

Finally a Newton diagram is *convenient* if it intersects both coordinate axes.

### 1.2. Discriminant curve

Let $F = (p, q) : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ be an analytic mapping given by $(u, v) = (p(x, y), q(x, y))$ such that $F^{-1}(0, 0) = \{(0, 0)\}$. Let $\operatorname{jac}(p, q) = 0$ be the equation of the critical locus of $F$, where $\operatorname{jac}(p,q) = \frac{\partial p}{\partial x}\frac{\partial q}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial q}{\partial x}$ is the usual jacobian determinant. The direct image of $\operatorname{jac}(p,q) = 0$ by $F$ is called the *discriminant curve* of $F$ (see Appendix).

Assume that $D(u,v) = 0$ is the discriminant curve. Then $\Delta_D$ is called the *jacobian Newton diagram* of $F$ (see [25]). We will write $\mathcal{N}_J(p,q)$ for the jacobian Newton diagram.

Below we give some formulas for jacobian Newton diagrams and discriminant curves.

FORMULA 1.1 (Teissier's formula [23]). *Assume that* $\operatorname{jac}(p,q) = h_1 \cdots h_r$, *where* $h_i$ *are irreducible series, not necessarily distinct, for* $1 \le i \le r$. *Then*

$$\mathcal{N}_J(p,q) = \sum_{i=1}^{r} \left\{ \frac{(q,h_i)_0}{(p,h_i)_0} \right\},$$

*where* $(f,g)_0$ *denotes the intersection multiplicity of* $f$ *and* $g$.

From now on we will only consider mappings

$$(l,f) : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0), \tag{1}$$

where $l$ is a regular function (i.e. $l = ax + by + \textit{higher order terms}$, $ax + by \not\equiv 0$) and $f$ is a singular series. Recall that a power series is called *singular* if its order is bigger than one. Under these assumptions $\operatorname{jac}(l,f) = 0$ is called the *polar curve of $f$ with respect to $l$*. The inclinations of the elementary diagrams of the jacobian Newton diagram $\mathcal{N}_J(l,f)$ are called *polar quotients*. These notions were studied by many authors (see for example [10, 20] for irreducible case, and [5, 7, 8, 9, 11, 15, 18, 19, 27] among others for the reduced case). See also the survey [13] for recent results. If the curves $l = 0$ and $f = 0$ are transverse, that is they do not share any tangent, then $\mathcal{N}_J(l,f)$ depends only on the equisingularity class of $f = 0$ (see [23]). Otherwise the jacobian Newton diagram may depend on relative position of curves $l = 0$ and $f = 0$ as the following example shows.

*Example* 1.2. Let $f = y^2 - x^5$ and let $l_1 = x$, $l_2 = y$, $l_3 = y - x^2$. Then $\operatorname{jac}(l_1,f) = 2y$, $\operatorname{jac}(l_2,f) = 5x^4$ and $\operatorname{jac}(l_3,f) = x(5x^3 - 4y)$. By Teissier's formula

$$\mathcal{N}_J(l_1,f) = \left\{ \frac{(f,y)_0}{(l_1,y)_0} \right\} = \left\{ \frac{5}{1} \right\}$$

$$\mathcal{N}_J(l_2,f) = 4\left\{ \frac{(f,x)_0}{(l_2,x)_0} \right\} = 4\left\{ \frac{2}{1} \right\} = \left\{ \frac{8}{4} \right\}$$

$$\mathcal{N}_J(l_3,f) = \left\{ \frac{(f,x)_0}{(l_3,x)_0} \right\} + \left\{ \frac{(f,5x^3-4y)_0}{(l_3,5x^3-4y)_0} \right\} = \left\{ \frac{2}{1} \right\} + \left\{ \frac{5}{2} \right\}.$$

For any local analytic diffeomorphism $\Phi : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ the substitution $(l_1, f_1) = (l \circ \Phi, f \circ \Phi)$ does not affect the equation of the discriminant curve. Hence without loss of generality we may assume that $l = x$ (take such a $\Phi$ that $l \circ \Phi = x$).

Formula 1.3. *If* $f(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x) \in \mathbf{C}\{x\}[y]$ *is a Weierstrass polynomial, i.e.* $a_i(0) = 0$ *for every* $i \in \{1, \dots, n\}$, *then the discriminant of the mapping* $(x, f) : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ *has an equation* $D(u, v) = 0$, *where*

$$D(u,v) = \mathrm{Discr}_y(f(u,y) - v) \tag{2}$$

*is the classical discriminant of a polynomial in one variable.*

*Proof.* The discriminant $\mathrm{Discr}_y(f(u,y) - v)$ is, up to an integer constant, equal to the resultant of polynomials $f(u,y) - v$ and $\frac{\partial f}{\partial y}(u,y)$. By the classical formula (see Theorem 10.10, Chapter I, [26]) the resultant of polynomials $P, Q \in K[Y]$, $Q = \prod_{i=1}^{s}(Y - \beta_i)$, where $K$ is a field is, up to a sign, a product $\prod_{i=1}^{s} P(\beta_i)$. We get $\mathrm{Discr}_y(f(u,y) - v) = c\prod_{j=1}^{n-1}[f(u, \gamma_j(u)) - v]$, where $c$ is a nonzero constant, $\gamma_j$ are Newton-Puiseux roots of the $y$-partial derivative and by Appendix Formula 1.3 follows. ∎

Formula 1.4. *Let* $f(x,y) = y^N + a_1(x)y^{N-1} + \cdots + a_N(x) \in \mathbf{C}\{x\}[y]$. *Assume that all nonzero roots of the polynomial* $f(0,y)$ *are simple. Then the discriminant of the mapping* $(x, f) : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ *is given by formula* (2).

*Proof.* Let $\frac{\partial f}{\partial y}(x,y) = N\prod_{i=1}^{N-1}(y - \gamma_i(x))$ be the Puiseux factorization of $y$-partial derivative. Take $\gamma_k(x)$ such that $\gamma_k(0) \neq 0$. Since $\gamma_k(0)$ is a root of $\frac{\partial f}{\partial y}(0, y)$ and all nonzero roots of $f(0,y)$ are simple one has $f(0, \gamma_k(0)) \neq 0$. We get

$$\mathrm{Discr}_y(f(u,y) - v) = \mathrm{const}\prod_{i=1}^{N-1}[v - f(u, \gamma_i(u))] = \mathrm{unit}\prod_{\gamma_i(0)=0}[v - f(u, \gamma_i(u))]$$

which is, up to a unit, equation (4) of Lemma 5.4. ∎

## 2. Jacobian Newton diagrams of irreducible series

In this section we consider mappings $(l, f) : (\mathbf{C}^2, 0) \to (\mathbf{C}^2, 0)$ under additional assumption that $f$ is an irreducible singular power series. Then the curve $f = 0$ is often called a *plane singular branch.*

Consider

$$S(f) = \{(f, g)_0 : g \in \mathbf{C}\{x, y\} \text{ and } f \text{ does not divide } g\}.$$

Clearly $0 \in S(f)$ (take $g = 1$) and if $a, b \in S(f)$ then $a + b \in S(f)$ since the intersection multiplicity is additive, so $S(f)$ is a semigroup, called the *semigroup of the branch* $f = 0$.

For any regular curve $l = 0$ the semigroup $S(f)$ has the $(f, l)_0$-*minimal system of generators* $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_h$ defined by conditions

(i) $\bar{b}_0 = (f, l)_0$,

(ii) $\bar{b}_k = \min(S(f) \setminus (\mathbf{N}\bar{b}_0 + \cdots + \mathbf{N}\bar{b}_{k-1}))$,

(iii) $S(f) = \mathbf{N}\bar{b}_0 + \cdots + \mathbf{N}\bar{b}_h$.

The sequence of generators can be characterized in purely arithmetical terms. Let us recall the next result (see [4, 28] for the generic case ($\bar{b}_0 = \operatorname{ord} f$) and [14] for the case when the curves $f = 0$, $l = 0$ are tangent).

THEOREM 2.1. *Let $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_h$ be a sequence of positive integers. Set $n_k = \gcd(\bar{b}_0, \dots, \bar{b}_{k-1})/\gcd(\bar{b}_0, \dots, \bar{b}_k)$ for $k \in \{1, \dots, h\}$. Then the following conditions are equivalent*:

(i) *there is a singular branch $f = 0$ and a regular curve $l = 0$ such that $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_h$ is the $(f, l)_0$-minimal system of generators of the semigroup $S(f)$*,

(ii) *the sequence $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_h$ satisfies the conditions*:

($Z_1$) *$n_k > 1$ for $k \in \{1, \dots, h\}$ and $n_1 \cdots n_h = \bar{b}_0$*,

($Z_2$) *$n_k \bar{b}_k < \bar{b}_{k+1}$ for $k \in \{1, \dots, h-1\}$.*

Now we can state the result proved in [22], [20], and [10] (we put by convention $n_0 = 1$).

THEOREM 2.2 (Smith–Merle–Ephraim). *Suppose that $f = 0$ is a singular branch and $l = 0$ is a regular curve. Let $\bar{b}_0, \dots, \bar{b}_h$ be the $(f, l)_0$-minimal system of generators of the semigroup $S(f)$. Then with the notation introduced above*

$$\mathcal{N}_J(l, f) = \sum_{k=1}^{h} \left\{ \frac{(n_k - 1)\bar{b}_k}{(n_k - 1)n_0 \cdots n_{k-1}} \right\}. \tag{3}$$

If $\bar{b}_0, \dots, \bar{b}_h$ is the sequence satisfying the conditions ($Z_1$) and ($Z_2$) of Theorem 2.1 then we will write $\mathcal{M}(\bar{b}_0, \dots, \bar{b}_h)$ for the Newton diagram (3) and we call it, following [13], the *Merle type diagram*. Note that the Newton diagram in formula (3) is written in the canonical form. Indeed, the quotients $\bar{b}_{k+1}/(n_k \bar{b}_k)$ of the inclinations of successive elementary Newton diagrams are greater than 1 by Theorem 2.1.

Let us look at Example 1.2 in the light of Theorem 2.2. The curve $f = 0$ has the semigroup $S(f) = \mathbf{N}2 + \mathbf{N}5$. One gets $(f, l_1)_0 = 2$, $(f, l_2)_0 = 5$, $(f, l_3)_0 = 4$ and is easy to verify that $\mathcal{N}_J(l_1, f) = \mathcal{M}(2, 5)$, $\mathcal{N}_J(l_2, f) = \mathcal{M}(5, 2)$ and $\mathcal{N}_J(l_3, f) = \mathcal{M}(4, 2, 5)$.

Theorem 2.3. *Let* $\Delta = \sum_{i=1}^{h} \left\{ \dfrac{L_i}{M_i} \right\}$ *be a convenient Newton diagram written in its canonical form. Put* $H_0 = 1$, $H_i = 1 + M_1 + \cdots + M_i$ *for* $i \in \{1, \ldots, h\}$ *and* $C_0 = H_h$, $C_i = H_{i-1}L_i/M_i$ *for* $i \in \{1, \ldots, h\}$. *Then* $\Delta$ *is a Merle type diagram if and only if the arithmetic conditions* (i)–(iii) *are satisfied*

(i) *the quotients* $H_i/H_{i-1}$ *are integers for* $i \in \{2, \ldots, h\}$,

(ii) *the quotients* $C_i$ *are integers for* $i \in \{1, \ldots, h\}$,

(iii) $\gcd(C_0, \ldots, C_i) = C_0/H_i$ *for* $i \in \{1, \ldots, h\}$.

*Moreover in such a case* $\Delta = \mathcal{M}(C_0, \ldots, C_h)$.

*Proof.* Assume that $\Delta$ is a Merle type diagram $\mathcal{M}(\bar{b}_0, \ldots, \bar{b}_h)$. Then $L_i = (n_i - 1)\bar{b}_i$ and $M_i = (n_i - 1)n_0 \cdots n_{i-1}$ for $i \in \{1, \ldots, h\}$. We have the equality $H_i = n_1 \cdots n_i$. Indeed $H_1 = 1 + M_1 = 1 + (n_1 - 1) = n_1$ and $H_{i+1} = H_i + M_{i+1} = n_1 \cdots n_i + (n_{i+1} - 1)n_1 \cdots n_i = n_1 \cdots n_{i+1}$ by the inductive hypothesis. It follows that $H_i/H_{i-1} = n_i$ hence condition (i) is satisfied.

It also follows that $C_i = \dfrac{H_{i-1}L_i}{M_i} = \dfrac{n_1 \cdots n_{i-1}(n_i - 1)\bar{b}_i}{(n_i - 1)n_1 \cdots n_{i-1}} = \bar{b}_i$. Hence condition (ii) is also satisfied.

It follows directly from the definition of the sequence $n_i$ that $\gcd(\bar{b}_0, \ldots, \bar{b}_i) = \bar{b}_0/(n_1 \cdots n_i)$ for $1 \le i \le h$. Moreover by condition $(Z_1)$ of Theorem 2.1 one has $n_1 \cdots n_h = \bar{b}_0$ which gives $C_0 = H_h = \bar{b}_0$. Thus $\gcd(C_0, \ldots, C_i) = C_0/H_i$ for $i \in \{1, \ldots, h\}$.

Now assume that conditions (i)–(iii) hold true for the Newton diagram $\Delta$. We will show that the sequence $C_0, \ldots, C_h$ satisfies arithmetical conditions of Theorem 2.1. It follows from (iii) that $n_i := \dfrac{\gcd(C_0, \ldots, C_{i-1})}{\gcd(C_0, \ldots, C_i)} = H_i/H_{i-1}$ for $i \in \{1, \ldots, h\}$. Thus $n_i > 1$ for $i \in \{1, \ldots, h\}$ and $n_1 \cdots n_h = C_0$.

Since $\Delta$ is written in canonical form one has $L_i/M_i < L_{i+1}/M_{i+1}$ for $i \in \{1, \ldots, h-1\}$. Multiplying these inequalities by $n_iH_{i-1} = H_i$ we get $n_iH_{i-1}L_i/M_i < H_iL_{i+1}/M_{i+1}$ which is equivalent with $n_iC_i < C_{i+1}$ for $i \in \{1, \ldots, h-1\}$.

Hence, the sequence $C_0, \ldots, C_h$ satisfies conditions $(Z_1)$ and $(Z_2)$ of Theorem 2.1. Moreover looking at the first part of the proof it is easy to see that $\Delta = \mathcal{M}(C_0, \ldots, C_h)$. ■

## 3. Discriminant criterion of irreducibility

Theorem 3.1. *Let* $f = 0$ *be a plane singular curve and let* $l = 0$ *be a regular curve. Then* $f$ *is irreducible if and only if* $\mathcal{N}_J(l, f)$ *is a Merle type diagram. Moreover, if* $\mathcal{N}_J(l, f) = \mathcal{M}(\bar{b}_0, \ldots, \bar{b}_h)$ *then* $f = 0$ *has the semigroup* $S(f) = \mathbf{N}\bar{b}_0 + \cdots + \mathbf{N}\bar{b}_h$.

*Example* 3.2. Let $l = x$ and $f = y^n - x^m$. Then by (2), $D(u, v) = (v + u^m)^{n-1}$, hence $\mathcal{N}_J(x, f) = \left\{ \dfrac{(n-1)m}{n-1} \right\}$. Under notation of Theorem 2.3 one gets $C_0 = H_1 = n$, $C_1 = m$ and conditions (i) and (ii) of Theorem 2.3 are clearly satisfied. Condition (iii) reduces to $\gcd(m, n) = 1$ and it is well-known that the curve $y^n - x^m = 0$ is irreducible if and only if $m$ and $n$ are co-prime.

The following two examples are taken from [17] (see also [2]).

*Example* 3.3. Let $f = (y^2 - x^3)^2 - x^7$. Then $\mathrm{jac}(x, f) = 4y(y^2 - x^3) = 4y(y - x^{3/2})(y + x^{3/2})$. By Lemma 5.4 we get $D(u, v) = (v - u^6 + u^7)(v + u^7)^2$. Hence $\mathcal{N}_J(x, f) = \{\frac{6}{1}\} + \{\frac{14}{2}\}$. Under notation of Theorem 2.3 one has $H_1 = 1 + 1 = 2$, $C_0 = H_2 = 1 + 1 + 2 = 4$, $C_1 = 6/1 = 6$, $C_2 = H_1 \cdot 14/2 = 14$ and since $\gcd(C_0, C_1, C_2) = 2 \neq 1$, it follows that $\mathcal{N}_J(x, f)$ is not a Merle type diagram. Therefore $f$ is not irreducible.

*Example* 3.4. Let $f(x, y) = (y^2 - x^3)^2 - x^5 y$. By Formula 1.3, $D(u, v) = -256v^3 + 256u^6v^2 + 288u^{13}v - 256u^{19} - 27u^{20}$ (we computed the discriminant using Sage) and the Newton diagram of the discriminant is $\mathcal{N}_J(x, f) = \{\frac{6}{1}\} + \{\frac{13}{2}\}$. It is easy to check that $\mathcal{N}_J(x, f)$ is a Merle type diagram $\mathcal{M}(4, 6, 13)$. Therefore $f$ is irreducible with semigroup $S(f) = \mathbf{N}4 + \mathbf{N}6 + \mathbf{N}13$.

*Example* 3.5. Let $f(x, y) = x^8 + (x^2 + y^3)^3$. The jacobian Newton diagram of $(x, f)$ is $\mathcal{N}_J(x, f) = \{\frac{12}{2}\} + \{\frac{48}{6}\}$ which is not a Merle type diagram. Note that in this example $x = 0$ is not transverse to $f(x, y) = 0$.

Corollary 3.6. *Let* $f(x, y) = y^N + a_1(x)y^{N-1} + \cdots + a_N(x) \in \mathbf{C}[x, y]$. *Assume that the curve* $f(x, y) = 0$ *intersects* $x = 0$ *only at the point* $(0, y_0)$. *Then the curve* $f(x, y) = 0$ *is analytically irreducible at* $(0, y_0)$ *if and only if the Newton diagram of* $\mathrm{Discr}_y(f(u, y) - v)$ *is a Merle type diagram.*

*Proof.* Put $\tilde{f}(x, y) = f(x, y + y_0)$. Then $f(x, y) = 0$ is analytically irreducible at $(0, y_0)$ if and only if $\tilde{f}(x, y) = 0$ is analytically irreducible at $(0, 0)$. Since $\mathrm{Discr}_y(f(u, y) - v) = \mathrm{Discr}_y(\tilde{f}(u, y) - v)$ the result follows from Formula 1.4. ∎

In the theorem below a Newton diagram of a formal power series in two variables appears, which is an obvious generalization of the usual Newton diagram.

Theorem 3.7. *Let* $f(x, y) \in \mathbf{C}[x, y]$, $f(0, 0) = 0$ *be a square free polynomial. Let* $T$ *be a variable and consider* $H(u, v) = \mathrm{Discr}_y(f(u + Ty, y) - v)$ *considered as an element of the formal power series ring* $\mathbf{C}[T][[u, v]]$. *Then the curve* $f(x, y) = 0$ *is analytically irreducible at the origin if and only if the Newton diagram* $\Delta_H$ *of* $H$ *is a Merle type diagram.*

*Proof.* A generic line passing through the origin intersects the curve $f(x, y) = 0$ transversally outside the origin. Hence for a generic constant $t \in \mathbf{C}$

the polynomial $\tilde{f}(x, y) = f(x + ty, y)$ multiplied by some nonzero constant satisfies the assumption of Formula 1.4. It follows from Theorem 3.1 that the curve $f(x, y) = 0$ has one branch at the origin if and only if the Newton diagram of $D_t(u, v) = \mathrm{Discr}_y(f(u + ty, y) - v) \in \mathbf{C}\{u, v\}$ is a Merle type diagram. Observe also that for sufficiently generic $t$ the Newton diagram $\Delta_{D_t}$ is equal to $\Delta_H$ which ends the proof. ∎

Using above theorem one can check the analytic irreducibility of an algebraic curve at any point. It is enough to choose a system of coordinates so that this point becomes the origin.

## 4. Proof of Theorem 3.1

The proof is based on Theorem 1 of [12] and the lemma following it:

THEOREM 4.1. *Let $f, g \in \mathbf{C}\{x, y\}$ be such that $\mathcal{N}_J(x, f) = \mathcal{N}_J(x, g)$. Assume that $x = 0$ is transverse to the curves $f = 0$ and $g = 0$. If $f$ is irreducible then $g$ is also irreducible.*

LEMMA 4.2. *Let $f$ be a convergent power series, and let $N$ be a positive integer. Write $f(0, y) = y^n +$ higher order terms. Put $\tilde{f}(x, y) = f(x^N, y)$. Then*

(i) *if $N$ and $n$ are coprime integers then $f$ is irreducible if and only if $\tilde{f}$ is irreducible,*
(ii) *if $N > n$ then $\tilde{f} = 0$ is transverse to $x = 0$,*
(iii) *$\mathcal{N}_J(x, \tilde{f}) = L(\mathcal{N}_J(x, f))$, where $L : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear automorphism given by $L(i, j) = (Ni, j)$.*

*Proof. Proof of* (i). Assume that $f = f_1 f_2$. Then $\tilde{f}(x, y) = f_1(x^N, y) \cdot f_2(x^N, y)$. It follows that if $\tilde{f}$ is irreducible then $f$ is irreducible.

Conversely, assume that $f$ is irreducible. Recall (see Theorem 2.1, Chapter IV, [26]) that the curve $f = 0$, with $\mathrm{ord}\, f(0, y) = n$, is a branch if and only if there exists a convergent power series $\phi(t)$ such that $f(t^n, \phi(t)) = 0$ and the greatest common divisor of the set $\{n\} \cup \mathrm{supp}\, \phi$ equals 1.

Let $\phi(t)$ be such a series and let $\tilde{\phi}(t) = \phi(t^N)$. Then $\tilde{f}(t^n, \tilde{\phi}(t)) = f(t^{nN}, \phi(t^N)) = 0$ and since $n$ and $N$ are co-prime the greatest common divisor of the set $\{n\} \cup \mathrm{supp}\, \tilde{\phi} = \{n\} \cup N \cdot \mathrm{supp}\, \phi$ equals 1. Consequently $\tilde{f} = 0$ is a branch.

*Proof of* (ii). By the assumption $N > n$ the homogeneous initial part of the series $f(x^N, y)$ is $y^n$. This gives (ii).

*Proof of* (iii). Let $\frac{\partial f}{\partial y}(x, y) = \mathrm{unit} \prod_{j=1}^{n-1} [y - \gamma_j(x)]$ be the Newton-Puiseux factorization of $\frac{\partial f}{\partial y}$. By Lemma 5.4 the discriminant of the mapping $(x, f)$ has an equation $D(u, v) = \prod_{j=1}^{n-1} [v - f(u, \gamma_j(u))]$. Since $\frac{\partial \tilde{f}}{\partial y}(x, y) = \frac{\partial f}{\partial y}(x^N, y)$

one has $\dfrac{\partial\tilde{f}}{\partial y}(x,y) = \text{unit}\prod_{j=1}^{n-1}[y - \gamma_j(x^N)]$ and consequently the discriminant of the mapping $(x,\tilde{f})$ has an equation $\tilde{D}(u,v) = \prod_{j=1}^{n-1}[v - \tilde{f}(u,\gamma_j(u^N))] = \prod_{j=1}^{n-1}[v - f(u^N,\gamma_j(u^N))] = D(u^N,v)$. Comparing $\Delta_D$ with $\Delta_{\tilde{D}}$ we get (iii). ∎

Now let us prove Theorem 3.1. Suppose that $\mathcal{N}_J(l,f) = \mathcal{N}_J(l,g)$, where $g$ is an irreducible power series. Applying an analytic change of coordinates we may assume that $l = x$. Take an integer $N > 0$ such that conclusions of (i) and (ii) of Lemma 4.2 are satisfied for $\tilde{f}(x,y) = f(x^N,y)$ and $\tilde{g}(x,y) = g(x^N,y)$. It follows from (iii) of Lemma 4.2 that $\mathcal{N}_J(x,\tilde{f}) = \mathcal{N}_J(x,\tilde{g})$. Since $\tilde{f}$ and $\tilde{g}$ satisfy assumptions of Theorem 4.1, $\tilde{f}$ is an irreducible power series. Hence by (i) of Lemma 4.2 $f$ is also irreducible.

## 5. Discriminant criterion of irreducibility at infinity

Let $p(x,y)$ be a complex polynomial of degree $n > 0$. Let $C \subset \mathbf{P}^2(\mathbf{C})$ be the projective closure of the curve $p(x,y) = 0$. Assume that $C$ intersects the line at infinity at only one point $Q$. The purpose of this section is to give a criterion for local analytical irreducibility of the curve $C$ at $Q$ without passing to local coordinates centered at $Q$. For this we need some terminology.

Let $g(x,y)$ be a polynomial of positive degree such that $g(x,0) \not\equiv 0$ and $g(0,y) \not\equiv 0$ (in other words its Newton diagram is convenient). Let $\mathscr{P}_0(g)$ be the boundary in $\mathbf{R}^2_+$ of $\Delta_g$ and $\mathscr{P}_\infty(g)$ be the boundary in $\mathbf{R}^2_+$ of $\Delta_\infty(g) =$ Convex Hull $(\operatorname{supp} g \cup \{(0,0)\})$. We call these sets the *Newton polygon of $g$ at zero* and the *Newton polygon of $g$ at infinity* respectively.

Theorem 5.1. *Let $p(x,y)$ be a complex polynomial of degree $n > 0$ without multiple factors and let $C$ be the projective closure of $p(x,y) = 0$. Assume that $C$ intersects the line at infinity at only one point $Q \neq (0:1:0)$. Put $D_\infty(x,t) :=$ $\mathrm{Discr}_y(p(x,y) - t)$ and let $L : \mathbf{Z}^2 \to \mathbf{Z}^2$ be the affine transformation defined by $L(i,k) = (n(n-1) - i - nk, k)$. Then the curve $C$ is analytically irreducible at $Q$ if and only if $L(\mathscr{P}_\infty(D_\infty))$ is the Newton polygon at zero of a Merle type diagram.*

*Proof.* Let $P(x,y,z) = z^n p\left(\dfrac{x}{z},\dfrac{y}{z}\right)$ be a homogeneous equation of the curve $C$. Assume that $C$ intersects the line at infinity only at $Q = (1 : y_0 : 0)$. Then $p(x,y) = P(x,y,1)$ and $f(y,z) := P(1,y,z) = 0$ is the affine equation of $C$ in coordinates $y$, $z$. In these coordinates the point $Q$ becomes $(y_0, 0)$. Since the curve $C$ intersects the line $z = 0$ only at $Q$ the polynomial $f(y,z)$ satisfies the assumptions of Corollary 3.6.

Put $D(x,z,t) = \mathrm{Disc}_y(P(x,y,z) - t)$. We have that $D_\infty = D(x,1,t)$ and $D_0 := \mathrm{Disc}_y(f(y,z) - t) = D(1,z,t)$. Since $P(x,y,z)$ is a homogeneous polynomial of degree $n$, giving to the variable $t$ the weight $n$, and the other variables the weight 1, the polynomial $D(x,z,t)$ is quasi-homogeneous of degree $n(n-1)$ (see

Theorem 10.9, Chapter I, [26]). In particular any term $c_{ijk}x^i z^j t^k$ of $D(x,z,t)$ corresponds to the point $(i,j,k)$ of the hyperplane $\Pi \equiv i+j+nk = n(n-1)$. Moreover such point determines the term $c_{ijk}x^i t^k$ of $D_\infty$ and the term $c_{ijk}z^j t^k$ of $D_0$. Put $L : \mathbf{Z}^2 \to \mathbf{Z}^2$ defined by $L(i,k) = (n(n-1)-i-nk, k)$. Then supp $D_0$ $= L(\operatorname{supp} D_\infty)$. The last relation gives $\mathscr{P}_0(D_0) = L(\mathscr{P}_\infty(D_\infty))$. By Corollary 3.6 the curve $C$ is analytically irreducible at $Q$ if and only if the Newton diagram $\Delta_{D_0}$ is a Merle type diagram. ■

*Remark* 5.2. *Let $p(x,y)$ be a polynomial of degree n which has one point at infinity different from* $(0:1:0)$. *Let us denote by q the maximal inclination of* $\mathscr{P}_0(D_0)$. *By* [21], *the Abhyankar-Moh inequality* (*see* [3]) *is equivalent to $q<n$. Note also that the Abhyankar-Moh inequality is equivalent to equisingularity at infinity of the family $p(x,y)-t=0$. By* [16] *this is also equivalent to the statement that all segments of $\mathscr{P}_\infty(D_\infty)$ have positive slopes.*

*Example* 5.3. Let $p(x,y) = x + (x+y^3)^3$ be a polynomial in $\mathbf{C}[x,y]$ which corresponds to the projective curve $C$ defined by $P(x,y,z) = xz^8 + (xz^2+y^3)^3$ $=0$. The only point at infinity of $C$ is $Q=(1:0:0)$. Moreover $D_\infty =$ $(x+x^3-t)^2(x-t)^6$ and $\mathscr{P}_\infty(D_\infty)$ has only two segments joining the point $(0,8)$ to $(6,6)$ and this one to $(12,0)$. The transformation of $\mathscr{P}_\infty(D_\infty)$ by $L(i,k) = (72-i-9k, k)$ is a polygon of two segments joining the point $(0,8)$ to $(12,6)$ and this one to $(60,0)$. This polygon is the Newton polygon of $\Delta =$ $\left\{\frac{12}{2}\right\} + \left\{\frac{48}{6}\right\}$. Since $\Delta$ is not a Merle type diagram, by Theorem 5.1 the curve $C$ is not analytically irreducible at $Q$. Observe that the local equation $P(1,y,z)$ $=0$ of the curve $C$ was studied in Example 3.5.

**Appendix**

The purpose of this section is to describe the direct image of a curve $h(x,y)=0$ by an analytic mapping $F : (\mathbf{C}^2,0) \to (\mathbf{C}^2,0)$, $F^{-1}(0,0) = \{0,0\}$. We restrict the discussion to the case $F=(x,f)$ and $h(0,y) \not\equiv 0$.

We will use the following properties of the direct image (see Preliminaries in [6]):

(A) Let $h(x,y)=0$ be the irreducible curve with analytic parametrization $t \to (t^m, \phi(t))$. The direct image $F_*(h=0)$ is the curve $g(u,v)=0$ characterized by two conditions:
   (i) there exist an irreducible curve $\tilde{g}(u,v)=0$ and an integer $d>0$ such that $\tilde{g}(t^m, f(\phi(t)))=0$ and $g=\tilde{g}^d$,
   (ii) the projection formula $(g,u)_0 = (h,x)_0$ holds.

(B) If $g_i=0$ are direct images of curves $h_i=0$ for $i=1,2$ then $g_1g_2=0$ is the direct image of the curve $h_1h_2=0$.

Lemma 5.4. *Let $F=(x,f) : (\mathbf{C}^2,0) \to (\mathbf{C}^2,0)$, $F^{-1}(0,0)=\{(0,0)\}$ be an analytic mapping and let $h(x,y) = \prod_{j=1}^n [y-\gamma_j(x)]$ be the Newton-Puiseux facto-*

*rization of the convergent power series* $h(x, y)$. *Then the direct image* $F_*(h = 0)$ *has the equation*

$$\prod_{j=1}^{n}[v - f(u, \gamma_j(u))] = 0. \tag{4}$$

*Proof.* By (B) it is enough to prove Lemma for irreducible convergent power series $h(x, y)$.

By Puiseux theorem there exists an integer $n > 0$ and a convergent power series $\phi(t) \in \mathbf{C}\{t\}$, $\phi(0) = 0$ such that $h(t^n, \phi(t)) = 0$ and $h(x, y) = \prod_{i=1}^{n}[y - \phi(\varepsilon_n^i x^{1/n})]$, where $\varepsilon_n$ is the $n$-th primitive root of unity. We want to show that $g(u, v) = \prod_{i=1}^{n}[v - f(\phi(\varepsilon_n^i u^{1/n}))]$ satisfies (A)(i) and (A)(ii).

Put $\psi(t) = f(\phi(t))$. Let $d$ be the greatest common divisor of $\{n\} \cup \operatorname{supp}\psi$. Then there exists a convergent power series $\psi_0$ such that $\psi(t) = \psi_0(t^d)$. Let $m = n/d$ and let $\varepsilon_m = \varepsilon_n^d$ be the $m$-th primitive root of unity. We get $g(u, v) = \prod_{i=1}^{n}[v - \psi(\varepsilon_n^i u^{1/n})] = \prod_{i=1}^{n}[v - \psi_0(\varepsilon_m^i u^{1/m})] = \left[\prod_{i=1}^{m}[v - \psi_0(\varepsilon_m^i u^{1/m})]\right]^d$. By Puiseux theorem $\tilde{g}(u, v) = \prod_{i=1}^{m}[v - \psi_0(\varepsilon_m^i u^{1/m})]$ is an irreducible convergent power series. One easily checks that $\tilde{g}(t^n, \psi(t)) \equiv 0$. Moreover $(g, u)_0 = (h, x)_0 = n$ which ends the proof. ∎

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