Z.-X. CHEN
KODAI MATH. J.
34 (2011), 244–256

ON PROPERTIES OF MEROMORPHIC SOLUTIONS FOR SOME DIFFERENCE EQUATIONS*

Zong-Xuan Chen

Abstract

In this paper, we investigate properties of finite order transcendental meromorphic solutions and rational solutions of difference Painlevé I and II equations.

1. Introduction and results

Painlevé and his colleagues [19] classified all equations of the Painlevé type of the form

$$\frac{d^2y}{dz^2} = F\left(z; y, \frac{dy}{dz}\right)$$

where F is rational in y and dy/dz and (locally) analytic in z. The first two of these are P_I and P_{II}:

$$\frac{d^2y}{dz^2} = 6y^2 + z \quad \text{and} \quad \frac{d^2y}{dz^2} = 2y^2 + zy + \alpha$$

where α is a constant. The differential Painlevé equations, discovered since the beginning of last century, have been an important research subject in the field of the mathematics and physics.

In the past 15 years, the discrete Painlevé equations became important research problems (see [1, 8]). For example, discrete Painlevé I equations

$$y_{n+1} + y_{n-1} = \frac{\alpha n + \beta}{y_n} + \gamma,$$

$$y_{n+1} + y_{n-1} = \frac{\alpha n + \beta}{y_n} + \frac{\gamma}{y_n^2},$$

2000 *Mathematics Subject Classification*. 30D35, 39A10.

Key words and phrases. Difference Painlevé equation, transcendental solution, rational solution.

*This project was supported by the National Natural Science Foundation of China (No. 10871076).

Received December 8, 2009; revised November 30, 2010.

and discrete Painlevé II equation

$$y_{n+1} + y_{n-1} = \frac{(\alpha n + \beta) y_n + \gamma}{1 - y_n^2},$$

where α, β and γ are constants, $n \in \mathbf{N}$.

Recently, a number of papers (including [1–7, 9–12, 14–17]) focused on complex difference equations and difference analogues of Nevanlinna's theory. As the difference analogues of Nevanlinna's theory are being investigated, many results on the complex difference equations are rapidly obtained.

Some results on existence of meromorphic solutions for certain difference equations were obtained by Shimomura [20] and Yanagihara [22].

Ablowitz et al [1] looked at difference equations of the type

$$w(z+1) + w(z-1) = R(z, w) \tag{1.1}$$

where R is rational in both of its arguments, and showed the following theorem.

THEOREM A ([1]). *If the second-order difference equation*

$$y(z+1) + y(z-1) = \frac{a_0(z) + a_1(z)y + \cdots + a_p(z)y^p}{b_0(z) + b_1(z)y + \cdots + b_q(z)y^q}$$

where a_i and b_i are polynomials, admits a non-rational meromorphic solution of finite order, then $\max(p, q) \le 2$.

Halburd and Korhonen [10–12] used value distribution theory and a reasoning related to the singularity confinement to single out the difference Painlevé I and II equations from difference equation (1.1).

They obtained that if (1.1) has an admissible meromorphic solutions of finite order, then either w satisfies a difference Riccati equation, or (1.1) can be transformed by a linear change in w to some classical difference equations, which include difference Painlevé I equations

$$w(z+1) + w(z-1) = \frac{az + b}{w(z)} + c, \tag{1.2}$$

$$w(z+1) + w(z-1) = \frac{az + b}{w(z)} + \frac{c}{w(z)^2} \tag{1.3}$$

and difference Painlevé II equation

$$w(z+1) + w(z-1) = \frac{(az + b)w + c}{1 - w^2}. \tag{1.4}$$

From the above, we see that difference Painlevé I and II equations are the development of the differential and discrete Painlevé I and II equations. So they are an important class of difference equations.

Chen and Shon [5] investigated some properties of meromorphic solutions of difference Painlevé *I* and *II* equations, and proved the following Theorems B and C.

In this paper, we assume the reader is familiar with basic notions of Nevanlinna's value distribution theory (see e.g. [13, 18, 21]). In addition, we use the notation $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, and $\lambda(f)$ and $\lambda\left(\frac{1}{f}\right)$ to denote, respectively, the exponents of convergence of zeros and poles of $f(z)$.

Theorem B ([5]). *Let a, b, c be constants with $a \neq 0$. If $f(z)$ is a finite order transcendental meromorphic solution of the difference Painlevé I equation* (1.2), *then*

(i) *f has at most one non-zero finite Borel exceptional value*;

(ii) $\lambda\left(\frac{1}{f}\right) = \lambda(f) = \sigma(f)$;

(iii) *$f(z)$ has infinitely many fixed points and satisfies $\tau(f) = \sigma(f)$.*

Theorem C ([5]). *Let a, b, c be constants with $ac \neq 0$. If $f(z)$ is a finite order transcendental meromorphic solution of the difference Painlevé II equation* (1.4), *then*

(i) *f has at most one non-zero finite Borel exceptional value*;

(ii) $\lambda\left(\frac{1}{f}\right) = \lambda(f) = \sigma(f)$;

(iii) *$f(z)$ has infinitely many fixed points and satisfies $\tau(f) = \sigma(f)$.*

In [5], they also consider properties of rational solutions of (1.2) and (1.4).

In this paper, we consider the other form of the difference Painlevé *I* equation, i.e. equation (1.3), and prove the following Theorems 1 and 3. For the difference Painlevé *II* equation (1.4), in [5], we consider the case $c \neq 0$, in this paper, we consider the case $c = 0$, i.e. equation

$$w(z+1) + w(z-1) = \frac{(az+b)w}{1-w^2}, \tag{1.5}$$

and prove the following Theorems 2 and 4.

Theorem 1. *Let a, b, c be constants such that $ac \neq 0$. Suppose that $w(z)$ is a finite order transcendental meromorphic solution of the difference Painlevé I equation* (1.3). *Then*

(i) *$w(z)$ has no any Borel exceptional value*;

(ii) *if $p(z)$ is a non-constant polynomial, then $w(z) - p(z)$ has infinitely many zeros and $\lambda(w - p) = \sigma(w)$.*

Theorem 2. *Let a, b be constants such that $|a| + |b| \neq 0$. Suppose that $w(z)$ is a finite order transcendental meromorphic solution of the difference Painlevé II equation* (1.5). *Then*

(i) *$w(z)$ has infinitely many poles, and satisfies $\lambda\left(\frac{1}{w}\right) = \sigma(w)$;*

(ii) *if $p(z)$ is a non-constant polynomial, then $w(z) - p(z)$ has infinitely many zeros and $\lambda(w - p) = \sigma(w)$;*

(iii) *if $a \neq 0$, then $w(z)$ assumes every non-zero value $d \in \mathbf{C}$ infinitely often, and satisfies $\lambda(w - d) = \sigma(w)$;*
if $a = 0$, then the Borel exceptional value of $w(z)$ can only come from a set $E = \left\{0, \left(1 - \frac{b}{2}\right)^{1/2}\right\}$.

Theorem 3. *Let a, b, c be constants such that $c \neq 0$ and $|a| + |b| \neq 0$. Then*

(i) *if $a = 0$, then equation* (1.3) *has nonzero constant solution $w(z) = B$ where B satisfies*

$$2B^3 - bB - c = 0,$$

the other rational solutions $w(z)$ satisfy $w(z) = B + \frac{S(z)}{H(z)}$ where $S(z)$ and $H(z)$ are relatively prime polynomials satisfying $\deg S(z) < \deg H(z)$;

(ii) *if $a \neq 0$ and $w(z) = \frac{S(z)}{H(z)}$ is a rational solution of* (1.3), *where*

$$S(z) = sz^k + s_{k-1}z^{k-1} + \cdots + s_0, \quad H(z) = hz^t + h_{t-1}z^{t-1} + \cdots + h_0$$

where $s, s_{k-1}, \ldots, s_0$ and $h, h_{t-1}, \ldots, h_0$ are constants such that $sh \neq 0$, then

$$t = k + 1 \quad \text{and} \quad s = -\frac{ch}{a}.$$

Theorem 4. *Consider rational solutions of difference Painlevé II equation* (1.5) *with a, b are constants.*

(i) *If $a \neq 0$, then* (1.5) *has no non-zero rational solution.*

(ii) *If $a = 0$ and $b = 2$, then a non-zero rational solution of* (1.5) *is of the form $w(z) = \frac{S(z)}{H(z)}$, where $S(z)$ and $H(z)$ are relatively prime polynomials satisfying* $\deg S(z) < \deg H(z)$.

(iii) *If $a = 0$ and $b \neq 2$ (and $b \neq 0$), then a non-zero rational solution of* (1.5) *is of the form*

$$w(z) = \left(1 - \frac{b}{2}\right)^{1/2} + \frac{S_1(z)}{H_1(z)},$$

where $S_1(z)$ and $H_1(z)$ are relatively prime polynomials satisfying $\deg S_1(z) < \deg H_1(z)$. $\left(1 - \frac{b}{2}\right)^{1/2}$ *are nonzero constant solutions of* (1.5).

Example 1. The constant solution $w(z) = B = \frac{1}{2}$ satisfies the difference Painlevé I equation

$$w(z+1) + w(z-1) = \frac{1}{w(z)} + \frac{-\frac{1}{4}}{w(z)^2}$$

and

$$2B^3 - bB - c = 0.$$

Example 2. The meromorphic function $w(z) = \tan\left(\frac{\pi}{4}z\right)$ satisfies the difference Painlevé II equation

$$w(z+1) + w(z-1) = \frac{4w}{1-w^2},$$

where $\sigma(w) = \lambda(w-p) = \lambda\left(\frac{1}{w}\right) = 1$, and where $p(z)$ is non-constant polynomial.

Example 3. The rational function $f(z) = \dfrac{1}{z+1}$ and the transcendental meromorphic function $f_1(z) = \dfrac{1}{e^{2\pi iz} + z + 1}$ satisfy the difference Painlevé II equation

$$f(z+1) + f(z-1) = \frac{2f}{1-f^2}.$$

The solution f_1 has infinitely many poles, but has no zero, and satisfies $\lambda\left(\frac{1}{f_1}\right) = \sigma(f_1) = 1$ and $\lambda(f_1) = 0$. This shows the Borel exceptional value of f_1 is in E.

2. Proof of Theorem 1

We need the following lemmas for the proof of Theorem 1.

Lemma 2.1 ([9, 17]). *Let f be a transcendental meromorphic solution of finite order σ of a difference equation of the form*

$$H(z,f)P(z,f) = Q(z,f)$$

where $H(z,f)$ is a difference product of total degree n in $f(z)$ and its shifts, and where $P(z,f)$, $Q(z,f)$ are difference polynomials so that the total degree of $Q(z,f)$ is $\le n$. Then, for each $\varepsilon > 0$,

$$m(r, P(z,f)) = O(r^{\sigma-1+\varepsilon}) + S(r,f),$$

possibly outside of an exceptional set of finite logarithmic measure.

In the remark of [10, P.15], it is pointed out that the following Lemma 2.2 holds. Or using the same reasoning as in the proof of Lemma 2.1 in [11], we can prove the following Lemma 2.2.

LEMMA 2.2. *Let f be a non-constant finite order meromorphic function. Then*

$$N(r+1, f) = N(r, f) + S(r, f)$$

outside of a possible exceptional set of finite logarithmic measure.

Remark 2.1. In [7], Chiang and Feng prove that let f be a meromorphic function with exponent of convergence of poles $\lambda(1/f) = \lambda < \infty$, $\eta \neq 0$ be fixed, then for each $\varepsilon > 0$,

$$N(r, f(z+\eta)) = N(r, f) + O(r^{\lambda-1+\varepsilon}) + O(\log r).$$

LEMMA 2.3 ([9, 17]). *Let $w(z)$ be a non-constant finite order meromorphic solution of*

$$P(z, w) = 0$$

where $P(z, w)$ is a difference polynomial in $w(z)$. If $P(z, a) \not\equiv 0$ for a meromorphic function a satisfying $\lim_{r\to\infty} \dfrac{T(r, a)}{T(r, w)} = 0$, *then*

$$m\left(r, \frac{1}{w-a}\right) = S(r, w)$$

outside of a possible exceptional set of finite logarithmic measure.

Proof of Theorem 1. Assume that $w(z)$ is a finite order transcendental meromorphic solution of (1.3).

(i) First, we prove $\lambda\left(\dfrac{1}{w}\right) = \sigma(w)$. By (1.3), we obtain that

$$w(z)\{[w(z+1) + w(z-1)]w(z)\} = (az+b)w(z) + c. \tag{2.1}$$

Applying Lemma 2.1 to (2.1), we see that for any given $\varepsilon > 0$, there is a subset $E \subset (1, \infty)$ having finite logarithmic measure such that for $|z| = r \notin [0, 1] \cup E$,

$$m(r, [w(z+1) + w(z-1)]w(z)) = O(r^{\sigma-1+\varepsilon}) + S(r, w), \tag{2.2}$$

where $\sigma = \sigma(w)$. On the other hand, by (1.3), we have

$$\begin{aligned}[w(z+1) + w(z-1)]w(z) &= w(z)\left[\frac{az+b}{w(z)} + \frac{c}{w(z)^2}\right] \\ &= (az+b) + \frac{c}{w(z)}.\end{aligned} \tag{2.3}$$

By Valiron-Mohon'ko lemma (see [18]) and (2.3), we see that

$$T(r, [w(z+1) + w(z-1)]w(z)) = T(r, w) + S(r, w). \tag{2.4}$$

Hence, by (2.2) and (2.4), we have that

$$N(r, [w(z+1) + w(z-1)]w(z)) = T(r, w) + S(r, w) + O(r^{\sigma-1+\varepsilon}). \tag{2.5}$$

If $w(z)$ has a pole of multiplicity k at z_0 and $|z_0| \le r - 1$, then $w(z+1)$ and $w(z-1)$ have poles at $z_0 - 1$, $z_0 + 1$ of multiplicity k respectively. Hence

$$N(r, [w(z+1) + w(z-1)]w(z)) \le 3N(r+1, w(z)). \tag{2.6}$$

By Lemma 2.2, we have

$$N(r+1, w(z)) = N(r, w(z)) + S(r, w). \tag{2.7}$$

Thus, by (2.5)–(2.7) we obtain that

$$\lambda\left(\frac{1}{w}\right) = \sigma(w). \tag{2.8}$$

Secondly, we show that for any finite value α, we have $\lambda(w - \alpha) = \sigma(w)$. Set $g(z) = w(z) - \alpha$. Substituting $w(z) = g(z) + \alpha$ into (1.3), we obtain that

$$g(z+1) + g(z-1) + 2\alpha = \frac{az+b}{g(z)+\alpha} + \frac{c}{(g(z)+\alpha)^2}. \tag{2.9}$$

It follows from (2.9) that

$$\begin{aligned} P(z, g) := {} & [g(z+1) + g(z-1) + 2\alpha](g(z)+\alpha)^2 \\ & - (az+b)(g(z)+\alpha) - c = 0. \end{aligned} \tag{2.10}$$

By (2.10), we have

$$P(z, 0) = 2\alpha^3 + \alpha(az+b) - c. \tag{2.11}$$

If $\alpha = 0$, then $P(z, 0) = -c \not\equiv 0$ since $c \neq 0$; if $\alpha \neq 0$, then

$$P(z, 0) = 2\alpha^3 + \alpha(az+b) - c \not\equiv 0$$

since $a \neq 0$. Thus by Lemma (2.3), we see that

$$m\left(r, \frac{1}{g}\right) = S(r, g)$$

outside of a possible exceptional set of finite logarithmic measure. Thus

$$\begin{aligned} N\left(r, \frac{1}{w-\alpha}\right) &= N\left(r, \frac{1}{g}\right) \\ &= T(r, g) + S(r, g) = T(r, w) + S(r, w) \end{aligned} \tag{2.12}$$

outside of a possible exceptional set of finite logarithmic measure. Hence, by (2.12) we have $\lambda(w-\alpha)=\sigma(w)$. Combing with (2.8), we see that $w(z)$ has no any Borel exceptional value.

(ii) Suppose that $p(z)$ is a non-constant polynomial. We use a similar method as above to prove $\lambda(w-p)=\sigma(w)$. Set $p(z)=d_kz^k+\cdots+d_0$ where $d_k,\dots,d_0$ are constants, $d_k\neq 0$ and $k\geq 1$, and $g_1(z)=w(z)-p(z)$. Substituting $w(z)=g_1(z)+p(z)$ into (1.3), we obtain that

$$\begin{aligned}(2.13)\qquad P_1(z,g_1)&=[g_1(z+1)+g_1(z-1)+p(z+1)+p(z-1)](g_1(z)+p(z))^2\\&\quad-(az+b)(g_1(z)+p(z))-c=0.\end{aligned}$$

Thus, since $d_k\neq 0$ and $k\geq 1$, it follows from (2.13) that

$$\begin{aligned}P_1(z,0)&=(p(z+1)+p(z-1))p(z)^2-(az+b)p(z)-c\\&=2d_k^3z^{3k}+\cdots\not\equiv 0.\end{aligned}$$

Continuing to use the same method as above, we obtain that $\lambda(w-p)=\sigma(w)$. Thus Theorem 1 is proved.

3. Proof of Theorem 2

Assume that $w(z)$ is a transcendental solution of (1.5) with $\sigma(w)<\infty$. We use a similar method as in the proof of Theorem 1.

(i) By (1.5), we obtain that

$$(3.1)\qquad w(z)\{w(z)[w(z+1)+w(z-1)]\}=w(z+1)+w(z-1)-(az+b)w(z).$$

Applying Lemma 2.1 to (3.1), we see that for any given $\varepsilon>0$, there is a subset $E\subset(1,\infty)$ having finite logarithmic measure such that for $|z|=r\notin[0,1]\cup E$,

$$(3.2)\qquad m(r,w(z)(w(z+1)+w(z-1)))=O(r^{\sigma-1+\varepsilon})+S(r,w),$$

where $\sigma=\sigma(w)$. By Valiron-Mohon'ko lemma (see [18]) and (1.5), obtain that

$$\begin{aligned}(3.3)\qquad T(r,w(z)(w(z+1)+w(z-1)))&=T\left(r,w(z)\frac{(az+b)w}{1-w^2}\right)\\&=2T(r,w)+S(r,w).\end{aligned}$$

Thus, by (3.2) and (3.3), we have

$$(3.4)\qquad N(r,w(z)(w(z+1)+w(z-1)))=2T(r,w)+S(r,w)+O(r^{\sigma-1+\varepsilon}).$$

On the other hand, if $w(z)$ has a pole of multiplicity m at z_0 and $|z_0|\leq r-1$, then $w(z+1)$ and $w(z-1)$ have poles at z_0-1, z_0+1 of multiplicity m respectively. Thus, by Lemma 2.2 and (3.4), we obtain that

$$\begin{aligned}3N(r, w(z)) &= 3N(r+1, w(z)) + S(r, w)\\ &\geq N(r, w(z)(w(z+1) + w(z-1))) + S(r, w)\\ &= 2T(r, w(z)) + S(r, w) + O(r^{\sigma-1+\varepsilon}).\end{aligned}$$

Hence $\lambda\left(\dfrac{1}{w}\right) = \sigma(w)$.

(ii) Use the same method as the proof of Theorem 1(ii), we get $\lambda(w-p) = \sigma(w)$.

(iii) By (1.5), we derive that

$$\begin{aligned}P_2(z, w) := &-w(z)^2(w(z+1) + w(z-1))\\ &- (az+b)w(z) + (w(z+1) + w(z-1)) = 0.\end{aligned} \tag{3.5}$$

For a non-zero value $d \in \mathbf{C}$, by (3.5), we have that

$$P_2(z, d) = -2d^3 - azd + (2-b)d. \tag{3.6}$$

We divide it into two cases to prove.

(a) If $a \neq 0$, then by $d \neq 0$ we have

$$P_2(z, d) = -2d^3 - azd + (2-b)d \not\equiv 0.$$

Using a similar method as above, we obtain that $\lambda(w-d) = \sigma(w)$.

(b) If $a = 0$ and $d \notin E$, then we obtain that

$$P_2(z, d) = -2d^3 + (2-b)d \not\equiv 0.$$

Using a similar method as above we obtain $\lambda(w-d) = \sigma(w)$. Hence, the Borel exceptional of $w(z)$ can only come from a set $E = \left\{0, \left(1 - \dfrac{b}{2}\right)^{1/2}\right\}$.

Thus Theorem 2 is proved.

4. Proof of Theorem 3

Assume that $w(z)$ is a rational solution of (1.3), and has poles $t_1, \ldots, t_k$. Consequently, we suppose that

$$\frac{d_{js_j}}{(z-t_j)^{s_j}} + \cdots + \frac{d_{j1}}{z-t_j} \quad (j = 1, \ldots, k)$$

are the principal parts of w at t_j respectively, where $d_{js_j}, \ldots, d_{j1}$ are constants, $d_{js_j} \neq 0$. Thus, $w(z)$ can be represented as

$$w(z) = \sum_{j=1}^{k}\left[\frac{d_{js_j}}{(z-t_j)^{s_j}} + \cdots + \frac{d_{j1}}{z-t_j}\right] + a_0 + a_1 z + \cdots + a_m z^m, \tag{4.1}$$

where $a_0, \ldots, a_m$ are constants.

We affirm that $a_m = \cdots = a_1 = 0$. Assume $a_m \neq 0$ $(m \geq 1)$. For sufficiently large z, by (4.1), we have

$$\begin{cases} w(z) = a_m z^m(1 + o(1)) \\ w(z \pm 1) = a_m(z \pm 1)^m(1 + o(1)) = a_m z^m(1 + o(1)) \end{cases} \tag{4.2}$$

By (1.3), we obtain that

$$(w(z+1) + w(z-1))w(z)^2 = (az + b)w(z) + c \tag{4.3}$$

Substituting (4.2) into (4.3), we obtain that

$$2a_m^3 z^{3m}(1 + o(1)) = (az + b)a_m z^m(1 + o(1)) + c. \tag{4.4}$$

Clearly, (4.4) is a contradiction since $a_m \neq 0$. Hence $a_m = 0$ $(m \geq 1)$.

(i) Suppose that $a = 0$. By observation for (1.3), we see that $w(z) = B$, where B satisfies $2B^3 - bB - c = 0$, is a nonzero constant solution of (1.3). Since $a_m = 0$ $(m \geq 1)$, $w(z)$ can be rewritten by (4.1) as

$$w(z) = \frac{S(z)}{H(z)} \tag{4.5}$$

where

$$S(z) = sz^k + s_{k-1}z^{k-1} + \cdots + s_0 \quad \text{and} \quad H(z) = hz^t + h_{t-1}z^{t-1} + \cdots + h_0, \tag{4.6}$$

where $s, s_{k-1}, \ldots, s_0$ and $h, h_{t-1}, \ldots, t_0$ are constants, $sh \neq 0$ and $k \leq t$. Suppose that $k < t$. Then substituting (4.5) into (1.3), we obtain that

$$\begin{aligned} &S(z+1)H(z-1)S(z)^2 + S(z-1)H(z+1)S(z)^2 \\ &\quad = bS(z)H(z)H(z+1)H(z-1) + cH(z)^2H(z+1)H(z-1). \end{aligned} \tag{4.7}$$

Thus, in (4.7) there only exists one term $cH(z)^2H(z+1)H(z-1)$ such that it's degree is highest. This contradiction show $k = t$. Thus, by (4.6) and (4.7), we obtain that

$$\begin{aligned} &\frac{s(z+1)^k + s_{k-1}(z+1)^{k-1} + \cdots + s_0}{h(z+1)^t + h_{t-1}(z+1)^{t-1} + \cdots + h_0} + \frac{s(z-1)^k + s_{k-1}(z-1)^{k-1} + \cdots + s_0}{h(z-1)^t + h_{t-1}(z-1)^{t-1} + \cdots + h_0} \\ &\quad = \frac{b(hz^t + h_{t-1}z^{t-1} + \cdots + h_0)}{sz^k + s_{k-1}z^{k-1} + \cdots + s_0} + \frac{c(hz^t + h_{t-1}z^{t-1} + \cdots + h_0)^2}{(sz^k + s_{k-1}z^{k-1} + \cdots + s_0)^2}. \end{aligned} \tag{4.8}$$

By (4.8), we obtain that as $z \to \infty$

$$2B^3 - bB - c = 0$$

where $B = \dfrac{s}{h}$. Hence, $w(z)$ can be rewritten as

$$w(z) = B + \frac{S_1(z)}{H_1(z)}$$

where $S_1(z)$ and $H_1(z)$ are polynomials with $\deg S_1(z) < \deg H_1(z)$, B is a constant satisfying $2B^3 - bB - c = 0$.

(ii) Suppose that $a \neq 0$. Above we have got $a_m = 0$ $(m \geq 1)$. Now assume $a_0 \neq 0$. Then for sufficiently large z, by (4.1), we see that

$$w(z) = a_0 + o(1), \quad w(z+1) = a_0 + o(1), \quad w(z-1) = a_0 + o(1). \tag{4.9}$$

By (4.3) and (4.9), we obtain that

$$(2a_0 + o(1))(a_0 + o(1))^2 = (az + b)(a_0 + o(1)) + c. \tag{4.10}$$

This is a contradiction since $a \neq 0$. Hence $a_0 = 0$. Thus, $f(z)$ can be rewritten as (4.5) with $\deg S(z) = k < \deg H(z) = t$. Substituting (4.5) into (1.3), we obtain that

$$\begin{aligned} (4.7)' \quad & S(z+1)H(z-1)S(z)^2 + S(z-1)H(z+1)S(z)^2 \\ & = (az+b)S(z)H(z)H(z+1)H(z-1) + cH(z)^2H(z+1)H(z-1). \end{aligned}$$

By observation for $(4.7)'$, we get

$$t = k + 1 \quad \text{and} \quad s = -\frac{ch}{a}.$$

Thus Theorem 3 is proved.

5. Proof of Theorem 4

We use a similar method as in the proof of Theorem 2 in the proof.

(i) Assume that $w(z)$ is a rational solution of (1.5), and has poles $t_1, \ldots, t_k$. Thus, $w(z)$ can be represented as (4.1). Using the same method as in the proof of Theorem 2, we obtain that $a_m = 0$ $(m \geq 1)$.

Now we prove $a_0 = 0$. If $a_0 \neq 0$, then for sufficiently large z, (4.9) holds. Substituting (4.9) into (1.5), we get

$$(az + b)(a_0 + o(1)) = -(a_0^2 + o(1))(2a_0 + o(1)) + (2a_0 + o(1)). \tag{5.1}$$

Since $a \neq 0$ and $a_0 \neq 0$, we see that (5.1) is a contradiction. Hence $a_0 = 0$. Thus, $w(z)$ can be rewritten by (4.1) as the form (4.5), and $S(z)$, $H(z)$ satisfy (4.6) with $k < t$. Substituting (4.5) into (1.5), we obtain that

$$\begin{aligned} (5.2) \quad & (az+b)H(z+1)H(z-1)H(z)S(z) \\ & = S(z+1)H(z-1)H(z)^2 - S(z+1)H(z-1)S(z)^2 \\ & \quad + S(z-1)H(z+1)H(z)^2 - S(z-1)H(z+1)S(z)^2. \end{aligned}$$

Thus, since $k < t$ and $a \neq 0$, we see that the degree of the right side of (5.2) $\leq 3t + k$; but the degree of the left side of (5.2) is equal to $3t + k + 1$. This is a contradiction. Hence, if $a \neq 0$, then (1.5) has no non-zero rational solution.

(ii) Suppose that $a = 0$, $b = 2$ and $w(z)$ is a non-zero rational solution of (1.5) and $w(z)$ can be represented as (4.1). Using the same method as above, we get that $a_m = 0$ $(m \geq 1)$ and $a_0 = 0$. Hence $w(z) = \dfrac{S(z)}{H(z)}$ with $\deg S < \deg H$.

(iii) Suppose that $a = 0$, $b \neq 2$, 0 and $w(z)$ is a non-zero rational solution of (1.5) and $w(z)$ can be represented as (4.1). Using the same method as above, we obtain that $a_m = 0$ $(m \geq 1)$.

By observation for (1.5), we easily see $\left(1 - \dfrac{b}{2}\right)^{1/2}$ are nonzero constant solutions of (1.5). Since $a_m = 0$ $(m \geq 1)$, $w(z)$ can be rewritten by (4.1) as the form (4.5), where $S(z)$ and $H(z)$ satisfy (4.6) with $k \leq t$.

We affirm $k = t = m$ in (4.6). Suppose that $k < t$. By (4.5), we obtain that

$$
\begin{aligned}
(5.2)' \qquad & bH(z+1)H(z-1)H(z)S(z) \\
&= S(z+1)H(z-1)H(z)^2 - S(z+1)H(z-1)S(z)^2 \\
&\quad + S(z-1)H(z+1)H(z)^2 - S(z-1)H(z+1)S(z)^2.
\end{aligned}
$$

Comparing coefficients and degrees of all terms of $(5.2)'$, by $k < t$, (4.6) and $(5.2)'$, we obtain that

$$bsh^3 = 2sh^3.$$

Sine $hs \neq 0$, we have $b = 2$. This contradicts the fact that $b \neq 2$. Hence $k = t = m$.

When $k = t = m$, again comparing coefficients of the highest degree terms (i.e. containing z^{4m}) of $(5.2)'$, we obtain that

$$(2 - b)h^3 s - 2s^3 h = 0,$$

i.e.

$$\frac{s}{h} = \left(1 - \frac{b}{2}\right)^{1/2}.$$

Hence $w(z)$ is of the form

$$w(z) = \left(1 - \frac{b}{2}\right)^{1/2} + \frac{H_1(z)}{G_1(z)}$$

where $H_1(z)$, $G_1(z)$ are polynomials satisfying $\deg H_1 < \deg G_1$. Thus Theorem 4 is proved.

Acknowledgements. The author is grateful to the referee for a number of helpful suggestions to improve the paper.

References

[1] M. Ablowitz, R. G. Halburd and B. Herbst, On the extension of Painlevé property to difference equations, Nonlinearty **13** (2000), 889–905.

[2] W. Bergweiler and J. K. Langley, Zeros of differences of meromorphic functions, Math. Proc. Camb. Phil. Soc. **142** (2007), 133–147.
[3] Z. X. Chen and Kwang Ho Shon, On zeros and fixed points of differencers of meromorphic functions, J. Math. Anal. Appl. **344** (2008), 373–383.
[4] Z. X. Chen and K. H. Shon, Estimates for zeros of differences of meromorphic functions, Science in China Series A. **52** (2009), 2447–2458.
[5] Z. X. Chen and K. H. Shon, Value distribution of meromorphic solutions of certain difference Painlevé equations, J. Math. Anal. Appl. **364** (2010), 556–566.
[6] Y. M. Chiang and S. J. Feng, On the growth of logarithmic differences, difference quotients and logarithmic derivatives of meromorphic functions, Trans. Amer. Math. Soc. **361** (2009), 3767–3791.
[7] Y. M. Chiang and S. J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, THe Ramanujan J. **16** (2008), 105–129.
[8] A. S. Fokas, From continuous to discrete Painlevé equations, J. Math. Anal. Appl. **180** (1993), 342–360.
[9] R. G. Halburd and R. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, J. Math. Anal. Appl. **314** (2006), 477–487.
[10] R. G. Halburd and R. Korhonen, Meromorphic solution of difference equation, integrability and the discrete Painlevé equations, J. Phys. A: Math. Theor. **40** (2007), 1–38.
[11] R. G. Halburd and R. Korhonen, Finite-order meromorphic solutions and the discrete Painlevé equations, Proc. Lond. Math. Soc. **94** (2007), 443–474.
[12] R. G. Halburd and R. Korhonen, Existence of finite-order meromorphic solutions as a detector of integrability in difference equations, Physica D. **218** (2006), 191–203.
[13] W. K. Hayman, Meromorphic functions, Clarendon Press, Oxford, 1964.
[14] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and K. Hohge, Complex difference equations of Malmquist type, Comput. Methods Funct. Theory **1** (2001), 27–39.
[15] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and J. Zhang, Value sharing results for shifts of meromorphic functions, and sufficient conditions for periodicity, J. Math. Anal. Appl. **355** (2009), 352–363.
[16] K. Ishizaki and N. Yanagihara, Wiman-Valiron method for difference equations, Nagoya Math. J. **175** (2004), 75–102.
[17] I. Laine and C. C. Yang, Clunie theorems for difference and q-difference polynomials, J. Lond. Math. Soc. **76(3)** (2007), 556–566.
[18] I. Laine, Nevanlinna theory and complex differential equations, W. de Gruyter, Berlin, 1993.
[19] P. Painlevé, Mémoire sur les équations différentielles dont l'integrale générale est uniforme, Bull. Soc. Math. France **28** (1900), 201–261.
[20] S. Shimomura, The first, the second and the fourth Painlevé transcendents are of finite order, Proc. Japan Acad. **77(A)** (2001), 42–45.
[21] L. Yang, Value distribution theory, Science Press, Beijing, 1993.
[22] N. Yanagihara, Meromorphic solutions of some difference equations, Funkcial. Ekvac. **23** (1980), 309–326.

Zong-Xuan Chen
School of Mathematical Sciences
South China Normal University
Guangzhou, 510631
P. R. China
E-mail: chzx@vip.sina.com