F. JIANG AND J. SHEN
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ASYMPTOTIC BEHAVIORS OF NONLINEAR NEUTRAL IMPULSIVE DELAY DIFFERENTIAL EQUATIONS WITH FORCED TERM*

Fangfang Jiang and Jianhua Shen†

Abstract

In this paper, we study the asymptotic behavior of solutions of a class of nonlinear neutral impulsive delay differential equations with forced term of the form

$$\begin{cases} [x(t)+c(t)x(t-\tau)]' + p(t)f(x(t-\delta)) = q(t), & t \ge t_0,\ t \ne t_k, \\ x(t_k) = b_k x(t_k^-) + (1-b_k)\int_{t_k-\delta}^{t_k} p(s+\delta)f(x(s))\,ds & \\ \qquad + (b_k-1)\int_{t_k}^{\infty} q(s)\,ds, & k \in \mathbf{Z}_+. \end{cases}$$

Sufficient conditions are obtained for every solution of the equations that tends to a constant as $t \to \infty$.

1. Introduction and preliminaries

The theory of impulsive differential equations is now being recognized to be not only richer than the corresponding theory of differential equations without impulses but also represents a more natural framework for mathematical models of many real-world phenomena [11]. The number of publications dedicated to its investigation has grown constantly in the recent years and a well-developed theory has taken in shape. See monographs [5, 11, 16] and references therein. However, the theory of impulsive functional differential equations has been less developed due to numerous theoretical and technical difficulties caused by their peculiarities. There are a few publications on qualitative theory. In particular, oscillation, stability theory and asymptotic behavior of solutions of some impulsive delay differential equations have been studied by several authors (see [4, 6, 7, 13, 15, 16, 18, 19]). Stability of some impulsive functional differential equations in more general form also has been studied by some authors (for example, see [1, 3, 8, 14, 17, 18]). However, by the best knowledge of author,

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†The corresponding author.

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there is little in the way of results for the asymptotic behavior of solutions of impulsive neutral delay differential equations [13, 16]. In particular, on the asymptotic behavior of solutions of nonlinear impulsive neutral differential equations with forced term.

On the other hand, it is well known that the asymptotic constancy is widely investigated for delay differential equations (with or without impulses). For example, see [2, 16] and reference therein.

In this paper, we consider the asymptotic behavior of solutions of a class of nonlinear neutral impulsive delay differential equation with forced term of the form

$$(1.1)\qquad \begin{cases} [x(t)+c(t)x(t-\tau)]' + p(t)f(x(t-\delta)) = q(t), & t \geq t_0,\ t \neq t_k, \\ x(t_k) = b_k x(t_k^-) + (1-b_k)\int_{t_k-\delta}^{t_k} p(s+\delta)f(x(s))\,ds & \\ \qquad\qquad + (b_k - 1)\int_{t_k}^{\infty} q(s)\,ds, & k \in \mathbf{Z}_+, \end{cases}$$

where $\tau > 0$, $\delta > 0$ $c(t), p(t) \in PC([t_0, \infty), \mathbf{R})$, $\mathbf{R}$ denotes the set of all real numbers, $p(t) \geq 0$, $q(t) \geq 0$ is a continuous function, $f : \mathbf{R} \to \mathbf{R}$ is also a continuous function. $\{t_k\}$, $k \in \mathbf{Z}_+$, denotes the impulsive sequence which satisfies $0 < t_k < t_{k+1} \uparrow \infty$ as $k \to \infty$, and b_k, $k \in \mathbf{Z}_+$, are constants, $\mathbf{Z}_+$ denotes the set of all positive integers, $PC([t_0, \infty), \mathbf{R})$ denotes the set of all functions $g : [t_0, \infty) \to \mathbf{R}$ such that g is continuous for $t_k \leq t < t_{k+1}$ and $\lim_{t \to t_k^-} g(t) = g(t_k^-)$ exists for all $k \geq 1$, $k \in \mathbf{Z}_+$.

In system (1.1), the impulsive term is also delayed, that is, it contains an integral term. When $q(t) = 0$, the corresponding results are obtained by Shen in [16], and more general form was consider in [12, 17]. Moreover, the existence and uniqueness of solutions and the stability were studied for the following more general impulsive differential equation

$$\begin{cases} x'(t) = f(t, x_t), & t \geq t_0,\ t \neq t_k, \\ \Delta x(t_k) = I_k(t, x_t), & t = t_k,\ k = 1, 2, 3 \ldots. \end{cases}$$

We note that in system (1.1), the impulse is a special form of the impulse term form, the method given in this paper, which comes from the idea of Shen [16], will mark this impulse term form. We should note that, the case of $q(t) = 0$ is just the paper [16], and note that when all $b_k = 1$, $k = 1, 2, \ldots,$ system (1.1) change into the following delay differential equation without impulses

$$[x(t) + c(t)x(t-\tau)]' + p(t)f(x(t-\delta)) = q(t), \quad t \geq t_0,$$

whose asymptotic behavior of solutions in some cases (for example, $c(t) \equiv 0$; $c(t) = c$; $f(x) = x$; and $c(t)$, $p(t)$ are continuous functions) have been investigated by several authors (see [9, 10]). Noting that, we can apply our theorems to systems without forced term and without impulses, and moreover, we improve the results in [9, 16].

With Eq. (1.1), one associates an initial condition of the form

$$(1.2)\qquad x_{t_0} = \varphi(s), \quad s \in [-\rho, 0], \quad \rho = \max\{\tau, \delta\},$$

where $x_{t_0} = x(t_0 + s)$ for $-\rho \le s \le 0$ and $\varphi \in PC([-\rho, 0], \mathbf{R}) \doteq \{\varphi : [-\rho, 0] \to \mathbf{R}$, φ is continuous everywhere except at the finite number of points t_k and $\varphi(t_k^-) = \lim_{t \to t_k^-} \varphi(t)$ and $\varphi(t_k^+) = \lim_{t \to t_k^+} \varphi(t)$ exists with $\varphi(t_k^+) = \varphi(t_k)\}$.

Definition 1.1. A function $x(t)$ is said to be a solution of Eq. (1.1) satisfying the initial condition (1.2), if

(1) $x(t) = \varphi(t - t_0)$ for $t_0 - \rho \le t \le t_0$, $x(t)$ is continuous for $t \ge t_0$, $t \ne t_k$, $k \in \mathbf{Z}_+$;

(2) $x(t) + c(t)x(t - \tau)$ is continuously differentiable for $t > t_0$, $t \ne t_k$, $t - \tau \ne t_k$, $t - \delta \ne t_k$, $k \in \mathbf{Z}_+$ and satisfies the first equation of system (1.1);

(3) $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^+) = x(t_k)$ hold and satisfy the second equation of system (1.1).

Definition 1.2. A solution of (1.1) is said to be non-oscillatory if it is eventually positive or eventually negative. Otherwise, it will be called oscillatory.

2. Main results

In connection with the nonlinear function $f(x)$, the impulsive perturbations b_k and the impulsive points t_k in (1.1), we assume that

(H_1) there is a constant $M > 0$ such that

$$|x| \le |f(x)| \le M|x|, \quad x \in \mathbf{R}, \qquad \text{and} \qquad xf(x) > 0, \quad x \ne 0; \tag{2.1}$$

(H_2) $t_k - \tau$ is not an impulsive point, and $0 < b_k \le 1$, $k \in \mathbf{Z}_+$, $\sum_{k=1}^{\infty}(1 - b_k) < \infty$;

Theorem 2.1. *Let (H_1) and (H_2) hold. Assume that $c(t_k) = b_k c(t_k^-)$, $\int_t^{\infty} q(s)\,ds \to 0$ as $t \to \infty$ and the following inequalities hold*

$$\lim_{t\to\infty} |c(t)| = \mu < 1, \tag{2.2}$$

$$\limsup_{t\to\infty} \left[\mu\left(1 + \frac{p(t + \tau + \delta)}{p(t + \delta)}\right) + \int_{t-\delta}^{t+\delta} p(s + \delta)\,ds \right] < \frac{2}{M}. \tag{2.3}$$

Then every solution of (1.1) tends to a constant as $t \to \infty$.

Proof. Let $x(t)$ be any solution of (1.1). We shall prove that the limit $\lim_{t\to\infty} x(t)$ exists and is finite. For this purpose, we rewrite (1.1) in the form

$$\begin{aligned} &\left[x(t) + c(t)x(t - \tau) - \int_{t-\delta}^{t} p(s + \delta) f(x(s))\,ds + \int_t^{\infty} q(s)\,ds \right]' \\ &\qquad + p(t + \delta) f(x(t)) = 0, \quad t \ge t_0,\ t \ne t_k, \end{aligned} \tag{2.4}$$

$$\begin{aligned} x(t_k) &= b_k x(t_k^-) + (1 - b_k) \int_{t_k-\delta}^{t_k} p(s + \delta) f(x(s))\,ds \\ &\quad + (b_k - 1) \int_{t_k}^{\infty} q(s)\,ds, \quad t = t_k,\ k \in \mathbf{Z}_+. \end{aligned} \tag{2.5}$$

From (2.2) and (2.3), we can select an $\varepsilon > 0$ sufficiently small, such that

$$\mu + \varepsilon < 1,$$

and

$$\limsup_{t\to\infty}\left[(\mu+\varepsilon)\left(1+\frac{p(t+\tau+\delta)}{p(t+\delta)}\right)+\int_{t-\delta}^{t+\delta} p(s+\delta)\,ds\right] < \frac{2}{M}. \tag{2.6}$$

On the other hand, we can also select a $t^* > t_0$ sufficiently large, such that

$$|c(t)| \le \mu + \varepsilon \quad \text{for } t \ge t^*. \tag{2.7}$$

From (2.1) and (2.7), we have that

$$\frac{f^2(x(t-\tau))}{x^2(t-\tau)} \ge 1 \ge \frac{|c(t)|}{\mu+\varepsilon}, \quad t \ge t^*.$$

Moreover,

$$|c(t)|x^2(t-\tau) \le (\mu+\varepsilon) f^2(x(t-\tau)), \quad t \ge t^*. \tag{2.8}$$

In the following part, for the sake of convenience, when we write a functional inequality without specifying its domain of validity, we mean that it holds for all sufficiently large t.

Let $V(t) = V_1(t) + V_2(t)$, where

$$V_1(t) = \left[x(t) + c(t)x(t-\tau) - \int_{t-\delta}^{t} p(s+\delta)f(x(s))\,ds + \int_t^\infty q(s)\,ds\right]^2,$$

$$\begin{aligned} V_2(t) = {} & \int_{t-\delta}^{t} p(s+2\delta)\int_s^t p(u+\delta)f^2(x(u))\,duds - 2\int_t^\infty q(s)\int_t^s p(u+\delta) \\ & \times f(x(u))\,duds + (\mu+\varepsilon)\int_{t-\tau}^{t} p(s+\delta+\tau)f^2(x(s))\,ds. \end{aligned}$$

For $t \ne t_k$, calculating, respectively, $\frac{\mathrm{dV_i}}{\mathrm{dt}}$ $(i = 1, 2)$ along the solution of (1.1) and using the inequality $a^2 + b^2 \ge 2ab$, we obtain

$$\begin{aligned} \frac{\mathrm{dV_1}}{\mathrm{dt}} &= 2\left[x(t) + c(t)x(t-\tau) - \int_{t-\delta}^{t} p(s+\delta)f(x(s))\,ds + \int_t^\infty q(s)\,ds\right] \\ &\quad \times (-p(t+\delta)f(x(t))) \\ &= -p(t+\delta)\left[2x(t)f(x(t)) + 2c(t)x(t-\tau)f(x(t)) - \int_{t-\delta}^{t} p(s+\delta)\right. \\ &\quad \left. \times (2f(x(s))f(x(t)))\,ds + 2\int_t^\infty q(s)f(x(t))\,ds\right] \end{aligned}$$

$$\leq -p(t+\delta)\Big[2x(t)f(x(t)) - |c(t)|x^2(t-\tau) - |c(t)|f^2(x(t))$$
$$- f^2(x(t))\int_{t-\delta}^{t} p(s+\delta)\,ds - \int_{t-\delta}^{t} p(s+\delta)f^2(x(s))\,ds\Big]$$
$$- 2p(t+\delta)f(x(t))\int_{t}^{\infty} q(s)\,ds,$$

$$\frac{\mathrm{dV}_2}{\mathrm{dt}} = -p(t+\delta)\int_{t-\delta}^{t} p(s+\delta)f^2(x(s))\,ds + p(t+\delta)f^2(x(t))$$
$$\times \int_{t-\delta}^{t} p(s+2\delta)\,ds + 2\int_{t}^{\infty} q(s)p(t+\delta)f(x(t))\,ds$$
$$+ (\mu+\varepsilon)p(t+\delta+\tau)f^2(x(t)) - (\mu+\varepsilon)p(t+\delta)f^2(x(t-\tau)).$$

Therefore, from the above two inequalities and (2.8), we obtain

$$\frac{\mathrm{dV}}{\mathrm{dt}} \leq -p(t+\delta)\Big[2x(t)f(x(t)) - |c(t)|f^2(x(t)) - f^2(x(t)) \tag{2.9}$$
$$\times \int_{t-\delta}^{t} p(s+\delta)\,ds - f^2(x(t))\int_{t-\delta}^{t} p(s+2\delta)\,ds\Big]$$
$$+ (\mu+\varepsilon)p(t+\delta+\tau)f^2(x(t))$$
$$\leq -p(t+\delta)f^2(x(t))\left[\frac{2x(t)}{f(x(t))} - |c(t)| - \int_{t-\delta}^{t+\delta} p(s+\delta)\,ds\right.$$
$$\left. - (\mu+\varepsilon)\frac{p(t+\delta+\tau)}{p(t+\delta)}\right]$$
$$\leq -p(t+\delta)f^2(x(t))\left[\frac{2}{M} - (\mu+\varepsilon)\left(1 + \frac{p(t+\delta+\tau)}{p(t+\delta)}\right)\right.$$
$$\left. - \int_{t-\delta}^{t+\delta} p(s+\delta)\,ds\right] \leq 0.$$

While for $t = t_k$, we have

$$V(t_k) = \left[x(t_k) + c(t_k)x(t_k-\tau) - \int_{t_k-\delta}^{t_k} p(s+\delta)f(x(s))\,ds + \int_{t_k}^{\infty} q(s)\,ds\right]^2 \tag{2.10}$$
$$+ \int_{t_k-\delta}^{t_k} p(s+2\delta)\int_{s}^{t_k} p(u+\delta)f^2(x(u))\,duds + (\mu+\varepsilon)$$
$$\times \int_{t_k-\tau}^{t_k} p(s+\delta+\tau)f^2(x(s))\,ds - 2\int_{t_k}^{\infty} q(s)\int_{t_k}^{s} p(u+\delta)f(x(u))\,duds$$

$$
\begin{aligned}
&= b_k^2\left[x(t_k^-) + c(t_k^-)x(t_k^- - \tau) - \int_{t_k-\delta}^{t_k} p(s+\delta)f(x(s))\,ds + \int_{t_k}^{\infty} q(s)\,ds\right]^2 \\
&\quad + \int_{t_k-\delta}^{t_k} p(s+2\delta)\int_s^{t_k} p(u+\delta)f^2(x(u))\,duds - 2\int_{t_k}^{\infty} q(s)\int_{t_k}^{s} p(u+\delta) \\
&\quad \times f(x(u))\,duds + (\mu+\varepsilon)\int_{t_k-\tau}^{t_k} p(s+\delta+\tau)f^2(x(s))\,ds \\
&\le V(t_k^-).
\end{aligned}
$$

From (2.6), (2.9) and (2.10), we can get

$$p(t+\delta)f^2((x(t)) \in L^1(t_0,\infty).$$

Hence for any $\rho > 0$, we have

$$\lim_{t\to\infty}\int_{t-\rho}^{t} p(s+\delta)f^2(x(s))\,ds = 0.$$

Since

$$
\begin{aligned}
&\int_{t-\delta}^{t} p(s+2\delta)\int_s^t p(u+\delta)f^2(x(u))\,duds \\
&\qquad \le \int_{t-\delta}^{t} p(s+2\delta)\,ds \int_{t-\delta}^{t} p(u+\delta)f^2(x(u))\,du \\
&\qquad \le \frac{2}{M}\int_{t-\delta}^{t} p(u+\delta)f^2(x(u))\,du \\
&\qquad \le 2\int_{t-\delta}^{t} p(u+\delta)f^2(x(u))\,du \to 0 \quad \text{as } t\to\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
(\mu+\varepsilon)\int_{t-\tau}^{t} p(s+\delta+\tau)f^2(x(s))\,ds &\le \frac{2}{M}\int_{t-\tau}^{t} p(s+\delta)f^2(x(s))\,ds \\
&\to 0 \quad \text{as } t\to\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_t^{\infty} q(s)\int_t^s p(u+\delta)f(x(u))\,duds &\le \int_t^{\infty} q(s)\int_t^{\infty} p(u+\delta)f(x(u))\,duds \\
&\to 0 \quad \text{as } t\to\infty,
\end{aligned}
$$

it follows that $\lim_{t\to\infty} V_2(t) = 0$. On the other hand, by (2.6), (2.9) and (2.10), we can find that $V(t)$ is eventually descreasing. In view of $V \ge 0$, we know

$\lim_{t\to\infty} V(t) = \beta$ exists and is finite. Thus, $\lim_{t\to\infty} V(t) = \lim_{t\to\infty} V_1(t) = \beta$, that is,

$$\lim_{t\to\infty}\left[x(t) + c(t)x(t-\tau) - \int_{t-\delta}^{t} p(s+\delta)f(x(s))\,ds + \int_{t}^{\infty} q(s)\,ds\right]^2 = \beta. \tag{2.11}$$

Next, we will show that the limit $\lim_{t\to\infty} x(t)$ exists and is finite. We let

$$y(t) = x(t) + c(t)x(t-\tau) - \int_{t-\delta}^{t} p(s+\delta)f(x(s))\,ds + \int_{t}^{\infty} q(s)\,ds,$$

then, in view of (2.11), we have

$$\lim_{t\to\infty} y^2(t) = \beta. \tag{2.12}$$

Furthermore, we have

$$\begin{aligned} y(t_k) &= x(t_k) + c(t_k)x(t_k-\tau) - \int_{t_k-\delta}^{t_k} p(s+\delta)f(x(s))\,ds + \int_{t_k}^{\infty} q(s)\,ds \\ &= b_k\left[x(t_k^-) + c(t_k^-)x(t_k^- - \tau) - \int_{t_k-\delta}^{t_k} p(s+\delta)f(x(s))\,ds + \int_{t_k}^{\infty} q(s)\,ds\right] \\ &= b_k y(t_k^-). \end{aligned}$$

Then system (2.4)–(2.5) can be rewritten as

$$\begin{cases} \dot{y}(t) + p(t+\delta)f(x(t)) = 0, & t \ge t_0,\ t \ne t_k, \\ y(t_k) = b_k y(t_k^-), & t = t_k,\ k \in \mathbf{Z}_+. \end{cases} \tag{2.13}$$

If $\beta = 0$, then $\lim_{t\to\infty} y(t) = 0$.

If $\beta > 0$, then there exists an enough large T_1 such that $y(t) \ne 0$ for any $t > T_1$. Therefore for $t_k > T_1$, $t \in [t_k, t_{k+1})$, we have $y(t) > 0$ or $y(t) < 0$, because $y(t)$ is continuous on $[t_k, t_{k+1})$, without loss of generally, we assume that $y(t) > 0$ on $[t_k, t_{k+1})$. It follows that $y(t_{k+1}) = b_k y(t_{k+1}^-) > 0$ thus $y(t) > 0$ on $[t_k, t_{k+1}]$. By simple induction to k, we conclude that $y(t) > 0$ on $[t_k, \infty)$. From (2.12), we have that

$$\begin{aligned} \lim_{t\to\infty} y(t) = \lim_{t\to\infty}\Bigg[x(t) + c(t)x(t-\tau) - \int_{t-\delta}^{t} p(s+\delta) \\ \times f(x(s))\,ds + \int_{t}^{\infty} q(s)\,ds\Bigg] = \lambda \end{aligned} \tag{2.14}$$

must exist and is finite. In view of (2.13), we have

$$\begin{aligned} \int_{t-\delta}^{t} p(s+\delta)f(x(s))\,ds &= y(t-\delta) - y(t) + \sum_{t-\delta<t_k<t} [y(t_k) - y(t_k^-)] \\ &= y(t-\delta) - y(t) - \sum_{t-\delta<t_k<t} (1-b_k)y(t_k^-). \end{aligned}$$

Letting $t \to \infty$, and noting that $\sum_{k=1}^{\infty}(1-b_k) < \infty$, then we obtain

$$\lim_{t\to\infty} \int_{t-\delta}^{t} p(s+\delta) f(x(s))\, ds = 0. \tag{2.15}$$

By condition $\lim_{t\to\infty} \int_t^{\infty} q(s)\, ds = 0$ and (2.14), we obtain

$$\lim_{t\to\infty} [x(t) + c(t)x(t-\tau)] = \lambda. \tag{2.16}$$

Next, we shall prove that

$$\lim_{t\to\infty} x(t) \text{ exists and is finite.} \tag{2.17}$$

To this end, we need to show that $|x(t)|$ is bounded. In fact, if $|x(t)|$ is unbounded, then there exists a sequence $\{s_n\}$ such that $s_n \to \infty$ and

$$|x(s_n^-)| \to \infty \quad \text{as } n \to \infty$$

and

$$|x(s_n^-)| = \sup_{t_0 \le t \le s_n} |x(t)|,$$

where, if s_n is not an impulsive point, then $x(s_n) = x(s_n^-)$. Thus, we have

$$\begin{aligned} |x(s_n^-) + c(s_n^-)x(s_n^- - \tau)| &\ge |x(s_n^-)| - |c(s_n^-)|\,|x(s_n^- - \tau)| \\ &\ge |x(s_n^-)|(1 - |c(s_n^-)|) \\ &\ge |x(s_n^-)|(1 - \mu - \varepsilon) \to \infty \quad \text{as } n \to \infty. \end{aligned}$$

This is a contradiction with (2.16). So $|x(t)|$ is bounded.

If $\mu = 0$, clearly, $\lim_{t\to\infty} x(t) = \lambda$, which shows that (2.17) holds.

If $0 < \mu < 1$, it is easily to see that $c(t)$ is eventually positive or negative. Otherwise, there is a sequence $\tau_1, \tau_2, \dots, \tau_k, \dots$, with $\tau_k \to \infty$ as $k \to \infty$ such that $c(\tau_k) = 0$, so $c(\tau_k) \to 0$, it is a contradiction with $\mu > 0$.

By condition (2.2), one can find a sufficiently large T_2 such that

$$|c(t)| < 1, \quad \text{for } t > T_2.$$

Set

$$\alpha = \liminf_{t\to\infty} x(t), \quad \beta = \limsup_{t\to\infty} x(t),$$

then we can choose two sequence $\{u_n\}$ and $\{v_n\}$, such that

$$u_n \to \infty, \quad v_n \to \infty \quad \text{as } n \to \infty,$$

and

$$\alpha = \lim_{t\to\infty} x(u_n), \quad \beta = \lim_{t\to\infty} x(v_n).$$

Now, we consider the following two possible cases

Case 1. $0 < c(t) < 1$ for $t > T_2$, we have

$$\lambda = \lim_{n\to\infty}[x(u_n) + c(u_n)x(u_n - \tau)] \le \alpha + \mu\beta$$

and

$$\lambda = \lim_{n\to\infty}[x(v_n) + c(v_n)x(v_n - \tau)] \ge \beta + \mu\alpha.$$

Thus, we get $\beta + \mu\alpha \le \lambda \le \alpha + \mu\beta$, that is, $\beta(1-\mu) \le \alpha(1-\mu)$.

Since $0 < \mu < 1$ and $\beta \ge \alpha$, it follows that $\beta = \alpha$. By (2.16), we obtain

$$\beta = \alpha = \lambda/(1+\mu),$$

which shows that (2.17) holds.

Case 2. $-1 < c(t) < 0$ for $t > T_2$, we have

$$\begin{aligned}\alpha = \lim_{n\to\infty} x(u_n) &= \lim_{n\to\infty}[x(u_n) + c(u_n)x(u_n - \tau) - c(u_n)x(u_n - \tau)]\\ &= \lim_{n\to\infty}[x(u_n) + c(u_n)x(u_n - \tau)] - \lim_{n\to\infty}[c(u_n)x(u_n - \tau)]\\ &= \lambda + \lim_{n\to\infty}[-c(u_n)x(u_n - \tau)] = \lambda + \mu\alpha,\end{aligned}$$

and

$$\begin{aligned}\beta = \lim_{n\to\infty} x(v_n) &= \lim_{n\to\infty}[x(v_n) + c(v_n)x(v_n - \tau) - c(v_n)x(v_n - \tau)]\\ &= \lim_{n\to\infty}[x(v_n) + c(v_n)x(v_n - \tau)] - \lim_{n\to\infty}[c(v_n)x(v_n - \tau)]\\ &= \lambda + \lim_{n\to\infty}[-c(v_n)x(v_n - \tau)] = \lambda + \mu\beta,\end{aligned}$$

therefore, we get $\alpha = \beta = \lambda/(1-\mu)$. This shows that (2.17) holds.

According to the discussion above, we conclude that (2.17) holds, and so the proof of Theorem 2.1 is complete. □

In the following theorem, we assume that the assumption $(H_2)^*$ holds which is different from the assumption (H_2), in the sense that the condition $\sum_{k=1}^{\infty}(1-b_k) < \infty$ is replaced by $t_k - t_{k-1} \ge \eta$ for all $k, k \in \mathbf{Z}_+$.

$(H_2)^*$ $t_k - \tau$ is not impulsive point for all $k \in \mathbf{Z}_+$ and $0 < b_k \le 1$ and there exists a $\eta > 0$ such that $t_k - t_{k-1} \ge \eta$ for all k.

Theorem 2.2. *Let* (H_1) *and* $(H_2)^*$ *hold. Assume that*

$$c(t_k) = b_k c(t_k^-), \quad \lim_{t\to\infty}\int_t^{\infty} q(s)\,ds = 0$$

and (2.2), (2.3) *hold. Then every solution of* (1.1) *tends to a constant as* $t \to \infty$.

Proof. From the proof of Theorem 2.1, we can also prove that (2.14) holds by using the conditions of Theorem 2.2. We also note that in the proof of

Theorem 2.1, the assumption $\sum_{k=1}^{\infty}(1-b_k)<\infty$ in (H_2) was used only in the proof of (2.15). Therefore, to complete the proof of Theorem 2.2, we only need to prove that (2.15) holds by using assumption $(H_2)^*$. In fact, since $t_k - t_{k-1} \geq \eta > 0$, it follows that the number of impulsive points in $(t-\delta, t)$ for $t \geq t_0 + \delta$ is at most $\left[\frac{\delta}{\eta}\right] = \mathrm{q}$. Without loss of generality, we set $t-\delta < t_i < t_{i+1} < \cdots < t_{i+q} < t$, $i = i(t)$. Then, in view of (2.14), we have

$$\lim_{t\to\infty} \sum_{t-\delta<t_k<t} \left[y(t_k) - y(t_k^-)\right] = \lim_{t\to\infty}[y(t_i) - y(t_i^-) + \cdots + y(t_{i+q}) - y(t_{i+q}^-)] = 0.$$

Due to

$$\int_{t-\delta}^{t} p(s+\delta)f(x(s))\,ds = y(t-\delta) - y(t) + \sum_{t-\delta<t_k<t}\left[y(t_k) - y(t_k^-)\right],$$

it follows that by passing to the limit both sides as $t \to \infty$, we conclude that (2.15) holds, the remaining proof of Theorem 2.2 follows from that of the Theorem 2.1, thus the proof of Theorem 2.2 is complete. □

By Theorem 2.1 and Theorem 2.2, we have the following asymptotic behavior results immediately.

THEOREM 2.3. *Either the conditions of Theorem* 2.1 *or the conditions of Theorem* 2.2 *imply that every oscillatory solution of* (1.1) *tends to zero as* $t \to \infty$.

COROLLARY 2.1. *Assume that*

$$\limsup_{t\to\infty} \int_{t-\delta}^{t+\delta} p(s+\delta)\,ds < 2, \quad \text{and} \quad \lim_{t\to\infty}\int_{t}^{\infty} q(s)\,ds = 0$$

hold, then every oscillatory solution of $x'(t) + p(t)x(t-\delta) = q(t)$ *tends to zero as* $t \to \infty$.

THEOREM 2.4. *The conditions in Theorem* 1.1 *together with*

$$\int_{t_0}^{\infty} p(t)\,dt = \infty, \tag{2.18}$$

imply that every solution of (1.1) *tends to zero as* $t \to \infty$.

Proof. By Theorem 2.3, we only have to prove that every non-oscillatory solution of (1.1) tends to zero as $t \to \infty$. Without loss of generality, let $x(t)$ be an eventually positive solution of (1.1), we shall prove $\lim_{t\to\infty} x(t) = 0$. As in the proof of Theorem 2.1, we can rewrite (1.1) in the form (2.13). Integrating from t_0 to t both sides of (2.13) yields

$$\int_{t_0}^{t} p(s+\delta)f(x(s))\,ds = y(t_0) - y(t) - \sum_{t_0<t_k<t}(1-b_k)y(t_k^-).$$

By using (2.14) and $\sum_{k=1}^{\infty}(1-b_k) < \infty$, we have

$$\int_{t_0}^{\infty} p(s+\delta)f(x(s))\,ds < \infty.$$

Which, together with (2.18) yields $\liminf_{t\to\infty} f(x(t))\,ds = 0$. We claim that

$$\liminf_{t\to\infty} x(t) = 0. \tag{2.19}$$

To this end, we let $\{S_m\}$ be a sequence, such that $S_m \to \infty$ as $m \to \infty$ and $\lim_{m\to\infty} f(x(S_m)) = 0$. We must have $\liminf_{m\to\infty} x(S_m) = c = 0$. In fact, if $c > 0$, then there exists a subsequence $\{S_{m_k}\}$ of $\{S_m\}$ such that $x(S_{m_k}) \geq \dfrac{c}{2}$ for k sufficiently large. By (H_1), we have that $f(x(S_{m_k})) \geq \xi$ for some $\xi > 0$ and sufficiently large k, which yields a contradiction because of $\lim_{k\to\infty} f(x(S_{m_k})) = 0$. Therefore, (2.19) holds.

On the other hand, by Theorem 2.1, we have that $\lim_{t\to\infty} x(t)$ exists. Therefore, $\lim_{t\to\infty} x(t) = 0$. The proof of Theorem 2.4 is then complete. □

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Fangfang Jiang
Department of Mathematics
Hangzhou Normal University
Hangzhou Zhejiang 310036
China
E-mail: jiangfangfang87@126.com

Jianhua Shen
Department of Mathematics
Hangzhou Normal University
Hangzhou Zhejiang 310036
China
E-mail: jhshen2ca@yahoo.com