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AN INEQUALITY OF FRANK, STEINMETZ AND WEISSENBORN

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Abstract

An inequality proved by Frank, Steinmetz and Weissenborn relates the frequency of poles of a function meromorphic in the plane to the frequency of zeros of a linear differential polynomial in that function with small coefficients. A version of this inequality is established in terms of the frequency of distinct zeros of the linear differential polynomial.

1. Introduction

The starting point is the following theorem due to Frank, Steinmetz and Weissenborn [1, 2, 10].

THEOREM 1.1 ([1, 2, 10]). *Let the function f be transcendental and meromorphic in the plane and let*

$$F = L(f), \quad L = D^p + a_{p-1}D^{p-1} + \cdots + a_0, \quad D = \frac{d}{dz}, \tag{1}$$

where $p \geq 2$ and the a_j are meromorphic in the plane with $T(r, a_j) = S(r, f)$. Let $\varepsilon > 0$. Then either (i) *f is a rational function in* (*local*) *solutions of the equation $L(w) = 0$, or* (ii)

$$N(r, F) \leq N(r, 1/F) + (2 + \varepsilon)N(r, f) + S(r, f). \tag{2}$$

Here $S(r, f)$ denotes as usual any quantity which is $o(T(r, f))$ as r tends to infinity outside a set of finite measure [4]. Theorem 1.1 was first proved by Frank and Weissenborn [2] (see also [3]) for the case where all the a_j are identically zero, in which case conclusion (i) is impossible. The result was then established by Steinmetz [10] when the a_j are rational functions, and the general case was completed by Frank [1]. The methods of [1, 2, 10] are related to Steinmetz' proof of the second fundamental theorem for small functions [9]. It is reasonable to ask whether some version of Theorem 1.1 holds with $N(r, 1/F)$ replaced by $\overline{N}(r, 1/F)$, and the aim of this note is to show that such a result does

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indeed follow from the approach of [1, 10]. However, the constants which arise are not so easy to control, and it is necessary to keep track of the orders of the differential operators which appear in the proof. The following theorem will be proved.

THEOREM 1.2. *Let* $1 < A \leq 2$ *and* $2 \leq p \in \mathbf{N}$ *and let* f *be a transcendental meromorphic function in the plane. Let* L *and* F *be given by* (1), *where* $a_0, \ldots, a_{p-1}$ *are functions meromorphic in the plane, and write*

$$T^*(r) = \log r + \sum_{j=0}^{p-1} T(r, a_j). \tag{3}$$

Then at least one of the following two conclusions holds:

(i) *the function* f *is a rational function in* (*local*) *solutions of the equation* $L(w) = 0$;

(ii) *the functions* f *and* F *satisfy*

$$N(r, F) \leq C\overline{N}(r, 1/F) + (1 + A)N(r, f) + S^*(r, F). \tag{4}$$

Here $S^*(r, F)$ *denotes any quantity which is* $O(T^*(r) + \log^+ T(r, F))$ *as* r *tends to infinity outside a set of finite measure*, *and* C *is a positive constant which may be chosen so that*

$$C \leq (1 + A) \exp\left(\frac{4(p-1)}{\log A}\right). \tag{5}$$

COROLLARY 1.1. *Let* $p \geq 2$ *and let the function* f *be transcendental and meromorphic in the plane. Assume that* $\overline{N}(r, 1/f^{(p)}) = S(r, f)$ *and that all but finitely many poles of* f *have multiplicity at most* $p - 1$. *Then* $N(r, f) = S(r, f)$.

Corollary 1.1 follows by taking $F = f^{(p)}$ and A close to 1 in (4) and observing that in this case $N(r, F) - N(r, f) = p\overline{N}(r, f)$ and $AN(r, f) \leq A(p-1)\overline{N}(r, f) + O(\log r)$.

2. Preliminaries

As in [1, 10] a key role is played by a result of Spigler [8] and a Wronskian identity [7].

THEOREM 2.1 ([8]). *Let* $a_0, \ldots, a_{m-1}, b_0, \ldots, b_{n-1}$ *be functions analytic on a simply connected plane domain* U, *and let* $u_1, \ldots, u_m$ *and* $v_1, \ldots, v_n$ *be fundamental solution sets in* U *of the equations*

$$u^{(m)}(z) + \sum_{j=0}^{m-1} a_j(z)u^{(j)}(z) = 0 \quad \textit{and} \quad v^{(n)}(z) + \sum_{j=0}^{n-1} b_j(z)v^{(j)}(z) = 0$$

respectively. Let H be the vector space over $\mathbf{C}$ generated by all the products $u_s v_t$ ($1 \le s \le m$, $1 \le t \le n$), and let $q \le mn$ be the dimension of H. Then there exist meromorphic functions $c_0, \dots, c_{q-1}$ on U, each of which is a rational function over $\mathbf{C}$ in the a_j, b_j and their derivatives, such that H is the solution space of

$$w^{(q)}(z) + \sum_{j=0}^{q-1} c_j(z) w^{(j)}(z) = 0.$$

Lemma 2.1 ([7]). *Let $f_1, \dots, f_k$ and $g_1, \dots, g_n$ be functions meromorphic on a plane domain U. Then the following identity holds on U:*

$$W(f_1, \dots, f_k, g_1, \dots, g_n) W(f_1, \dots, f_k)^{n-1} = W(h_1, \dots, h_n), \quad h_j = W(f_1, \dots, f_k, g_j).$$

3. Proof of Theorem 1.2

The proof mainly follows [1, 10], but requires additional detail in places. Let A, p, L and the a_j be as in the statement of the theorem, and let Λ denote the field consisting of those meromorphic functions which are rational functions, with coefficients in $\mathbf{C}$, in the a_j and their derivatives. Then $T(r, b) = O(T^*(r))$ as r tends to infinity outside a set of finite measure, for every $b \in \Lambda$, by (3). Write $\Lambda[D]$ for the set of homogeneous linear differential operators with coefficients in Λ. Let $U \subseteq \mathbf{C}$ be a simply connected domain on which all the a_j are analytic, and let $w_1, \dots, w_p$ be linearly independent solutions of $L(w) = 0$ on U. For $s \in \mathbf{N}$ let M_s denote the vector space over $\mathbf{C}$ generated by all elements of the form

$$w_1^{\alpha_1} \cdots w_p^{\alpha_p}, \quad 0 \le \alpha_j \in \mathbf{Z}, \quad \alpha_1 + \cdots + \alpha_p = s. \tag{6}$$

Choose $s \in \mathbf{N}$, let $n = \dim(M_s)$ and $k = \dim(M_{s+1})$ and let $u_1, \dots, u_n$ be a basis for M_s, and $U_1, \dots, U_k$ be a basis for M_{s+1}. It is clear that $p \le n \le k$. For the time being let f be any function meromorphic on U, and set

$$\Omega = \Omega(f) = \frac{W(U_1, \dots, U_k, u_1 f, \dots, u_n f)}{W(U_1, \dots, U_k) W(u_1, \dots, u_n)}. \tag{7}$$

Then Lemma 2.1 and standard properties of Wronskians [6] give

$$\Omega = \frac{W(W(U_1, \dots, U_k, u_1 f), \dots, W(U_1, \dots, U_k, u_n f))}{W(U_1, \dots, U_k)^n W(u_1, \dots, u_n)}$$

and so

$$\Omega = \frac{W(K(u_1 f), \dots, K(u_n f))}{W(u_1, \dots, u_n)}, \quad \text{where } K(w) = \frac{W(U_1, \dots, U_k, w)}{W(U_1, \dots, U_k)}. \tag{8}$$

Now $U_1, \dots, U_k$ are linearly independent solutions of $K(w) = 0$, but by Theorem 2.1 they also solve a kth order equation with coefficients in Λ, from which it follows that $K \in \Lambda[D]$ in (8).

Lemma 3.1. *There exist operators $N_{q,\mu} \in \Lambda[D]$, for $0 \le q \le n-1$, $0 \le \mu \le n-1$, each of order at most $k+n$, such that*

$$\Omega = \Omega(f) = \det(N_{q,\mu}(F)), \quad F = L(f). \tag{9}$$

Proof. It is shown in [1, 10] that $\Omega(f)$ is a homogeneous differential polynomial in F, of degree n, with coefficients in Λ, but (4) requires a bound for the orders of the $N_{q,\mu}$, and so the argument will be sketched. The u_j solve a homogeneous nth order linear differential equation $Q(w) = 0$ over Λ, by Theorem 2.1. Hence Leibniz' rule and the division algorithm [5, p. 126] give operators L_μ, M_μ, P_μ in $\Lambda[D]$, of orders at most k, $k-p$ and $p-1$ respectively, such that

$$K(u_j f) = \sum_{\mu=0}^{n-1} u_j^{(\mu)} L_\mu(f) = \sum_{\mu=0}^{n-1} u_j^{(\mu)} (M_\mu(F) + P_\mu(f)), \quad F = L(f), \tag{10}$$

for $j = 1, \dots n$. But $u_j w_v \in M_{s+1}$ and $L(w_v) = 0$ and so $0 = \sum_{\mu=0}^{n-1} u_j^{(\mu)} P_\mu(w_v)$ for $j = 1, \dots n$ and $v = 1, \dots, p$. Since the u_j are linearly independent it follows that $P_\mu(w_v) = 0$ for $\mu = 0, \dots, n-1$ and $v = 1, \dots, p$. But the P_μ have order at most $p-1$ and the w_v are linearly independent, and so each P_μ is the zero operator. Now differentiating (10) and using the equation for the u_j gives

$$(K(u_j f))^{(q)} = u_j N_{q,0}(F) + \dots + u_j^{(n-1)} N_{q,n-1}(F)) \quad \text{for } 0 \le q \le n-1,\ 1 \le j \le n,$$

with $N_{q,\mu} \in \Lambda[D]$, of order at most $k+q \le k+n$. But then $(K(u_j f))^{(q)}$ is the dot product of the vector $(N_{q,0}(F), \dots, N_{q,n-1}(F))$ with the jth column of the Wronskian matrix of $u_1, \dots, u_n$. Thus (8) leads to (9). □

Henceforth let f be transcendental and meromorphic in the plane, and let $F = L(f)$.

Lemma 3.2. *The functions f, Ω and F satisfy*

$$N(r, \Omega(f)) \le (n+k) N(r, f) + S^*(r, F). \tag{11}$$

Proof. The u_j solve a homogeneous linear differential equation $Q(w) = 0$ in U, while the U_j solve $K(w) = 0$, where Q and K are elements of $\Lambda[D]$. As in [1], writing (7) in the form

$$\Omega(f) = \frac{f^{n+k} W(U_1/f, \dots, U_k/f, u_1, \dots, u_n)}{W(U_1, \dots, U_k) W(u_1, \dots, u_n)}$$

takes care of poles of Ω arising from poles of f at which all coefficients of L, Q, K and the $N_{q,\mu}$ are analytic. On the other hand if at least one coefficient from L, Q, K or some $N_{q,\mu}$ has a pole at z_1, let σ be the largest multiplicity among these poles at z_1. Then $N_{q,\mu}(F)$ has at most a pole of multiplicity $\gamma + \tau + 2\sigma$ at z_1, where $\gamma \ge 0$ is the multiplicity of the pole of f at z_1 and τ depends only on p, k and n. □

LEMMA 3.3. *If* $\Omega(f) \not\equiv 0$ *then* F *and* $\Omega(f)$ *satisfy*

$$N(r, 1/F) - \frac{1}{n} N(r, 1/\Omega(f)) \le (k+n)\overline{N}(r, 1/F) + S^*(r, F). \tag{12}$$

Proof. Write (9) in the form

$$\Omega(f) = F^n G, \quad \frac{1}{F^n} = \frac{G}{\Omega(f)}, \quad \text{where } G = \det\left(\frac{N_{q,\mu}(F)}{F}\right).$$

The operators $N_{q,\mu}$ all have order at most $k+n$, by Lemma 3.1, and $F^{(j)}/F$ has a pole of multiplicity at most j at a zero of F. This proves (12) and Lemma 3.3. □

If $\Omega(f) \equiv 0$ then (7) implies that $U_1, \dots, U_k$, $u_1 f, \dots u_n f$ are linearly dependent on U and so f is a rational function of the U_j and u_j and hence of the w_j, which gives conclusion (i) of the theorem. Assume henceforth that $\Omega(f)$ does not vanish identically. Then (9) gives

$$nm(r, 1/F) = m(r, 1/F^n) \le m(r, 1/\Omega) + m(r, \Omega/F^n) \le m(r, 1/\Omega) + S^*(r, F). \tag{13}$$

But (9), (13) and the first fundamental theorem now lead to

$$\begin{aligned} nm(r, 1/F) + N(r, 1/\Omega(f)) &\le m(r, \Omega(f)) + N(r, \Omega(f)) + S^*(r, F) \\ &\le nm(r, F) + N(r, \Omega(f)) + S^*(r, F) \\ &\le nm(r, F) + (n+k)N(r, f) + S^*(r, F), \end{aligned}$$

using Lemma 3.2. Dividing through by n and adding $N(r, F) + N(r, 1/F)$ to both sides gives

$$N(r, F) + \frac{1}{n} N(r, 1/\Omega(f)) \le N(r, 1/F) + \left(1 + \frac{k}{n}\right) N(r, f) + S^*(r, F). \tag{14}$$

This inequality, but without the term involving zeros of Ω, is used in [1, 10] to prove Theorem 1.1, based on the fact that $\inf\{k/n\} = 1$ [9]. The presence of $N(r, 1/\Omega(f))$ in (14) yields, using Lemma 3.3,

$$N(r, F) \le (k+n)\overline{N}(r, 1/F) + \left(1 + \frac{k}{n}\right) N(r, f) + S^*(r, F). \tag{15}$$

It remains only to determine s in order to ensure that $k/n \le A$, while keeping a reasonable bound on $k+n$. As in [9] the dimension $l(s) = \dim(M_s)$ is at most the number of distinct products (6). This yields, since $\log(1 + s/x)$ is decreasing and $\log(1+x) \le \sqrt{x}$ for $x > 0$,

$$\begin{aligned} \log l(s) &\le \log\left(\frac{(s+p-1)!}{s!(p-1)!}\right) = \sum_{j=1}^{p-1} \log\left(1 + \frac{s}{j}\right) \\ &\le \int_0^{p-1} \log\left(1 + \frac{s}{x}\right) dx \le \int_0^{p-1} \sqrt{\frac{s}{x}}\, dx = 2\sqrt{s(p-1)}. \end{aligned} \tag{16}$$

With $[x]$ the greatest integer not exceeding $x \in \mathbf{R}$, set

$$t = \left[\frac{4(p-1)}{(\log A)^2}\right] + 1 \geq 2, \tag{17}$$

and assume that

$$\frac{l(s+1)}{l(s)} > A \quad \text{for } s = 1, \dots, t-1. \tag{18}$$

Since $l(1) = p \geq 2$ and $A \leq 2$ this gives $l(t) \geq pA^{t-1} \geq A^t$ and, using (16),

$$t \log A \leq \log l(t) \leq 2\sqrt{t(p-1)}, \quad t \leq \frac{4(p-1)}{(\log A)^2},$$

which contradicts (17). Hence the assumption (18) must be false, and so by (16) and (17) there exists s with $1 \leq s \leq t-1$ such that

$$\frac{k}{n} = \frac{k_s}{n_s} = \frac{l(s+1)}{l(s)} \leq A \tag{19}$$

and

$$k + n \leq (1+A)l(s), \quad \log l(s) \leq 2\sqrt{s(p-1)} \leq 2\sqrt{(t-1)(p-1)} \leq \frac{4(p-1)}{\log A},$$

which on combination with (15) gives (4) and (5). This completes the proof of Theorem 1.2. □

Remark. When $\varepsilon = A - 1$ is small and positive a better bound for C is obtained as follows. For fixed $p \geq 2$ define μ and t by

$$\varepsilon \sim \log A = \int_\mu^\infty \frac{\log(1+u)}{u^2}\, du \sim \frac{\log \mu}{\mu}, \quad t = [\mu(p-1)] + 1,$$

as $\varepsilon \to 0+$. Then $t/(p-1) > \mu$ and (16) gives

$$\log l(t) \leq \int_0^{p-1} \log\left(1 + \frac{t}{x}\right) dx = t \int_{t/(p-1)}^\infty \frac{\log(1+u)}{u^2}\, du$$
$$< t \log A \sim \mu(p-1) \log A \sim \mu(p-1)\varepsilon \sim (p-1)\log(1/\varepsilon).$$

Hence the argument following (18) shows that there exists s with $1 \leq s \leq t-1$ such that (19) holds, as well as

$$C \leq k + n \leq 2l(s+1) \leq 2l(t) \leq 2\varepsilon^{-p}.$$

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