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ON HEISENBERG'S INEQUALITY AND BELL'S INEQUALITY

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1. Introduction

(i) Heisenberg's uncertainty principle in quantum mechanics is formulated, following Kennard (1927), as

"The product of deviation of the x-component of position and deviation of the x-component of momentum is not smaller than $\frac{1}{2}\hbar$".

This inequality allows no exception. We will call this "Heisenberg-Kennard inequality".

There is a well-known proof of Heisenberg-Kennard inequality by Robertson (1929), who used Schwarz's inequality.

We will, however, prove that Heisenberg-Kennard inequality follows from Robertson's inequality only in a very special case, but doesn't in general.

More precisely, let $\psi(t,x) = e^{R(t,x)+iS(t,x)}$ be a solution of Schrödinger equation

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\sigma^2\triangle\psi - V(t,x)\psi = 0.$$

Then Heisenberg-Kennard inequality follows from Robertson's inequality if and only if

$$\int \nabla R \cdot \nabla S \mu_t \, dx = 0,$$

where $\mu_t = |\psi(t,x)|^2$, or equivalently

$$\int\left\{\frac{\partial R}{\partial t} + \frac{1}{2}\sigma^2\Delta S\right\}\mu_t \, dx = 0.$$

Hence, there exists no proof for Heisenberg-Kennard inequality up to now except for the above special case. Moreover, we will indicate that there is a counter example against Heisenberg-Kennard inequality. Therefore, Heisenberg's uncertainty principle in quantum mechanics is false.

(ii) Bell's claim "no local hidden variable model can explain the quantum mechanical correlation" in Bell (1964) will be shown to be false. Bell's claim

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was based on Bell's inequality. We will show that Bell's inequality concerns neither locality nor non-locality at all. Hence Bell's claim doesn't follow from Bell's inequality. We will show in addition that Bell's inequality holds only under Bell's additional dependence condition, but doesn't in general. We will moreover give a local spin correlation model, which is a counter-example against Bell's claim.

2. The expectation and variation

(i) The expectation

In probability theory the expectation $P[X]$ of a random variable $X(\omega)$ is defined by

$$P[X] = \int X(\omega)\, dP.$$

In quantum mechanics, for a self-adjoint operator A, the inner product of ψ and $A\psi$

$$\langle \psi, A\psi \rangle$$

is interpreted as the expectation of A, where ψ is an element of a Hilbert space H with $\|\psi\| = 1$.

If we handle a single bounded operator A, then there exists a random variable $h_A(\omega)$ and a probability measure μ such that

$$\langle \psi, A\psi \rangle = \int h_A(\omega)\, d\mu.$$

In fact, if **B** is a Banach algebra of bounded commutative self-adjoint operators, then there exist a compact Hausdorff space Ω and an isometry θ from the Banach algebra **B** onto the space $\mathrm{C}(\Omega)$. Namely for each $A \in \mathbf{B}$ there exists $h_A \in \mathrm{C}(\Omega)$ such that $h_A = \theta A$ and $\|h_A\| = \|A\|$, where the norm of h_A is $\|h_A\| = \sup_{\omega \in \Omega} |h_A(\omega)|$, cf. Nagasawa (1959). Moreover, for a linear functional defined by

$$m(h_A) = \langle f, Af \rangle, \quad h_A \in \mathrm{C}(\Omega),$$

there exists a probability measure μ such that

$$m(h_A) = \int h_A(\omega)\, d\mu,$$

by Riesz-Markov-Kakutani theorem, cf. Yosida (1965).

Hence the interpretation of the inner product $\langle \psi, A\psi \rangle$ as the expectation of A is reasonable.

However, if we treat a set of (non-commutable) self-adjoint operators, we need further consideration.

Definition 2.1. Let H be a Hilbert space. For $\psi \in H$ with the norm $\|\psi\| = 1$, and a set $\boldsymbol{F}$ of self-adjoint operators (physical quantities), if there exists a probability space $\{\Omega, \mathscr{F}, P\}$ and a random variable $h_A(\omega)$ such that

$$\langle \psi, A\psi \rangle = \int h_A(\omega) P[d\omega], \quad \text{for any } A \in \boldsymbol{F}, \tag{2.1}$$

then we will call the random variable $h_A(\omega)$ a *function representation* (*hidden variable*) of the self-adjoint operator $A \in \boldsymbol{F}$, and $\{(\Omega, \mathscr{F}, P); h_A, A \in \boldsymbol{F}\}$ will be called a *function representation* (*hidden variable*) of $\{\boldsymbol{F}, \psi\}$.

It should be remarked here that the function representation is not uniquely determined. In fact, if a random variable $g(\omega)$ is with mean zero, we can take $h_A(\omega) + g(\omega)$ as another function representation instead of $h_A(\omega)$.

Therefore, when sets of physical quantities are involved in a context, we must carefully choose hidden variables so that they are physically meaningful, and consistent with physical quantities we handle.

(ii) The variance

In probability theory the variance of a random variable X is defined by

$$Var(X) = \int \left(X(\omega) - \int X \, d\mu \right)^2 d\mu. \tag{2.2}$$

In quantum mechanics Robertson (1929) gave a definition of variance of a self-adjoint operator A *by*

$$\langle \psi, (A - \langle \psi, A\psi \rangle)^2 \psi \rangle. \tag{2.3}$$

However, first of all, it is not at all clear what physical quantity the operator $(A - \langle \psi, A\psi \rangle)^2$ is, and no reasonable physical explanation for this has been given. Moreover, the inner product by equation (2.3) is not variance of A, but something else, except for a very special case, as will be shown below.

To clarify this point we consider:

(iii) The case of Schrödinger operator

On a Hilbert space $H = \mathrm{L}^2(\mathbf{R}^3)$ we consider Schrödinger operator (momentum)

$$\boldsymbol{p} = \frac{\hbar}{i} \nabla.$$

We take a set of operators

$$\boldsymbol{F} = \{x, \boldsymbol{p}, \boldsymbol{p}^2\},$$

and a Schrödinger function (wave function)

$$\psi_t(x) = e^{R(t,x) + iS(t,x)},$$

with $\|\psi_t\| = 1$, and consider a function representation of $\{\boldsymbol{F}, \psi_t\}$.

We first take the operator $\boldsymbol{p}$. Then

$$\begin{aligned}\langle\psi_t, \boldsymbol{p}\psi_t\rangle &= \hbar\int \bar{\psi}_t(x)\frac{1}{i}\nabla\psi_t(x)\,dx \\ &= \hbar\int \bar{\psi}_t(x)\left(\frac{1}{i}\nabla R(t,x) + \nabla S(t,x)\right)\psi_t(x)\,dx \\ &= \int \hbar\left(\frac{1}{i}\nabla R(t,x) + \nabla S(t,x)\right)\mu_t(x)\,dx,\end{aligned}$$

where

$$\mu_t(x) = \bar{\psi}_t(x)\psi_t(x) = e^{2R(t,x)},$$

as usual in quantum mechanics. Since $\boldsymbol{p}$ is a self-sdjoint operator, the inner product $\langle\psi_t, \boldsymbol{p}\psi_t\rangle$ is real, and hence

$$\int \hbar(\nabla R(t,x))\mu_t(x)\,dx = 0. \tag{2.4}$$

Therefore, we can write an equality

$$\langle\psi_t, \boldsymbol{p}\psi_t\rangle = \int \hbar(\nabla R(t,x) + \nabla S(t,x))\mu_t(x)\,dx, \tag{2.5}$$

and take

$$h_{\boldsymbol{p}} = \hbar(\nabla R(t,x) + \nabla S(t,x)) \tag{2.6}$$

as a function representation of the momentum operator $\boldsymbol{p} = \frac{\hbar}{i}\nabla$.

Remark. Because of equation (2.4) one can take $\hbar\nabla S(t,x)$ as a function representation of the momentum operator $\boldsymbol{p} = \frac{\hbar}{i}\nabla$. But we won't. The reason is $\hbar(\nabla R(t,x) + \nabla S(t,x))$ has a good physical meaning as the momentum of stochastic processes of dynamic theory of random motion in quantum physics, cf. Nagasawa (1993, 2000, 2002, 2007). But $\hbar\nabla S(t,x)$ has no such a good physical meaning.

We now take $\boldsymbol{p}^2 = \left(\frac{\hbar}{i}\nabla\right)^2$. Then

$$\begin{aligned}\int \bar{\psi}_t(x)(\boldsymbol{p}^2\psi_t(x))\,dx &= \int \hbar^2\nabla\bar{\psi}_t(x)\nabla\psi_t(x)\,dx \\ &= \int \hbar^2\bar{\psi}_t(x)((\nabla R(t,x))^2 + (\nabla S(t,x))^2)\psi_t(x)\,dx \\ &= \int \hbar^2((\nabla R(t,x))^2 + (\nabla S(t,x))^2)\mu_t(x)\,dx,\end{aligned}$$

where $\mu_t = \bar{\psi}_t \psi_t$. Thus we have

$$\langle \psi_t, \boldsymbol{p}^2 \psi_t \rangle = \int \hbar^2 ((\nabla R)^2 + (\nabla S)^2) \mu_t(x)\, dx.$$

Therefore, we can take

$$h_{\boldsymbol{p}^2} = \hbar^2((\nabla R)^2 + (\nabla S)^2) \tag{2.7}$$

as a function representation of the operator $\boldsymbol{p}^2 = \left(\frac{\hbar}{i}\nabla\right)^2$.

Remark. There is a good reason to do so, since

$$\frac{1}{2m}\hbar^2((\nabla R)^2 + (\nabla S)^2)$$

is the kinetic energy $\frac{1}{2m}\boldsymbol{p}^2$ of stochastic processes of dynamic theory of random motion in quantum physics, cf. Nagasawa (1993, 2000, 2002, 2007).

THEOREM 2.1 (Nagasawa (2009)). *Let*

$$\psi_t(x) = e^{R(t,x)+iS(t,x)}$$

be a Schrödinger function (*wave function*). *Let* $h_{\boldsymbol{p}} = \hbar(\nabla R + \nabla S)$ *be a function representation of the momentum operator* $\boldsymbol{p}$, *and* $h_{\boldsymbol{p}^2} = \hbar^2((\nabla R)^2 + (\nabla S)^2)$ *be a function representation of* $\boldsymbol{p}^2$.

If

$$\int \nabla R \cdot \nabla S \mu_t(x)\, dx \neq 0,$$

i.e. if ∇R *and* ∇S *has correlation*, *then*

$$\int (h_{\boldsymbol{p}})^2(t,x)\mu_t(x)\, dx \neq \int h_{\boldsymbol{p}^2}(t,x)\mu_t(x)\, dx, \tag{2.8}$$

where $\mu_t(x) = e^{2R(t,x)}$.

Proof. Since the function representation of momentum operator $\boldsymbol{p}$ is $h_{\boldsymbol{p}}(t,x) = \hbar\nabla(R(t,x) + S(t,x))$, we have

$$(h_{\boldsymbol{p}})^2 = \hbar^2((\nabla R)^2 + (\nabla S)^2) + 2\hbar^2 \nabla R \cdot \nabla S. \tag{2.9}$$

By comparing equation (2.9) with equation (2.7), we can complete the proof.

Remark. If $\nabla R \nabla S \neq 0$, then $(h_{\boldsymbol{p}})^2 \neq h_{\boldsymbol{p}^2}$.

THEOREM 2.2. *Let* $\psi(t,x) = e^{R(t,x)+iS(t,x)}$ *satisfy the Schrödinger equation*

$$i\frac{\partial \psi}{\partial t} + \frac{1}{2}\sigma^2 \triangle \psi - V(t,x)\psi = 0. \tag{2.10}$$

Then the equality

$$\int (h_{\boldsymbol{p}})^2(t,x)\mu_t(x)\,dx = \int h_{\boldsymbol{p}^2}(t,x)\mu_t(x)\,dx$$

holds, if and only if

(2.11) $$\int\left\{\frac{\partial R}{\partial t} + \frac{1}{2}\sigma^2\Delta S\right\}\mu_t\,dx = 0.$$

Proof. Substitute $\psi(t,x) = e^{R(t,x)+iS(t,x)}$ in equation (2.10), and divide the resulting equation by ψ. Then it is easy to see that from the imaginary part we have

$$\frac{\partial R}{\partial t} + \frac{1}{2}\sigma^2\Delta S + \sigma^2\nabla S\cdot\nabla R = 0.$$

Therefore, by Theorem 2.1 we can complete the proof.

Remark. The equality

$$(h_{\boldsymbol{p}})^2 = h_{\boldsymbol{p}^2}$$

holds, if and only if

(2.12) $$\frac{\partial R}{\partial t} + \frac{1}{2}\sigma^2\Delta S \equiv 0.$$

THEOREM 2.3. *Let $\psi_t(x) = e^{R(t,x)+iS(t,x)}$ be a Schrödinger function. Then*:

(i) *Only when ∇R and ∇S has no correlation, or equivalently* (2.11) *holds*,

(2.13) $$\langle\psi_t, (\boldsymbol{p} - \langle\psi_t, \boldsymbol{p}\psi_t\rangle)^2\psi_t\rangle$$

gives the variance of the function representation $h_{\boldsymbol{p}} = \hbar(\nabla R + \nabla S)$ of the momentum operator $\boldsymbol{p}$.

(ii) *Hence the expectation of the operator $(\boldsymbol{p} - \langle\psi_t, \boldsymbol{p}\psi_t\rangle)^2$ in* (2.13) *is in general not variance, but something else.*

Proof. For the momentum operator $\boldsymbol{p}$

$$\begin{aligned}\langle\psi_t, (\boldsymbol{p} - \langle\psi_t, \boldsymbol{p}\psi_t\rangle)^2\psi_t\rangle &= \langle\psi_t, \boldsymbol{p}^2\psi_t\rangle - (\langle\psi_t, \boldsymbol{p}\psi_t\rangle)^2\\ &= \int h_{\boldsymbol{p}^2}(t,x)\mu_t(x)\,dx - \left(\int h_{\boldsymbol{p}}(t,x)\mu_t(x)\,dx\right)^2,\end{aligned}$$

the right hand side of which is, if $\int\nabla R\nabla S\mu_t\,dx \neq 0$,

$$\neq \int (h_{\boldsymbol{p}})^2(t,x)\mu_t(x)\,dx - \left(\int h_{\boldsymbol{p}}(t,x)\mu_t(x)\,dx\right)^2,$$

by Theorem 2.1. Namely, $\langle \psi_t, (\boldsymbol{p} - \langle \psi_t, \boldsymbol{p}\psi_t \rangle)^2 \psi_t \rangle$ is not equal to the variance of the function representation $h_{\boldsymbol{p}} = \hbar(\nabla R + \nabla S)$ of the momentum operator $\boldsymbol{p}$, when ∇R and ∇S has correlation. This completes the proof.

3. Heisenberg's inequality

Robertson (1929) put

$$X = (x - \langle \psi_t, x\psi_t \rangle)\psi_t, \quad Y = (p_x - \langle \psi_t, p_x\psi_t \rangle)\psi_t$$

in Schwarz's inequality

$$\sqrt{\langle X, X \rangle}\sqrt{\langle Y, Y \rangle} \geq \frac{1}{2}|\langle X, Y \rangle - \langle Y, X \rangle|, \tag{3.1}$$

and gets

$$\sqrt{\langle \psi_t, (x - \langle \psi_t, x\psi_t \rangle)^2 \psi_t \rangle}\sqrt{\langle \psi_t, (p_x - \langle \psi_t, p_x\psi_t \rangle)^2 \psi_t \rangle} \geq \frac{1}{2}\hbar, \tag{3.2}$$

(Robertson (1929)). Robertson claimed that the inequality (3.2) implies the Heisenberg-Kennard inequality:

"The product of deviation of the x-component of position and deviation of the x-component of momentum is not smaller than $\frac{1}{2}\hbar$",

that is, Heisenberg's uncertainty principle (Kennard (1927)).

However, Robertson's claim is false, because

$$\sqrt{\langle \psi_t, (p_x - \langle \psi_t, p_x\psi_t \rangle)^2 \psi_t \rangle}$$

in Robertson's inequality (3.2) coincides with the deviation of the x-component of momentum only for very special ψ_t by Theorem 2.3.

Therefore, there is no proof for the Heisenberg-Kennard inequality.

Moreover we have

Theorem 3.1 (Nagasawa (2009)). *Heisenberg's uncertainty principle doesn't hold. In other words, there is no positive minimum for the product of the deviation $\sqrt{Var(x)}$ of the x-component of position and the deviation $\sqrt{Var(h_{p_x})}$ of the x-component p_x of momentum in general.*

Proof. Let $\psi(t,x) = e^{R(t,x)+iS(t,x)}$ and fix $R(t,x)$. Then $\sqrt{Var(x)}$ is fixed. The function representation of p_x is

$$h_{p_x} = \hbar\frac{\partial R}{\partial x} + \hbar\frac{\partial S}{\partial x},$$

and the variance of h_{p_x} is

$$Var(h_{p_x}) = \int \hbar^2 \left(\frac{\partial R}{\partial x} + \frac{\partial S}{\partial x}\right)^2 \mu_t \, dx - \left(\int \hbar \frac{\partial S}{\partial x}\mu_t \, dx\right)^2, \tag{3.3}$$

where $\mu_t = \bar{\psi}_t\psi_t = e^{2R}$ is the distribution density.

If we choose, for instance, $S = -\varepsilon R$, then $\psi(t,x) = e^{R(t,x)+iS(t,x)}$ is a solution of the Schrödinger equation with an appropriately chosen potential function. Then, since the second integral of the right hand side of equation (3.3) vanishes, we have

$$Var(h_{p_x}) = (1-\varepsilon)^2 \int e^{2R}\hbar^2 \left(\frac{\partial R}{\partial x}\right)^2 dx,$$

which can be arbitrary small, by choosing ε. Therefore, $\sqrt{Var(x)}\sqrt{Var(h_{p_x})}$ can be arbitrary small. This complete the proof.

4. Inequality for the product of the expectation of kinetic energy and the variance of position

As an application of Schwarz's inequality we have

THEOREM 4.1. *Let* $\boldsymbol{x} = (x, y, z)$, *and* $\boldsymbol{p} = (p_x, p_y, p_z)$. *Then for each coordinate* x, y, z

$$\langle \psi_t, (x - \langle \psi_t, x\psi_t \rangle)^2 \psi_t \rangle \left\langle \psi_t, \frac{1}{2m}\boldsymbol{p}^2\psi_t \right\rangle \geq \frac{1}{4}\frac{1}{2m}\hbar^2, \tag{4.1}$$

that is, the product of the expectation of kinetic energy and the variance of each component of position is bounded away from zero by a positive constant $\frac{1}{4}\frac{1}{2m}\hbar^2$.

Proof. We set $X = (x - \langle \psi_t, x\psi_t \rangle)\psi_t$ and $Y = p_x\psi_t$ in Schwarz's inequality (3.1). Then

$$\langle \psi_t, (x - \langle \psi_t, x\psi_t \rangle)^2 \psi_t \rangle \langle \psi_t, p_x^2\psi_t \rangle \geq \frac{1}{4} |\langle \psi_t, (xp_x - p_x x)\psi_t \rangle|^2.$$

Since

$$(p_x x - xp_x)\psi_t = \frac{\hbar}{i}\psi_t,$$

for arbitrary ψ_t, we have

$$\langle \psi_t, (x - \langle \psi_t, x\psi_t \rangle)^2 \psi_t \rangle \langle \psi_t, p_x^2\psi_t \rangle \geq \frac{1}{4}\hbar^2.$$

Moreover, since

$$\langle \psi_t, (x - \langle \psi_t, x\psi_t \rangle)^2 \psi_t \rangle \langle \psi_t, p_y^2\psi_t \rangle = 0,$$

and

$$\langle \psi_t, (x - \langle \psi_t, x\psi_t \rangle)^2 \psi_t \rangle \langle \psi_t, p_z^2\psi_t \rangle = 0,$$

by adding the above three equations, we get inequality (4.1). This completes the proof.

Inequality (4.1) has significant physical applications. In fact, the inequality shows that if the kinetic energy becomes small (say, in the ground state), then there is a lower limit for the valiance of position.

As an example, we consider Bose-Einstein condensation, cf. Mewes et al. (1996), D. S. Jin (1996). Suppose the condensation goes on and the kinetic energy approaches the ground state. Then we can predict that explosion occurs, since the variance of position can't be small by inequality (4.1). The explosion is actually observed, for this see Cornish et al. (2001), Ueda-Saito (2002).

5. Locality of function representations

We consider a pair of particles with spin created at the origin. They fly to locations A and B which are far away separated of each other, and are detected by Stern-Gerlach magnets at the locations A and B. We denote the orientation of Stern-Gerlach magnets at A and B by $\{\mathbf{a}, \mathbf{b}\}$, where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are vectors with the norm 1.

Let $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ be the spin matrix, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and set

$$S(\mathbf{a}, \mathbf{1}) = \sigma\mathbf{a} \otimes \mathbf{1}, \quad S(\mathbf{1}, \mathbf{b}) = \mathbf{1} \otimes \sigma\mathbf{b}.$$

Then $S(\mathbf{a}, \mathbf{1}) = \sigma\mathbf{a} \otimes \mathbf{1}$ and $S(\mathbf{1}, \mathbf{b}) = \mathbf{1} \otimes \sigma\mathbf{b}$ are commutative matrices on a Hilbert space $H = \mathbf{R}^2 \times \mathbf{R}^2$.

We set

$$\psi = \frac{u^+ \otimes u^- - u^- \otimes u^+}{\sqrt{2}}, \tag{5.1}$$

where

$$u^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In physics the operators $S(\mathbf{a}, \mathbf{1}) = \sigma\mathbf{a} \otimes \mathbf{1}$ and $S(\mathbf{1}, \mathbf{b}) = \mathbf{1} \otimes \sigma\mathbf{b}$ are interpreted as the spin of particles detected at the locations A and B, respectively.

We give a physical interpretation of our model as follows.

We consider a pair of particles with spin created at the origin. They fly to locations A and B which are far away separated of each other.

The operator $S(\mathbf{a}, \mathbf{1}) = \sigma\mathbf{a} \otimes \mathbf{1}$ represents spin of a particle detected by Stern-Gerlach magnets settled at the locations A.

The operator $S(\mathbf{1}, \mathbf{b}) = \mathbf{1} \otimes \sigma\mathbf{b}$ represents spin of another particle detected by Stern-Gerlach magnets settled at the locations B.

The vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are orientations of Stern-Gerlach magnets at the locations A and B, respectively.

After a pair of particles are created at the origin, spin of the particles will not change.

Equation (5.1) means that spin is up $\uparrow$ or down $\downarrow$.

We now compute $\langle \psi, \sigma\mathbf{a} \otimes \mathbf{1}\psi \rangle$ and $\langle \psi, \mathbf{1} \otimes \sigma\mathbf{b}\psi \rangle$. We have

$$u^+ \sigma\mathbf{a} u^+ = (1, 0)\begin{pmatrix} a_3 & a_1 - a_2 i \\ a_1 + a_2 i & -a_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a_3,$$

$$u^- \sigma\mathbf{a} u^- = (0, 1)\begin{pmatrix} a_3 & a_1 - a_2 i \\ a_1 + a_2 i & -a_3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -a_3,$$

$$u^+ \sigma\mathbf{a} u^- = (1, 0)\begin{pmatrix} a_3 & a_1 - a_2 i \\ a_1 + a_2 i & -a_3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a_1 - a_2 i,$$

$$u^- \sigma\mathbf{a} u^+ = (0, 1)\begin{pmatrix} a_3 & a_1 - a_2 i \\ a_1 + a_2 i & -a_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a_1 + a_2 i.$$

Therefore

$$\langle \psi, \sigma\mathbf{a} \otimes \mathbf{1}\psi \rangle = 0, \quad \langle \psi, \mathbf{1} \otimes \sigma\mathbf{b}\psi \rangle = 0. \tag{5.2}$$

This means that the expectation of spins of the particles detected at the locations A and B are equal to zero.

Moreover, we have

$$\langle \psi, (\sigma\mathbf{a} \otimes \mathbf{1})(\mathbf{1} \otimes \sigma\mathbf{b})\psi \rangle = -\mathbf{ab}, \tag{5.3}$$

where $\mathbf{ab} = a_1 b_1 + a_2 b_2 + a_3 b_3$. This means that operators $S(\mathbf{a}, \mathbf{1}) = \sigma\mathbf{a} \otimes \mathbf{1}$ and $S(\mathbf{1}, \mathbf{b}) = \mathbf{1} \otimes \sigma\mathbf{b}$ have correlation. This correlation of spins of the pair of particles was induced at the origin, when the particles were created.

An important point in our discussion is the so-called locality of the function representation of $\{\boldsymbol{F}, \psi\}$. This was the key point of Einstein-Podolsky-Rozen (1935) (cf. also [2] of Bell (1964).)

Definition 5.1. Let $\boldsymbol{F}$ be a set of three operators

$$\boldsymbol{F} = \{S(\mathbf{a}, \mathbf{1}), S(\mathbf{1}, \mathbf{b}), S(\mathbf{a}, \mathbf{1})S(\mathbf{1}, \mathbf{b})\},$$

and ψ be given by (5.1). A function representation

$$\{\Omega, P; h_{\mathbf{a},1}, h_{1,\mathbf{b}}, h_{\mathbf{a},1}h_{1,\mathbf{b}}\}$$

of $\{\boldsymbol{F}, \psi\}$ is *local*, and $h_{\mathbf{a},1}$ and $h_{1,\mathbf{b}}$ are *local hidden variables*, if the following locality condition is satisfied:

(L) The random variable $h_{\mathbf{a},1}$ does not depend on the orientation $\mathbf{b}$ of Stern-Gerlach magnet at the location B which is separated from the location A, and the random variable $h_{1,\mathbf{b}}$ does not depend on the orientation $\mathbf{a}$ of Stern-Gerlach magnet at the location A which is separated from the location B.

We emphasize here that locality is defined for *a single experiment* with fixed orientations $\{\mathbf{a}, \mathbf{b}\}$ of Stern-Gerlach magnets at the locations A and B. If several experiments with different orientations of Stern-Gerlach magnets are involved in a context, we require locality for each experiment.

We now consider Bell's discussion in Bell (1964) on spin correlations.

Bell's discussion involves the spin correlations of the following four experiments with different orientations of Stern-Gerlach magnets at the locations A and B:

(Experiment 1) The orientations of Stern-Gerlach magnets are $\{\mathbf{a}, \mathbf{b}\}$.
(Experiment 2) The orientations of Stern-Gerlach magnets are $\{\mathbf{a}, \mathbf{c}\}$.
(Experiment 3) The orientations of Stern-Gerlach magnets are $\{\mathbf{b}, \mathbf{c}\}$.
(Experiment 4) The orientations of Stern-Gerlach magnets are $\{\mathbf{b}, \mathbf{b}\}$.

We assume that these experiments are independently done and have no influence of each-other.

To discuss the experiments above we take a set of operators

$$\begin{aligned} \boldsymbol{F} = \{ & S^{\{1\}}(\mathbf{a}, \mathbf{1}), S^{\{1\}}(\mathbf{1}, \mathbf{b}), S^{\{1\}}(\mathbf{a}, \mathbf{1})S^{\{1\}}(\mathbf{1}, \mathbf{b}); \\ & S^{\{2\}}(\mathbf{a}, \mathbf{1}), S^{\{2\}}(\mathbf{1}, \mathbf{c}), S^{\{2\}}(\mathbf{a}, \mathbf{1})S^{\{2\}}(\mathbf{1}, \mathbf{c}); \\ & S^{\{3\}}(\mathbf{b}, \mathbf{1}), S^{\{3\}}(\mathbf{1}, \mathbf{c}), S^{\{3\}}(\mathbf{b}, \mathbf{1})S^{\{3\}}(\mathbf{1}, \mathbf{c}); \\ & S^{\{4\}}(\mathbf{b}, \mathbf{1}), S^{\{4\}}(\mathbf{1}, \mathbf{b}), S^{\{4\}}(\mathbf{b}, \mathbf{1})S^{\{4\}}(\mathbf{1}, \mathbf{b})\}, \end{aligned} \tag{5.4}$$

where $\{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$ indicate Experiments 1, 2, 3 and 4, respectively, and consider $\{\boldsymbol{F}, \psi\}$ with ψ given in (5.1).

We then consider a function representation of $\{\boldsymbol{F}, \psi\}$:

$$\{\Omega, P; h_A, A \in \boldsymbol{F}\}. \tag{5.5}$$

As we have already remarked, such a function representation is not uniquely determined. Hence, we must carefully settle function representations, by distinguishing the above four experiments.

We will introduce function representations (hidden variables) to fix notations in the following, where non-uniquness of function representations will have no influence. As a matter of fact, explicit forms of function representations will be given in section 7, in which the existence of local function representations $\{\Omega, P; h_A, A \in \boldsymbol{F}\}$ will be discussed.

For Experiment 1, we denote function representations of operators $S^{\{1\}}(\mathbf{a}, 1)$ and $S^{\{1\}}(1, \mathbf{b})$ as

$$h^{\{1\}}_{\mathbf{a},1}(\omega), \quad h^{\{1\}}_{1,\mathbf{b}}(\omega).$$

We assume they take values ± 1.

Since $h^{\{1\}}_{\mathbf{a},1}(\omega)$ and $h^{\{1\}}_{1,\mathbf{b}}(\omega)$ are function representations of $S^{\{1\}}(\mathbf{a},1)$ and $S^{\{1\}}(1,\mathbf{b})$, we have

$$P[h^{\{1\}}_{\mathbf{a},1}] = 0, \quad P[h^{\{1\}}_{1,\mathbf{b}}] = 0,$$

because of equation (5.2). Moreover, we have

$$P[h^{\{1\}}_{\mathbf{a},1} h^{\{1\}}_{1,\mathbf{b}}] = -\mathbf{ab},$$

because of equation (5.3).

For Experiment 2, we denote function representations of operators $S^{\{2\}}(\mathbf{a},1)$ and $S^{\{2\}}(1,\mathbf{c})$ as

$$h^{\{2\}}_{\mathbf{a},1}(\omega), \quad h^{\{2\}}_{1,\mathbf{c}}(\omega).$$

We assume they take values ± 1.

Since $h^{\{2\}}_{\mathbf{a},1}(\omega)$ and $h^{\{2\}}_{1,\mathbf{c}}(\omega)$ are function representations of $S^{\{2\}}(\mathbf{a},1)$ and $S^{\{2\}}(1,\mathbf{c})$, we have

$$P[h^{\{2\}}_{\mathbf{a},1}] = 0, \quad P[h^{\{2\}}_{1,\mathbf{c}}] = 0,$$

because of equation (5.2). Moreover, we have

$$P[h^{\{2\}}_{\mathbf{a},1} h^{\{2\}}_{1,\mathbf{c}}] = -\mathbf{ac},$$

because of equation (5.3).

Since Experiments 1 and 2 are independently done, function representations $h^{\{1\}}_{\mathbf{a},1}$ and $h^{\{2\}}_{\mathbf{a},1}$ are independent of each other, and

$$h^{\{1\}}_{\mathbf{a},1} \neq h^{\{2\}}_{\mathbf{a},1}. \tag{5.6}$$

For Experiment 3, we denote function representations of operators $S^{\{3\}}(\mathbf{b},1)$ and $S^{\{3\}}(1,\mathbf{c})$ as

$$h^{\{3\}}_{\mathbf{b},1}(\omega), \quad h^{\{3\}}_{1,\mathbf{c}}(\omega).$$

We assume they take values ± 1. Then we have

$$P[h^{\{3\}}_{\mathbf{b},1}] = 0, \quad P[h^{\{3\}}_{1,\mathbf{c}}] = 0,$$

because of equation (5.2). Moreover, we have

$$P[h^{\{3\}}_{\mathbf{b},1} h^{\{3\}}_{1,\mathbf{c}}] = -\mathbf{bc},$$

because of equation (5.3).

Since Experiments 2 and 3 are independently done, function representations $h^{\{2\}}_{1,\mathbf{c}}$ and $h^{\{3\}}_{1,\mathbf{c}}$ are independent of each other, and

$$h^{\{2\}}_{1,\mathbf{c}} \neq h^{\{3\}}_{1,\mathbf{c}}. \tag{5.7}$$

For Experiment 4, we denote function representations of operators $S^{\{4\}}(\mathbf{b},1)$ and $S^{\{4\}}(1,\mathbf{b})$ as

$$h^{\{4\}}_{\mathbf{b},1}(\omega), \quad h^{\{4\}}_{1,\mathbf{b}}(\omega).$$

We assume they take values ± 1 and

$$h^{\{4\}}_{\mathbf{b},1}(\omega)h^{\{4\}}_{1,\mathbf{b}}(\omega) = -1.$$

Then we have

$$P[h^{\{4\}}_{\mathbf{b},1}] = 0, \quad P[h^{\{4\}}_{1,\mathbf{b}}] = 0,$$

because of equation (5.2), and

$$P[h^{\{4\}}_{\mathbf{b},1}h^{\{4\}}_{1,\mathbf{b}}] = -1,$$

because of equation (5.3).

Since Experiments 3 and 4 are independently done, function representations $h^{\{3\}}_{\mathbf{b},1}$ and $h^{\{4\}}_{\mathbf{b},1}$ are independent of each other; and also $h^{\{4\}}_{1,\mathbf{b}}$ and $h^{\{1\}}_{1,\mathbf{b}}$ are independent of each other, since Experiments 4 and 1 are independently done. Therefore

$$h^{\{3\}}_{\mathbf{b},1} \neq h^{\{4\}}_{\mathbf{b},1} \quad \text{and} \quad h^{\{4\}}_{1,\mathbf{b}} \neq h^{\{1\}}_{1,\mathbf{b}}. \tag{5.8}$$

We assume that hidden variables

$$\{h^{\{1\}}_{\mathbf{a},1}, h^{\{1\}}_{1,\mathbf{b}}\}, \quad \{h^{\{2\}}_{\mathbf{a},1}, h^{\{2\}}_{1,\mathbf{c}}\}, \quad \{h^{\{3\}}_{\mathbf{b},1}, h^{\{3\}}_{1,\mathbf{c}}\} \quad \text{and} \quad \{h^{\{4\}}_{\mathbf{b},1}, h^{\{4\}}_{1,\mathbf{b}}\} \tag{5.9}$$

of Experiments 1, 2, 3 and 4 satisfy locality.

6. Bell's inequality

Bell (1964) used hidden variables given by (5.9) in discussing correlations of hidden variables for Experiments 1, 2, 3 and 4, but he made an additional assumption on relations between hidden variables of these experiments.

Namely, Bell required

$$h^{\{2\}}_{\mathbf{a},1} = h^{\{1\}}_{\mathbf{a},1}, \quad h^{\{2\}}_{1,\mathbf{c}} = h^{\{3\}}_{1,\mathbf{c}}, \quad h^{\{4\}}_{\mathbf{b},1} = h^{\{3\}}_{\mathbf{b},1} \quad \text{and} \quad h^{\{4\}}_{1,\mathbf{b}} = h^{\{1\}}_{1,\mathbf{b}}. \tag{6.1}$$

By (6.1) Bell rejects all of (5.6), (5.7) and (5.8). This means that he requires dependence between hidden variables of Experiments 1, 2, 3, and 4.

We will call (6.1) *Bell's additional condition*, or *Bell's dependence*.

(As a matter of fact, in his paper Bell (1964) he simply neglected and didn't write the superscript $\{1\}$, $\{2\}$, $\{3\}$ and $\{4\}$ of hidden variables in (6.1). Moreover, he confused Bell's dependence in (6.1) with locality. This was his error.)

We note that *Bell's dependence* in (6.1) *has nothing to do with locality* (L) of Definition 5.1. Locality is, in fact, a notion on each *single experiment*.

Proposition 6.1. *Assume Bell's dependence in* (6.1). *Then*

$$|P[h^{\{1\}}_{\mathbf{a},1}h^{\{1\}}_{1,\mathbf{b}}] - P[h^{\{2\}}_{\mathbf{a},1}h^{\{2\}}_{1,\mathbf{c}}]| \leq 1 + P[h^{\{3\}}_{\mathbf{b},1}h^{\{3\}}_{1,\mathbf{c}}]. \tag{6.2}$$

Proof. Since $h^{\{4\}}_{\mathbf{b},1}h^{\{4\}}_{1,\mathbf{b}} = -1$, we have

$$h^{\{1\}}_{\mathbf{a},1}h^{\{1\}}_{1,\mathbf{b}} - h^{\{2\}}_{\mathbf{a},1}h^{\{2\}}_{1,\mathbf{c}} = h^{\{1\}}_{\mathbf{a},1}h^{\{1\}}_{1,\mathbf{b}} + h^{\{2\}}_{\mathbf{a},1}h^{\{2\}}_{1,\mathbf{c}}h^{\{4\}}_{\mathbf{b},1}h^{\{4\}}_{1,\mathbf{b}} \tag{6.3}$$

By Bell's dependence in (6.1), equation (6.3) can be rewritten as

$$h^{\{1\}}_{\mathbf{a},1}h^{\{1\}}_{1,\mathbf{b}} - h^{\{2\}}_{\mathbf{a},1}h^{\{2\}}_{1,\mathbf{c}} = h^{\{1\}}_{\mathbf{a},1}h^{\{1\}}_{1,\mathbf{b}} + h^{\{1\}}_{\mathbf{a},1}h^{\{3\}}_{1,\mathbf{c}}h^{\{3\}}_{\mathbf{b},1}h^{\{1\}}_{1,\mathbf{b}}$$

and hence we have

$$= h^{\{1\}}_{\mathbf{a},1}h^{\{1\}}_{1,\mathbf{b}}(1 + h^{\{3\}}_{\mathbf{b},1}h^{\{3\}}_{1,\mathbf{c}}), \quad P - a.e..$$

Because $h^{\{1\}}_{\mathbf{a},1}h^{\{1\}}_{1,\mathbf{b}} \leq 1$,

$$h^{\{1\}}_{\mathbf{a},1}h^{\{1\}}_{1,\mathbf{b}} - h^{\{2\}}_{\mathbf{a},1}h^{\{2\}}_{1,\mathbf{c}} \leq 1 + h^{\{3\}}_{\mathbf{b},1}h^{\{3\}}_{1,\mathbf{c}}, \quad P - a.e..$$

Taking the expectation of both sides, we get (6.2). This complete the proof.

Inequality (6.2) is called Bell's inequality. In the proof of Proposition 6.1, the locality (L) of Definition 5.1 is not used, but applied is Bell's dependence in (6.1). Hence, Bell's inequality has nothing to do with locality at all.

PROPOSITION 6.2. *Bell's dependence in* (6.1) *is inconsistent with the random variables* $\{h^{\{1\}}_{\mathbf{a},1}, h^{\{1\}}_{1,\mathbf{b}}, h^{\{2\}}_{\mathbf{a},1}, h^{\{2\}}_{1,\mathbf{c}}, \ldots\}$ *in* (5.9) *to be function representations of Experiments* 1, 2, 3 *and* 4 *of spin correlation.*

Proof. Let $\{h^{\{1\}}_{\mathbf{a},1}, h^{\{1\}}_{1,\mathbf{b}}, h^{\{2\}}_{\mathbf{a},1}, h^{\{2\}}_{1,\mathbf{c}}, \ldots\}$ in (5.9) be a function representation of four experiments of spin correlation, and assume Bell's dependence in (6.1). We then get Bell's inequality (6.2), and reach a contradiction. In fact, we take vectors $\mathbf{b} = (b_1, b_2, b_3)$ and $\mathbf{c} = (c_1, c_2, c_3)$ being orthogonal. Then

$$P[h^{\{3\}}_{\mathbf{b},1}h^{\{3\}}_{1,\mathbf{c}}] = -\mathbf{bc} = \mathbf{0}.$$

Therefore, the right hand side of inequality (6.2) is equal to 1. Since

$$P[h^{\{1\}}_{\mathbf{a},1}h^{\{1\}}_{1,\mathbf{b}}] = -\mathbf{ab}, \quad P[h^{\{2\}}_{\mathbf{a},1}h^{\{2\}}_{1,\mathbf{c}}] = -\mathbf{ac},$$

if we set $\mathbf{a} = \dfrac{\mathbf{b}-\mathbf{c}}{\|\mathbf{b}-\mathbf{c}\|}$, we get

$$P[h^{\{1\}}_{\mathbf{a},1}h^{\{1\}}_{1,\mathbf{b}}] - P[h^{\{2\}}_{\mathbf{a},1}h^{\{2\}}_{1,\mathbf{c}}] = -\mathbf{a}(\mathbf{b}-\mathbf{c}) = -\frac{\|\mathbf{b}-\mathbf{c}\|^2}{\|\mathbf{b}-\mathbf{c}\|}$$

$$= -\|\mathbf{b}-\mathbf{c}\| = -\sqrt{2}.$$

Thus, by (6.2), we reach a contradiction $\sqrt{2} \leq 1$. This completes the proof.

Remark. Aspect-Dalibard-Roger (1982) showed experimentally that Bell's inequality doesn't hold. Their experiment implies that Bell's dependence in (6.1) is inconsistent with their experiment.

Bell claimed that, if hidden variables are local, they must satisfy Bell's inequality (6.2); therefore "no local hidden variable model can explain the quantum mechanical correlation", since inequality (6.2) induces a contradiction.

However, Bell's claim is false, because of Proposition 6.1. In fact, Bell's inequality (6.2) concerns neither locality nor non-locality at all. It follows from Bell's dependence assumption in (6.1) which has nothing to do with locality.

7. A local spin correlation model

We consider

(Experiment 1) the orientations of Stern-Gerlach magnets are $\{\mathbf{a}, \mathbf{b}\}$;
(Experiment 2) the orientations of Stern-Gerlach magnets are $\{\mathbf{a}', \mathbf{b}'\}$;
(Experiment 3) the orientations of Stern-Gerlach magnets are $\{\mathbf{a}'', \mathbf{b}''\}$;

and so on.

We assume that these experiments are independently done, and have no influence of each other.

To discuss these experiments we take a set of operators

$$\boldsymbol{F} = \{S(\mathbf{a}, \mathbf{1}), S(\mathbf{1}, \mathbf{b}), S(\mathbf{a}, \mathbf{1})S(\mathbf{1}, \mathbf{b}); \{\mathbf{a}, \mathbf{b}\} \ \textit{is arbitrary}\}. \tag{7.1}$$

An important point here is the so-called locality of the function representation of $\{\boldsymbol{F}, \psi\}$, cf. Einstein-Podolsky-Rozen (1935), Bell (1964).

We denote a function representation of $\{\boldsymbol{F}, \psi\}$ as

$$\{\Omega, P; h_{\mathbf{a},\mathbf{1}}^{\{\mathbf{a},\mathbf{b}\}}, h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}, h_{\mathbf{a},\mathbf{1}}^{\{\mathbf{a},\mathbf{b}\}} h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}; \{\mathbf{a}, \mathbf{b}\} \ \textit{is arbitrary}\} \tag{7.2}$$

where superscript $\{\mathbf{a}, \mathbf{b}\}$ of $h_{\mathbf{a},\mathbf{1}}^{\{\mathbf{a},\mathbf{b}\}}$ and $h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}$ distinguishes an experiment with vectors **a** and **b** at the location A and B from others, i.e. each superscript $\{\mathbf{a}, \mathbf{b}\}$ indicates each independent experiment. We repeat the definition of locality of the function representation in (7.2).

Definition 7.1. Let $\boldsymbol{F}$ be the set of operators given by (7.1) and ψ be given by (5.1). A function representation in (7.2) of $\{\boldsymbol{F}, \psi\}$ is *local*, if it satisfies the following locality condition: For each experiment with $\{\mathbf{a}, \mathbf{b}\}$,

(L) the random variable $h_{\mathbf{a},\mathbf{1}}^{\{\mathbf{a},\mathbf{b}\}}$ does not depend on the vector **b** of Stern-Gerlach magnet at the location B, and the random variable $h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}$ does not depend on the vector **a** of Stern-Gerlach magnet at the location A. Namely, $h_{\mathbf{a},\mathbf{1}}^{\{\mathbf{a},\mathbf{b}\}}$ and $h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}$ are local hidden variables.

Theorem 7.1. *Let* $\boldsymbol{F}$ *be the set of operators given by* (7.1) *and* ψ *be given by* (5.1). *Then there exists a local function representation of* $\{\boldsymbol{F}, \psi\}$ *such as in* (7.2).

Proof. We first set $W = \{0, 1\} \times \{0, 1\}$, and define functions $h_1(i, j)$ and $h_2(i, j)$ on W by

$$h_1(0, j) = 1, \quad h_1(1, j) = -1, \quad j = 0, 1, \tag{7.3}$$

and

$$h_2(i, 0) = -1, \quad h_2(i, 1) = 1, \quad i = 0, 1. \tag{7.4}$$

The function $h_1(i,j)$ does not depend on the second variable, and the function $h_2(i,j)$ does not depend on the first variable. Namely they are locally defined.

We then define the product space

$$\Omega = \prod_{\{\mathbf{a},\mathbf{b}\}} \Omega_{\{\mathbf{a},\mathbf{b}\}},$$

where $\Omega_{\{\mathbf{a},\mathbf{b}\}} = W$. An element of Ω is

$$\omega = (\ldots, \omega_{\mathbf{a},\mathbf{b}}, \ldots, \omega_{\mathbf{a}',\mathbf{b}'}, \ldots),$$

where $\ldots, \omega_{\mathbf{a},\mathbf{b}} \in \Omega_{\{\mathbf{a},\mathbf{b}\}}, \ldots, \omega_{\mathbf{a}',\mathbf{b}'} \in \Omega_{\{\mathbf{a}',\mathbf{b}'\}}, \ldots$.

For each $\{\mathbf{a},\mathbf{b}\}$, we define functions $h^{\{\mathbf{a},\mathbf{b}\}}_{\mathbf{a},1}(\omega)$ and $h^{\{\mathbf{a},\mathbf{b}\}}_{1,\mathbf{b}}(\omega)$ on the product space Ω by

$$\begin{aligned} h^{\{\mathbf{a},\mathbf{b}\}}_{\mathbf{a},1}(\omega) &= h_1(\omega_{\mathbf{a},\mathbf{b}}), \quad \text{where } \omega = (\ldots, \omega_{\mathbf{a},\mathbf{b}}, \ldots, \omega_{\mathbf{a}',\mathbf{b}'}, \ldots), \\ h^{\{\mathbf{a},\mathbf{b}\}}_{1,\mathbf{b}}(\omega) &= h_2(\omega_{\mathbf{a},\mathbf{b}}), \quad \text{where } \omega = (\ldots, \omega_{\mathbf{a},\mathbf{b}}, \ldots, \omega_{\mathbf{a}',\mathbf{b}'}, \ldots), \end{aligned} \tag{7.5}$$

where the functions h_1 and h_2 are given by (7.3) and (7.4), respectively.

The random variable $h^{\{\mathbf{a},\mathbf{b}\}}_{\mathbf{a},1}(\omega)$ depends only on $\omega_{\mathbf{a},\mathbf{b}} \in \Omega_{\{\mathbf{a},\mathbf{b}\}}$ and equals to h_1 given by (7.3), and hence $h^{\{\mathbf{a},\mathbf{b}\}}_{\mathbf{a},1}(\omega)$ does not depend on vector $\mathbf{b}$. The random variable $h^{\{\mathbf{a},\mathbf{b}\}}_{1,\mathbf{b}}(\omega)$ depends only on $\omega_{\mathbf{a},\mathbf{b}} \in \Omega_{\{\mathbf{a},\mathbf{b}\}}$ and equals to h_2 given by (7.4), and hence $h^{\{\mathbf{a},\mathbf{b}\}}_{1,\mathbf{b}}(\omega)$ does not depend on vector $\mathbf{a}$. Therefore, the random variables $h^{\{\mathbf{a},\mathbf{b}\}}_{\mathbf{a},1}$ and $h^{\{\mathbf{a},\mathbf{b}\}}_{1,\mathbf{b}}$ satisfy the locality condition (L) of Definition 7.1.

Let $p = \{p_{ij}; i,j = 0,1\}$ be a probability measure on $W = \{0,1\} \times \{0,1\}$ given by

$$p_{00} = p_{11} = \frac{1 + \mathbf{ab}}{4}, \quad p_{01} = p_{10} = \frac{1 - \mathbf{ab}}{4}. \tag{7.6}$$

We define the product measure

$$P = \prod_{\{\mathbf{a},\mathbf{b}\}} p_{\{\mathbf{a},\mathbf{b}\}}, \quad \text{where } p_{\{\mathbf{a},\mathbf{b}\}} = p, \tag{7.7}$$

on the product space $\Omega = \prod_{\{\mathbf{a},\mathbf{b}\}} \Omega_{\{\mathbf{a},\mathbf{b}\}}$, cf. eg. Halmos (1950), p. 158 (2).

The probability measure P is common for all pairs $\{\mathbf{a},\mathbf{b}\}$, and a *universal* probability measure for discussing the problem of spin correlations.

Since the measure P is the direct product, if we change the orientation of Stern-Gerlach magnets at the locations A and B, namely, if $\{\mathbf{a},\mathbf{b}\} \neq \{\mathbf{a}',\mathbf{b}'\}$, then a set of random variables

$$\{h^{\{\mathbf{a},\mathbf{b}\}}_{\mathbf{a},1}, h^{\{\mathbf{a},\mathbf{b}\}}_{1,\mathbf{b}}, h^{\{\mathbf{a},\mathbf{b}\}}_{\mathbf{a},1} h^{\{\mathbf{a},\mathbf{b}\}}_{1,\mathbf{b}}\}$$

and a set of random variables

$$\{h^{\{\mathbf{a}',\mathbf{b}'\}}_{\mathbf{a}',1}, h^{\{\mathbf{a}',\mathbf{b}'\}}_{1,\mathbf{b}'}, h^{\{\mathbf{a}',\mathbf{b}'\}}_{\mathbf{a}',1} h^{\{\mathbf{a}',\mathbf{b}'\}}_{1,\mathbf{b}'}\}$$

are independent.

Moreover, the expectation of the random variables $h_{\mathbf{a},1}^{\{\mathbf{a},\mathbf{b}\}}(\omega)$ and $h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}(\omega)$ given by (7.5) are

$$\int h_{\mathbf{a},1}^{\{\mathbf{a},\mathbf{b}\}}(\omega)P[d\omega] = 0, \quad \int h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}(\omega)P[d\omega] = 0,$$

and the correlation is

$$\int h_{\mathbf{a},1}^{\{\mathbf{a},\mathbf{b}\}}(\omega)h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}(\omega)P[d\omega] = -\mathbf{ab},$$

where P is the universal probability measure given by (7.7).

Therefore, comparing them with equations (5.2) and (5.3), we have, for each pair $\{\mathbf{a},\mathbf{b}\}$,

$$\langle\psi, \sigma\mathbf{a}\otimes\mathbf{1}\psi\rangle = \int h_{\mathbf{a},1}^{\{\mathbf{a},\mathbf{b}\}}(\omega)P[d\omega],$$

$$\langle\psi, \mathbf{1}\otimes\sigma\mathbf{b}\psi\rangle = \int h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}(\omega)P[d\omega], \tag{7.8}$$

$$\langle\psi, (\sigma\mathbf{a}\otimes\mathbf{1})(\mathbf{1}\otimes\sigma\mathbf{b})\psi\rangle = \int h_{\mathbf{a},1}^{\{\mathbf{a},\mathbf{b}\}}(\omega)h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}(\omega)P[d\omega].$$

Thus we have shown

$$\{\Omega, P; h_{\mathbf{a},1}^{\{\mathbf{a},\mathbf{b}\}}, h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}, h_{\mathbf{a},1}^{\{\mathbf{a},\mathbf{b}\}}h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}; \{\mathbf{a},\mathbf{b}\}\ \textit{arbitrary}\}$$

is a function representation of $\{\boldsymbol{F},\psi\}$, where $\boldsymbol{F}$ is given by (7.1) and ψ is given by (5.1).

We have already shown that the random variables $h_{\mathbf{a},1}^{\{\mathbf{a},\mathbf{b}\}}(\omega)$ and $h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}(\omega)$ given by (7.5) are local hidden variables. Therefore, our function representation in (7.2) is local. This completes the proof.

Remark. What the equality in (7.8) means is that the left and right hand sides of the equation coincide for each pair $\{\mathbf{a},\mathbf{b}\}$. One can see also that the right hand side follows from the left hand side. In fact, let ψ be given by (5.1). Then the left hand side of equation (7.8) is

$$\langle\psi, (\sigma\mathbf{a}\otimes\mathbf{1})(\mathbf{1}\otimes\sigma\mathbf{b})\psi\rangle = \frac{1}{2}\{(u^{+}\sigma\mathbf{a}u^{+})(u^{-}\sigma\mathbf{b}u^{-}) + (u^{-}\sigma\mathbf{a}u^{-})(u^{+}\sigma\mathbf{b}u^{+}) - (u^{+}\sigma\mathbf{a}u^{-})(u^{-}\sigma\mathbf{b}u^{+}) - (u^{-}\sigma\mathbf{a}u^{+})(u^{+}\sigma\mathbf{b}u^{-})\}$$

where $u^{+} = \begin{pmatrix}1\\0\end{pmatrix}$, $u^{-} = \begin{pmatrix}0\\1\end{pmatrix}$,

$$= -\mathbf{ab}$$

which can be decomposed as

$$= -\frac{1+\mathbf{ab}}{4} + \frac{1-\mathbf{ab}}{4} + \frac{1-\mathbf{ab}}{4} - \frac{1+\mathbf{ab}}{4}$$

and with the functions $h_1(i,j)$ and $h_2(i,j)$ given by equations (7.3) and (7.4), respectively, one gets

$$= h_1(0,0)h_2(0,0)\frac{1+\mathbf{ab}}{4} + h_1(0,1)h_2(0,1)\frac{1-\mathbf{ab}}{4}$$

$$+h_1(1,0)h_2(1,0)\frac{1-\mathbf{ab}}{4} + h_1(1,1)h_2(1,1)\frac{1+\mathbf{ab}}{4}$$

moreover, by definition (7.6) of the probability measure $p = \{p_{ij}; i,j = 0,1\}$

$$= h_1(0,0)h_2(0,0)p_{00} + h_1(0,1)h_2(0,1)p_{01}$$

$$+h_1(1,0)h_2(1,0)p_{10} + h_1(1,1)h_2(1,1)p_{11}$$

then according to definitions (7.5) and (7.7)

$$= \int h_{\mathbf{a},1}^{\{\mathbf{a},\mathbf{b}\}}(\omega)h_{1,\mathbf{b}}^{\{\mathbf{a},\mathbf{b}\}}(\omega)P[d\omega],$$

which is the right hand side of equation (7.8).

About the relation of Theorem 7.1 to other hidden variable theories, cf. Nagasawa (1997), and Nagasawa-Schröder (1997).

As a special case of Theorem 7.1 we have

THEOREM 7.2. *For Experiments* 1, 2, 3 *and* 4 *take the set of operators* $\boldsymbol{F}$ *given in* (5.4), *and* ψ *in* (5.1). *Then there exists a local function representation*

$$\{\Omega, P; h_{\mathbf{a},\mathbf{1}}^{\{1\}}, h_{\mathbf{1},\mathbf{b}}^{\{1\}}, h_{\mathbf{a},\mathbf{1}}^{\{1\}}h_{\mathbf{1},\mathbf{b}}^{\{1\}}; h_{\mathbf{a},\mathbf{1}}^{\{2\}}, h_{\mathbf{1},\mathbf{c}}^{\{2\}}, h_{\mathbf{a},\mathbf{1}}^{\{2\}}h_{\mathbf{1},\mathbf{c}}^{\{2\}}; h_{\mathbf{b},\mathbf{1}}^{\{3\}}, h_{\mathbf{1},\mathbf{c}}^{\{3\}}, h_{\mathbf{b},\mathbf{1}}^{\{3\}}h_{\mathbf{1},\mathbf{c}}^{\{3\}}; h_{\mathbf{b},\mathbf{1}}^{\{4\}}, h_{\mathbf{1},\mathbf{b}}^{\{4\}}, h_{\mathbf{b},\mathbf{1}}^{\{4\}}h_{\mathbf{1},\mathbf{b}}^{\{4\}}\}$$

of $\{\boldsymbol{F}, \psi\}$. *Pairs* $\{h_{\mathbf{a},\mathbf{1}}^{\{1\}}, h_{1,\mathbf{b}}^{\{1\}}\}, \{h_{\mathbf{a},\mathbf{1}}^{\{2\}}, h_{\mathbf{1},\mathbf{c}}^{\{2\}}\}, \{h_{\mathbf{b},\mathbf{1}}^{\{3\}}, h_{\mathbf{1},\mathbf{c}}^{\{3\}}\}$ *and* $\{h_{\mathbf{b},\mathbf{1}}^{\{4\}}, h_{\mathbf{1},\mathbf{b}}^{\{4\}}\}$ *satisfy locality* (L) *of Definition* 7.1, *that is, they are local hidden variables.*

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