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HIGHER CODIMENSIONAL EUCLIDEAN HELIX SUBMANIFOLDS

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Abstract

A submanifold of $\mathbf{R}^n$ whose tangent space makes constant angle with a fixed direction d is called a *helix*. Helix submanifolds are related with the eikonal PDE equation. We give a method to find every solution to the eikonal PDE on a Riemannian manifold locally. As a consequence we give a local construction of arbitrary Euclidean helix submanifolds of any dimension and codimension. Also we characterize the ruled helix submanifolds and in particular we describe those which are minimal.

1. Introduction

In our work [6] we give a method to construct locally all the helix hypersurfaces in the Euclidean space with respect to some constant unitary direction d. They are always ruled by straight line segments and these segments are the integral curves of the orthogonal projection of d on TM. In this article we work with higher codimensional and dimensional immersed helix submanifolds M in $\mathbf{R}^n$, i.e. submanifolds whose tangent space makes a constant angle with respect to a constant unitary direction d called a helix direction. We denote the unitary tangent and normal components of d by T and ξ, respectively. We called them tangent and normal helix directions, respectively. The integral curves of T are classic helices in the ambient with respect to the same direction d, so we call them helix lines. Some properties of helix submanifolds have been investigated in other Riemannian ambients, see for example [7], [8], [11]. Some other motivations for the study of helix submanifolds comes from the physics of interfaces of liquid crystals (see [12] for details), and that they appear contained in the shadow boundary of a submanifold (see [11], [10]).

In higher codimension the helix lines are just geodesics of the helix submanifold (Proposition 2.4). When we have the condition $\nabla^{\perp}_{T}\xi = 0$, they are also

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geodesics in the ambient, where $\nabla^{\perp}$ is the connection induced in $TM^{\perp}$ (Theorem 4.3). In Theorems 4.4 and 4.6 we explain how to construct and reconstruct the helix submanifolds whose helix lines are segments of straight lines, we call them *ruled helix* submanifolds. These results are local and they are the natural extension from codimension one to higher codimension. In Theorem 7.1, we prove that these kind of submanifolds are minimal if and only if its non empty intersections with hyperplanes orthogonal to d are also minimal in the Euclidean ambient. So there are many examples of minimal ruled helix submanifold, of any codimension greater or equal that one, and dimension greater or equal than two.

In Theorem 3.3, we proved that a Riemannian product submanifold is a helix if and only if their factors are also helix. In Section 5 we study the problem of local construction and reconstruction of an arbitrary helix submanifold with any dimension and codimension. We solved this by using eikonal functions f on a Riemannian manifold i.e. $\|\nabla f\|$ is constant. In Theorems 5.2 and 5.3, there is a method to find locally any solution of the eikonal PDE in any Riemannian manifold. By our previous work in [6] we have a concrete method to construct locally all the helix submanifolds by finding an eikonal function on a Riemannian manifold.

In Section 6, we classify the class of strong r-helix submanifolds (See Definition 6.1) which were introduced in our work [6]. There we asked for the classification of those helix which have r linearly independent (called weak r-helix) helix directions d_j whose normal helix direction ξ_j are parallel with the normal connection. The latter condition says that $\nabla^{\perp}\xi_j = 0$, which implies that $\nabla^{\perp}_{T_j}\xi_j = 0$ where T_j is the tangent helix direction of d_j. So, these class of helix has a straight line segment for each direction d_j. In Theorem 6.5 we see that these submanifolds are strong r-helix, i.e. they are helix with respect to any direction in a r-dimensional subspace of $\mathbf{R}^n$. The first author proved in [5] the existence of helix submanifolds in Euclidean space which are weak r-helix but not strong r-helix. The aforementioned classification is explained in Theorem 6.6.

2. Preliminaries

In this article a manifold M is assumed to be C^{∞} and connected.

Definition 2.1. Given a submanifold $M \subset \mathbf{R}^n$ and an unitary vector $d \neq 0$ in $\mathbf{R}^n$, we say that M is a *helix with respect to* d if for each $q \in M$ the angle between d and T_qM is constant.

Let us recall that a unitary vector d can be decomposed in its tangent and orthogonal components along the submanifold M, i.e. $d = \cos(\theta)\mathrm{T} + \sin(\theta)\xi$ with $\|\mathrm{T}\| = \|\xi\| = 1$, where $\mathrm{T} \in TM$ and $\xi \in \nu(M)$. The angle between T_qM and d is constant if and only if the tangential component of d has constant length $\|\cos(\theta)T\| = \cos(\theta)$.

If $\theta = \frac{\pi}{2}$, M is contained in a hyperplane orthogonal to d. In the case that $\theta = 0$ then M is a Riemannian product with one factor with direction parallel to d. So, we can assume that $0 < \theta < \frac{\pi}{2}$. Under these conditions we can say that M is a helix of angle θ.

We will call T and ξ the tangent and normal directions of the helix submanifold M. We can call d the helix direction of M and we will assume d always to be unitary.

Definition 2.2. Let $M \subset \mathbf{R}^n$ be a helix submanifold of angle $\theta \neq \frac{\pi}{2}$ w.r. to the direction $d \in \mathbf{R}^n$. We will call the integral curves of the tangent direction T of the helix M, *helix lines* of M w.r. to d.

The helix lines are classical helices. Since d and ξ are orthogonal to $M \cap H$ so is T. Then the helix lines are orthogonal to the level sets $H \cap M$, where H is any hyperplane orthogonal to d if $\theta \neq \frac{\pi}{2}$.

Definition 2.3. We say that a helix submanifold M is a *ruled helix*, if all the helix lines of M are straight lines.

Recall that if $d = \cos(\theta)\mathrm{T} + \sin(\theta)\xi$ is the decomposition of d in its normal and tangent components, we say that ξ is parallel normal in the direction T if

$$\nabla^{\perp}_{\mathrm{T}}\xi = 0.$$

Here $\nabla^{\perp}$ denotes the normal connection of M induced by the standard covariant derivative of the Euclidean ambient. Let us denote by D the standard covariant derivative in $\mathbf{R}^n$ and by ∇ the induced covariant derivative in M. Since M is full, i.e. not contained in any hyperplane, we have $\cos(\theta) \neq 0$. Let A^{ξ} and α be the shape operator and the second fundamental form of $M \subset \mathbf{R}^n$.

Taking the covariant derivative D with respect to X in both hands of the equation

$$d = \cos(\theta)\mathrm{T} + \sin(\theta)\xi$$

we obtain:

$$0 = \cos(\theta)D_X\mathrm{T} + \sin(\theta)D_X\xi.$$

Introducing the normal component and the tangential one (in M) we get:

$$\cos(\theta)(\nabla_X\mathrm{T} + \alpha(X,\mathrm{T})) + \sin(\theta)(-A^{\xi}(X) + \nabla^{\perp}_X\xi) = 0.$$

This implies

$$\mathscr{H} = \begin{cases} \cos(\theta)\nabla_X\mathrm{T} - \sin(\theta)A^{\xi}(X) = 0, \\ \cos(\theta)\alpha(X,\mathrm{T}) + \sin(\theta)\nabla^{\perp}_X\xi = 0. \end{cases}$$

Let us observe that the conditions given by these two formulas are also sufficient, i.e. if there exist a constant θ, a unitary tangent vector field T on M and a normal unitary vector field ξ such that they satisfy these equations, then the vector field $d := \cos(\theta)\mathrm{T} + \sin(\theta)\xi$ is constant in $\mathbf{R}^n$. Therefore M is a helix with respect to d.

Assume $\sin(\theta) \neq 0$. Then the first equation of the system $\mathscr{H}$ is $A^{\xi}(X) = \cot(\theta)\nabla_X \mathrm{T}$. Notice that if $\sin(\theta) = 0$ then $\mathrm{T} = \dfrac{d}{\|d\|}$ and in this case M splits since the constant direction d is tangent to M.

Proposition 2.4. *The helix lines of a helix submanifold $M \subset \mathbf{R}^n$ are geodesics in M.*

Proof. The equation $A^{\xi}(X) = \cot(\theta)\nabla_X \mathrm{T}$ implies $A^{\xi}(\mathrm{T}) = 0$. Indeed, for any vector field X we have

$$\langle A^{\xi}(\mathrm{T}), X\rangle = \langle A^{\xi}(X), \mathrm{T}\rangle = \cot(\theta)\langle \nabla_X \mathrm{T}, \mathrm{T}\rangle = 0,$$

since T has unit length. So we get that $\nabla_{\mathrm{T}}\mathrm{T} = \cot(\theta)A^{\xi}(\mathrm{T}) = 0$. $\square$

Remark 2.5. Let us observe that for any helix euclidean submanifold M, the conditions $\alpha(T, X) = 0$ and $\nabla^{\perp}_X \xi = 0$ are equivalent for every $X \in TM$. So, in particular $\alpha(T, T) = 0$ and $\nabla^{\perp}_T \xi = 0$ are equivalent.

Definition 2.6. The *relative nullity space* of the second fundamental form α (of $M \subset \mathbf{R}^n$) is the subset

$$\{X \in TM \mid \alpha(X, Y) = 0, \text{ for every } Y \in TM\}.$$

In the special case when ξ is parallel normal, i.e. $\nabla^{\perp}\xi = 0$, we can say more:

Proposition 2.7. *Let M be a helix with parallel normal direction. Then the tangent direction T is in the relative nullity of M.*

Proof. This is a direct consequence of the second equation of the system $\mathscr{H}$. $\square$

3. Product of helices

We will use the next well known result of linear algebra.

Lemma 3.1. *Let $V \subset \mathbf{R}^n$ be a linear subspace. Let $v_1, \dots, v_k$ be any basis of V. Then the orthogonal projection of $\mathbf{R}^n$ onto V is given by $\pi : \mathbf{R}^n \to V$, $\pi(v) = A(A^tA)^{-1}A^tv^t$, where A is the matrix with the vector v_j^t as jth-column. Here $v = (x_1, \dots, x_n) \in \mathbf{R}^n$.*

In particular, if the basis of V is orthonormal, then $\pi(v) = AA^tv^t$.

An easy consequence of this Lemma is the following corollary.

Corollary 3.2. *If M_j is a helix submanifold of $\mathbf{R}^{n_j}$, $j = 1,2$ w.r. to the direction unitary d_j, then the Riemannian product $M = M_1 \times M_2 \subset \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ is a helix submanifold w.r. to the direction $d = 1/\sqrt{2}(d_1, d_2) \in \mathbf{R}^{n_1+n_2}$.*

Proof. The matrix A (of Lemma 3.1) consist of the 2×2-matrix blocks A_{11} and A_{22} and a zero matrix elsewhere. Each block A_{jj} is determined by a frame field (local orthonormal basis) of TM_j for $j = 1,2$. The projection $\pi : \mathbf{R}^n \to T_pM_1 \oplus T_qM_2$ is given by the orthogonal decomposition, $\pi(d_1, d_2) = A_{11}A_{11}^t d_1^t + A_2 A_{22}^t d_2^t$, which has constant length. □

In the next result, we see that the reciprocal of this Corollary 3.2, is valid.

Theorem 3.3. *Let $M_1 \subset \mathbf{R}^{n_1}$, $M_2 \subset \mathbf{R}^{n_2}$ be two submanifolds. If the product $M_1 \times M_2 \subset \mathbf{R}^{n_1+n_2}$ is a helix submanifold then both $M_1 \subset \mathbf{R}^{n_1}$, $M_2 \subset \mathbf{R}^{n_2}$ are helix submanifolds.*

Proof. We will denote by $d = d_1 + d_2 \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} = \mathbf{R}^{n_1+n_2}$ the helix direction of $M_1 \times M_2$, where $d_1 \in \mathbf{R}^{n_1}$ and $d_2 \in \mathbf{R}^{n_2}$. Let T as before, the unitary orthogonal projection of d on $T(M_1 \times M_2)$. Let T_j be the unitary orthogonal projection of d_j on TM_j, for $j = 1,2$. We will use the natural identification $T(M_1 \times M_2) = TM_1 \oplus TM_2$. Under this identification, $T = T_1 + T_2$.

First, we will see that T_1 does not depend on M_2, i.e. $T_1(p,y) = T_1(p,z)$, where $p \in M_1$, $y, z \in M_2$. This equality should be interpreted in terms of the identification of each tangent space of $M_1 \times M_2$ with a linear subspace of $\mathbf{R}^{n_1+n_2}$.

Let us observe that $T_1(p,y)$ is the orthogonal projection of d on $T_{(p,y)}(M_1 \times \{y\})$ and similarly $T_1(p,z)$ is the projection of d on $T_{(p,z)}(M_1 \times \{z\})$. But $M_1 \times \{y\}$ is a translation of $M_1 \times \{z\}$ in $\mathbf{R}^{n_1+n_2}$. Since d is invariant under translation in $\mathbf{R}^{n_1+n_2}$, $T_{(p,y)}(M_1 \times y)$ is a translation of $T_{(p,z)}(M_1 \times z)$. Then $T_1(p,y)$ is a translation of $T_1(p,z)$ in $\mathbf{R}^{n_1+n_2}$, i.e. they are equal. In particular T_1 has constant length on $\{p\} \times M_2$. Analogously T_2 has constant length on $M_1 \times \{q\}$ for any $q \in M_2$.

Now, let $p \in M_1$ be any point and let us consider the slice $\{p\} \times M_2 \subset M_1 \times M_2$. Then the vector field T_2 on $\{p\} \times M_2$ has constant length:

For every $y \in M_2$ we have the next equality

$$\begin{aligned}
&\langle T_1(p,y), T_1(p,y)\rangle + \langle T_2(p,y), T_2(p,y)\rangle \\
&\quad = \langle T_1(p,y) + T_2(p,y), T_1(p,y) + T_2(p,y)\rangle = \langle T(p,y), T(p,y)\rangle.
\end{aligned}$$

Since $M_1 \times M_2$ is a helix, this sum $\langle T_1(p,y), T_1(p,y)\rangle + \langle T_2(p,y), T_2(p,y)\rangle$ does not depend on y. Now, to see that T_2 has constant length on $\{p\} \times M_2$, just apply the fact that T_1 has constant length on $\{p\} \times M_2$.

We deduce that for $p \in M_1$ the slice $\{p\} \times M_2$ is a helix with respect to the direction $0 + d_2 \in \{0\} \oplus TM_2$. Therefore, M_2 is a helix with respect to d_2. In the similar way, M_1 is a helix with respect to d_1. □

4. Construction of helices

4.1. Helix curves of $\mathbf{R}^n$

When $\dim M = 1$, we can describe all the classic helix curves (i.e. the tangent vector of the curve makes constant angle with respect to a fixed direction) $\gamma \subset \mathbf{R}^n$ with respect to $d = (0, \dots, 0, 1)$ as follows. Let $\alpha : I \subset \mathbf{R} \to \mathbf{R}^{n-1}$, then $\gamma(t) = (\alpha(t), at + b)$, where $a, b \in \mathbf{R}$ are constants. The parameter t of γ should be its arc-length, so, the curve α satisfies $\|\alpha'\|^2 = 1 - a^2$ and $|a| \leq 1$.

Let us describe also the Euclidean helices of dimension one in terms of the Serret-Frenet formulas. Let $M \subset \mathbf{R}^n$ be an immersed 1-dimensional submanifold (a regular curve). Let T be an unitary tangent vector field on M ($p \in M$, $T(p) \in T_pM$ with $\|T\| = 1$).

It is well known ([13] pages 30–33) that there exists an orthonormal basis

$$\{T,\ \xi_2 = D_T T/\|D_T T\|,\ \xi_3, \dots, \xi_n\}$$

of $T\mathbf{R}^n_{|M} = TM \oplus TM^{\perp}$ such that $D_T T = k_1 \xi_2$, $D_T \xi_2 = -k_1 T + k_2 \xi_3, \dots, D_T \xi_{n-1} = -k_{n-2} \xi_{n-2} + k_{n-1} \xi_n$, $D_T \xi_n = -k_{n-1} \xi_{n-1}$. Where D is the standard connection of $\mathbf{R}^n$ and we are assuming that $k_1 = \|D_T T\| \neq 0$, $k_j \neq 0$ for $j = 2, \dots, n$ (This is without loss of generality). This is equivalent to say that M is full, i.e. it is not contained in a hyperplane of the ambient $\mathbf{R}^n$ (See Spivak Vol. 4, [13] page 38, for details). These equations are called *Serret-Frenet* formulas and we can call the frame, a *Serret-Frenet frame*.

We are going to describe the necessary and sufficient conditions for M to be a helix.

Let $h_3 := \dfrac{k_1}{k_2}$, $h_4 := \dfrac{1}{k_3}\left(\dfrac{k_1}{k_2}\right)'$ (Let us observe that $h_3' = k_3 h_4$) and

$$h_m := \frac{1}{k_{m-1}}(h'_{m-1} + k_{m-2} h_{m-2}), \quad \text{for } 5 \leq m \leq n.$$

THEOREM 4.1. *Let $M^1 \subset \mathbf{R}^n$ be a full curve. The following conditions are equivalent*:

(i) *$M^1 \subset \mathbf{R}^n$ is a helix,*

(ii) *$h_n \neq 0$ and $h_3^2 + h_4^2 + \cdots + h_n^2$ is a nonzero constant c^2 with $c > 0$.*

Proof. Assume that condition (ii) holds. We can define $\theta := \tan^{-1}(c)$ which is a constant in the interval $(0, \pi/2)$. Let $\xi := \cot(\theta)(h_3 \xi_3 + \cdots + h_n \xi_n)$, therefore $\|\xi\|^2 = \cot^2(\theta)(h_3^2 + h_4^2 + \cdots + h_n^2) = \cot^2(\theta) c^2 = \cot^2(\theta)\tan^2(\theta) = 1$. Consider the vector field $Z := \cos(\theta) T + \sin(\theta)\xi = \cos(\theta)(T + h_3 \xi_3 + \cdots + h_n \xi_n)$. We will verify that Z is constant along M, i.e. $D_T Z = 0$. Observe that $\|Z\| = 1$ and $\langle T, Z \rangle$ is the constant $\cos(\theta)$ (this means that the angle between TM and Z is constant). We know from Section 2, that M is a helix if and only if $\cos(\theta)\nabla_X T - \sin(\theta) A^{\xi}(X) = 0$, $\cos(\theta)\alpha(X, \mathrm{T}) + \sin(\theta)\nabla_X^{\perp}\xi = 0$. Since $\dim M = 1$, $\nabla_T T = 0$ and thus $D_T T = \alpha(T, T)$.

So a 1-dimensional immersed submanifold $M \subset \mathbf{R}^n$ is a helix with respect to the direction $Z = \cos(\theta)T + \sin(\theta)\xi$ if and only if

$$D_T T = -\tan(\theta)\nabla_T^{\perp}\xi. \tag{1}$$

The next step, is to check that our vector fields T and ξ satisfy this equation $D_T T + \tan(\theta)\nabla_T^{\perp}\xi = 0$. Using the relation $\langle \nabla_T^{\perp}\xi_j, \xi_i \rangle = \langle D_T \xi_j, \xi_i \rangle$ and the Serret-Frenet's formulas we deduce that

$$\nabla_T^{\perp}\xi_2 = k_2\xi_3, \quad \nabla_T^{\perp}\xi_j = -k_{j-1}\xi_{j-1} + k_j\xi_{j+1}, \quad \nabla_T^{\perp}\xi_n = -k_{n-1}\xi_{n-1},$$

where $3 \leq j \leq n-1$. It will be important the above recursive formula $k_{m-1}h_m = h'_{m-1} + k_{m-2}h_{m-2}$ where $5 \leq m \leq n$. It is convenient the next equivalent equation: $h'_m = k_m h_{m+1} - k_{m-1}h_{m-1}$ where $4 \leq m \leq n-1$ together with $h'_3 = k_3 h_4$. We need an expression for h'_n: since the expression in (ii) is constant, $h_n h'_n = -h_3 h'_3 - h_4 h'_4 - \cdots - h_{n-1}h'_{n-1}$. From the previous formula we obtain that $h_3 h'_3 = k_3 h_3 h_4$ and $h_m h'_m = k_m h_m h_{m+1} - k_{m-1}h_{m-1}h_m$ for $4 \leq m \leq n-1$. An algebraic calculus shows that $h_n h'_n = -k_{n-1}h_{n-1}h_n$. Since $h_n \neq 0$, we get the relation $h'_n = -k_{n-1}h_{n-1}$.

Then

$$\tan(\theta)\nabla_T^{\perp}\xi = h'_3\xi_3 + h_3\nabla_T^{\perp}\xi_3 + \sum_{m=4}^{n-1}(h'_m\xi_m + h_m\nabla_T^{\perp}\xi_m) + h'_n\xi_n + h_n\nabla_T^{\perp}\xi_n.$$

Let us calculate the latter summation in two parts. First one:

$$\begin{aligned}
&\sum_{m=4}^{n-1}(h'_m\xi_m + h_m\nabla_T^{\perp}\xi_m) \\
&\quad = \sum_{m=4}^{n-1}((k_m h_{m+1} - k_{m-1}h_{m-1})\xi_m + h_m(-k_{m-1}\xi_{m-1} + k_m\xi_{m+1})) \\
&\quad = (k_4 h_5 - k_3 h_3)\xi_4 + (k_{n-1}h_n - k_{n-2}h_{n-2})\xi_{n-1} \\
&\qquad + \sum_{m=5}^{n-2}((k_m h_{m+1} - k_{m-1}h_{m-1})\xi_m - h_4 k_3\xi_3 - h_5 k_4\xi_4 \\
&\qquad - \sum_{m=5}^{n-2} h_{m+1}k_m\xi_m + \sum_{m=5}^{n-2} h_{m-1}k_{m-1}\xi_m + h_{n-2}k_{n-2}\xi_{n-1} + h_{n-1}k_{n-1}\xi_n \\
&\quad = -h_4 k_3\xi_3 - k_3 h_3\xi_4 + k_{n-1}h_n\xi_{n-1} + h_{n-1}k_{n-1}\xi_n
\end{aligned}$$

and we get

$$\sum_{m=4}^{n-1}(h'_m\xi_m + h_m\nabla_T^{\perp}\xi_m) = -h_4 k_3\xi_3 - k_3 h_3\xi_4 + k_{n-1}h_n\xi_{n-1} + h_{n-1}k_{n-1}\xi_n.$$

Let us observe that we arranged the index in the second and third summation from 5 to $n-2$. Second part:

$$\begin{aligned} &h_3'\xi_3 + h_3\nabla_T^\perp\xi_3 + h_n'\xi_n + h_n\nabla_T^\perp\xi_n \\ &\quad = k_3h_4\xi_3 - h_3k_2\xi_2 + h_3k_3\xi_4 - k_{n-1}h_{n-1}\xi_n - h_nk_{n-1}\xi_{n-1}. \end{aligned}$$

So, by the combination of these two parts, we have that $\tan(\theta)\nabla_T^\perp\xi = -h_3k_2\xi_2 = -k_1\xi_2$. The latter equality shows that $D_TT = k_1\xi_2 = -(h_3'\xi_3 + h_3\nabla_T^\perp\xi_3 + \cdots + h_n'\xi_n + h_n\nabla_T^\perp\xi_n) = -\tan(\theta)\nabla_T^\perp\xi$ (Let us observe that $D_TZ = 0$, which proves that Z is constant).

This is the equality (1), which proves that M is a helix with respect to the direction

$$Z = \cos(\theta)(T + h_3\xi_3 + \cdots + h_n\xi_n).$$

This proves (ii) $\to$ (i).

To show (i) $\to$ (ii) assume that M^1 is a helix with respect to a constant direction d. Then $\langle d, T\rangle = \cos(\theta)$ is constant and we have that $d = \cos(\theta)T + \sin(\theta)\xi = \cos(\theta)T + \langle d,\xi_2\rangle\xi_2 + \cdots + \langle d,\xi_n\rangle\xi_n$. So

$$\langle d,\xi_2\rangle^2 + \cdots + \langle d,\xi_n\rangle^2$$

is a constant. So, we need to find the relation between $\langle d,\xi_j\rangle$ and the curvatures $k_1, k_2, \ldots, k_n$:

$0 = T\langle d,T\rangle = \langle d, D_TT\rangle = \langle d,\xi_2\rangle k_2$, then $\langle d,\xi_2\rangle = 0$. Now we can take the derivative again, $0 = T\langle d,\xi_2\rangle = \langle d, D_T\xi_2\rangle = -\cos(\theta)k_1 + \langle d,\xi_3\rangle k_2$. Therefore, $\langle d,\xi_3\rangle = \dfrac{k_1}{k_2}\cos(\theta)$. Taking the derivative of $\langle d,\xi_4\rangle$ and $\langle d,\xi_5\rangle$, we obtain that $\langle d,\xi_4\rangle = \dfrac{1}{k_3}\left(\dfrac{k_1}{k_2}\right)'\cos(\theta)$, $\langle d,\xi_5\rangle = \dfrac{1}{k_4}\left[\left(\dfrac{1}{k_3}\left(\dfrac{k_1}{k_2}\right)'\right)'\cos(\theta) + \dfrac{k_1k_3}{k_2}\cos(\theta)\right]$.

With this process we get the next recursive formula: for $5 \le m \le n$, $\langle d,\xi_m\rangle = h_m\cos(\theta)$, where h_m is as before,

$$\langle d,\xi_2\rangle^2 + \cdots + \langle d,\xi_n\rangle^2 = \cos(\theta)(h_3^2 + h_4^2 + \cdots + h_n^2)$$

is constant.

If $h_n = 0$ then $\cos(\theta)h_n = \langle d,\xi_n\rangle = 0$. Thus, $0 = \langle d, D_T\xi_n\rangle = -k_{n-1}\langle d,\xi_n\rangle = -k_{n-1}\cos(\theta)h_{n-1}$. We deduce that $h_{n-1} = 0$. Continuing this process, we get that $h_j = 0$ for $j = n, n-1, \ldots, 4$. Therefore, $\cos(\theta)h_4 = \langle d,\xi_4\rangle = 0$. Then $0 = \langle d, D_T\xi_4\rangle = -k_3\langle d,\xi_3\rangle = -k_3\cos(\theta)h_3$, which proves that $h_3 = 0$. Let us recall that $h_3 = k_1/k_2$, thus we have a contradiction because all the curvatures are nowhere zero since M^1 was assumed to be full. $\square$

For example, a full M is a helix in $\mathbf{R}^3$ if and only $\dfrac{k_1}{k_2}$ is a nonzero constant. In $\mathbf{R}^4$ the conditions are

$$\left(\frac{k_1}{k_2}\right)^2 + \left(\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right)^2 \text{ is constant and } \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)' \neq 0.$$

Remark 4.2. For $n = 3$, that is to say, for curves in the Euclidean space $\mathbf{R}^3$ Theorem 4.1 is a classical result stated by M. A. Lancret, a pupil of Monge, in 1802 and first proved by B. de Saint Venant in 1845 (see [14] for details).

To finish with curves, let us notice that condition (ii) in Theorem 4.1 is really necessary since there exist full curves of $\mathbf{R}^4$ such that

$$\left(\frac{k_1}{k_2}\right)^2 + \left(\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\right)^2 \quad \text{is constant}$$

and

$$\frac{1}{k_3}\left(\frac{k_1}{k_2}\right)' = 0$$

which are not helix. Indeed, such curves are constructed by using the existence theorem with k_1, k_2, k_3 such that $\dfrac{k_1}{k_2}$ is a non zero constant and k_3 is an arbitrary but not zero function.

4.2. Construction and reconstruction of any ruled helix

Theorem 4.3. *Let $M \subset \mathbf{R}^n$ be a full submanifold which is a helix with respect to the direction d. Let ξ be the normal component of d, i.e. $d = \cos(\theta)\mathrm{T} + \sin(\theta)\xi$. Then M is a ruled helix if and only if ξ is $\nabla^{\perp}_{T}\xi = 0$.*

Proof. Assume that $\nabla^{\perp}_{T}\xi = 0$. Then the second equation of the system $\mathscr{H}$ implies $\alpha(\mathrm{T},\mathrm{T}) = 0$ since M is full. Then $D_{\mathrm{T}}\mathrm{T} = \nabla_{\mathrm{T}}\mathrm{T} + \alpha(\mathrm{T},\mathrm{T}) = 0$ by Proposition 2.4. So we get that the helix lines are straight lines of $\mathbf{R}^n$.

If M is a ruled helix, by definition its helix lines are (segments of) straight lines in the ambient $\mathbf{R}^n$, then $0 = D_T d = \cos(\theta)D_T T + \sin(\theta)D_T\xi = \sin(\theta)D_T\xi$. So the Weingarten formula implies that $0 = D_T\xi = -A^{\xi}(T) + \nabla^{\perp}_{T}\xi$. In particular, $\nabla^{\perp}_{T}\xi = 0$. $\square$

Now, we will see a method to construct locally all the ruled helix submanifolds $M \subset \mathbf{R}^n$ of codimension $n - k$.

First, we begin with a immersed submanifold $L^{k-1} \subset \mathbf{R}^{n-1}$ and an unitary normal vector field η of $L \subset \mathbf{R}^{n-1}$. Without loss of generality we can assume that d is the vector $(0,\dots,0,1) \in \mathbf{R}^n$. We can immerse L in $\mathbf{R}^n$ in a canonical way. That is, $L \subset \mathbf{R}^{n-1} \times \{0\} \subset \mathbf{R}^n = \mathbf{R}^{n-1} \times \mathbf{R}$.

Now, we define the vector field $\mathrm{T}(x) := \sin(\theta)\eta(x) + \cos(\theta)d$, where $x \in L$ (recall that η is normal to L). So T is a vector field defined along the submanifold L.

Finally, we are ready to describe the immersion of M in $\mathbf{R}^n$.

The immersion $f =: L \times \mathbf{R} \to \mathbf{R}^n$ is as follows:

$$f(x,s) := x + s\mathrm{T}(p).$$

For $-\varepsilon < s < \varepsilon$ enough small, f is an immersion.

THEOREM 4.4. *The immersed submanifold $M = f(L \times (-\varepsilon, \varepsilon))$ is a helix of angle θ with respect to d.*

Proof. As a first case, we will verify that the immersion is a helix at the points $f(x,0)$. Let $Y_1, \dots, Y_{k-1}$ be a frame field of L at x. Then $Y_1, \dots, Y_{k-1}, T$ is a frame field of $f(M)$ at $f(x,0)$. This determine a matrix $A = (Y_1^t \cdots Y_{k-1}^t T^t)$, so by Lemma 3.1, the orthogonal projection of d into $T_{(x,0)}M$ is $\pi(d) = AA^t(d^t) = A(0, \cos(\theta))^t = \cos(\theta)T^t$.

Now, let $s \in (-\varepsilon, \varepsilon)$.

Affirmation: T is orthogonal to $T_{(x,s)}M$. Let $Y \in T_xL$ any vector field generated by the frame field Y_j's of L as before at $x \in L$. Let us denote by D the standard covariant derivative of $\mathbf{R}^n$, by $\nabla^\perp$ the normal connection and by A the shape operator of $L \subset \mathbf{R}^n$. So we can calculate the derivative of the immersion f at (x,s) as follows:

$$f_*(Y) = D_Y f = Y + sD_Y T = Y + s\nabla_Y^\perp T - A_T(Y).$$

Since $\|T\| = 1$, $T \in T^\perp L$ and $A_T(Y) \in TL$,

$$\langle T, f_*(Y)\rangle = s\langle T, \nabla_Y^\perp T\rangle = 0.$$

Let us observe that if we extend the vector field of L, η, into M by translation, then

$$\langle \eta, f_*(Y)\rangle = s\langle \eta, \nabla_Y^\perp T\rangle = s\sin(\theta)\langle \eta, \nabla_Y^\perp \eta\rangle = 0.$$

Let observe that $L_s := \{f(y,s) \in M \mid y \in L\} = M \cap H$, where H is a orthogonal hyperplane to d. So, we have again the same conditions as in the first case of this proof: the tangent direction T is orthogonal to $L_s = f(L \times \{s\})$ and is valid the decomposition $\mathrm{T}(y,s) := \sin(\theta)\eta(y,s) + \cos(\theta)d$, where $T(y,s) = T(y)$, $\eta(y,s) = \eta(y)$. This means that we extend the vector fields T, η (along L) into vector fields along M by translation along the lines generated by the original T. Therefore, $\eta(x,s)$ and $T(x,s)$ are orthogonal to L_s. $\square$

Remark 4.5. A different proof of Theorem 4.4 can be given by showing directly that the vector field $d - \langle d, \mathrm{T}\rangle\mathrm{T} = d - \cos(\theta)\mathrm{T}$ is normal to M.

Let M be a helix obtained by the construction above. If the normal component ξ of M is not parallel with respect to the normal connection of L, then it is not necessarily parallel with respect to the normal connection of M. This is interesting because they are the first examples with such property.

Now we will see that any helix submanifold of angle $\theta \neq \dfrac{\pi}{2}$, such that their helix lines (See Definition 2.2) are straight lines, has a local structure as the construction of the last theorem.

Let us denote by $H_{p,d}$ the orthogonal hyperplane to the direction d through the point $p \in \mathbf{R}^n$. The normalized projection of d onto $H_{p,d}$ is just $\dfrac{\pi(d)}{\|\pi(d)\|}$, where π is the orthogonal projection onto $H_{p,d}$.

Theorem 4.6. *Let M be a ruled helix submanifold of angle $\theta \neq \frac{\pi}{2}$ with respect to $d = (0, \dots, 0, 1)$. Then M is locally as in the latter construction, using as η the normalized projection of T onto $H_{p,d}$.*

Proof. If $\theta = 0$, M is locally the cylinder $M \times \mathbf{R}$ and we are done. So we can assume that $\theta \neq 0$. Let T and ξ be the (unitary) tangent and normal direction of M. So, $d = \cos(\theta)T + \sin(\theta)\xi$. Let $p = (a_1, \dots, a_n) \in M \subset \mathbf{R}^n$ and let $s := a_n \in \mathbf{R}$. Finally, let us denote by L_s the submanifold $M \cap H$, where $H = H_{p,d}$. We define the unitary vector field $\eta : M \to \mathbf{R}^n$ by $\eta = \sin(\theta)T - \cos(\theta)\xi$. Let us observe that $\langle \sin(\theta)\eta, T - \sin(\theta)\eta \rangle = 0$, $\langle \sin(\theta)\eta, d \rangle = 0$. Moreover, we have to that $\eta_{|L_s} : L_s \to \mathbf{R}^n$ is orthogonal to L_s. But this is clear because η can be a linear combination of T and d (ξ is a combination of T and d), which are orthogonal to $H \cap M$. These properties proves that $\sin(\theta)\eta$ is the orthogonal projection of T onto $H_{p,d}$. So, η is just the normalized projection of T.

Since $\langle \eta, d \rangle = 0$, we can conclude that in fact $\eta_{|L_s} : L_s \to \mathbf{R}^{n-1} \times \{0\} \subset \mathbf{R}^n$. So L_s is a $k-1$-dimensional submanifold of $H = \mathbf{R}^{n-1} \times \{s\} \subset \mathbf{R}^n$ and $\eta_{|L_s} : L_s \to T^{\perp}L_s \subset H$ is smooth unitary normal vector field. Here $k = \dim M$.

Finally, let us observe that $T = \sin(\theta)\eta + \cos(\theta)d$, as in the construction. □

Let d, T and ξ the helix direction, tangent and normal directions respectively of M.

Corollary 4.7. *If $\nabla^{\perp}_{T}\xi = 0$, the helix submanifold M can be locally constructed as in Theorem* 4.4.

Proof. By Theorem 4.3, M is a ruled helix. Then apply Theorem 4.6. □

5. Non ruled helix

In the previous sections we obtained some general properties of an arbitrary helix submanifold in Euclidean space: Let M be any helix submanifold in $\mathbf{R}^n$, with respect to the unitary helix direction d and whose tangent and normal helix directions are T and ξ respectively. Let us consider the next two basic properties,

- the helix lines of M are orthogonal to $H \cap M$ (See the observation after Definition 2.2),
- the helix lines are geodesics of M (Proposition 2.4).

So, an arbitrary helix submanifold looks like a hypersurface $H \cap M$ in M with orthogonal classical general helices curves, in $\mathbf{R}^n$, through it (the helix lines of M which are geodesics of M).

In this section we will describe the precise way to glue the helix lines to the hypersurface $H \cap M$ of M orthogonal to d. This will be possible with the help of eikonal functions which will be constructed by using Fermi coordinates. In particular, we will know how to construct examples of non ruled helix submanifolds.

Definition 5.1. Let (N, g) be a Riemannian manifold. Let $f : N \to \mathbf{R}$ be a function and let ∇f be its gradient i.e. $df(X) = g(\nabla f, X)$. We say that f is *eikonal* if $\|\nabla f\|$ is constant.

Theorem 5.2 (A). *Let (N, g) be a n-dimensional connected Riemannian manifold. Let $L \subset N$ be an isometrically immersed submanifold of codimension one. Then around any point of p there exist an open neighbourhood U and a nonconstant eikonal function f on U.*

Proof. Let $p \in N$ and let $U \subset N$ be a normal neighbourhood of N around p such that on $V := U \cap L$, a neighbourhood of L around p, the map $\exp_{|TV^{\perp}} : TV^{\perp} \to U \subset N$ is a diffeomorphism.

Let us consider Fermi coordinates on U:

Let $Z : V \to TV^{\perp}$ be an unitary local normal field on V. We define $F(t, x_1, \ldots, x_{n-1}) = \exp(tZ(x_1, \ldots, x_{n-1}))$. Now we can define a function $f : U \to \mathbf{R}$ by $f(\exp(tZ(x_1, \ldots, x_{n-1}))) = t$ which is a submersion. Since f is nonconstant and N is connected, using Proposition 2 of [9] we can deduce that f is Eikonal. □

Theorem 5.3 (B). *Let (N, g) be a n-dimensional connected Riemannian manifold and let f a nonconstant Eikonal function on (N, g). Then for every point of $x \in N$, there exist an open neighbourhood U around x and a hypersurface $L \subset N$ isometrically immersed submanifold of U, such that $f_{|U}$ is given by the Fermi coordinates around a neighbourhood of L.*

Proof. In Proposition 2 of [9], A. E. Fischer proved that f is a Riemannian submersion. So, if $s_0 := f(x)$ then $L := f^{-1}(s_0)$ is a hypersurface in N. We give any level hypersurface $L_s := f^{-1}(s)$ the induced Riemannian metric of N. We choose the open neighbourhood U in N around x so that $\exp_{|TV^{\perp}} : TV^{\perp} \to U \subset N$ is a diffeomorphism, where $V := U \cap L$. Now we will prove that for s small, $L_s \cap U$ is orthogonal to every geodesic orthogonal to $L \cap U$. Let us observe that the integral curves of ∇f are geodesics of M and they are orthogonal to the level sets L_s. It follows that the level sets are equidistant. So, f looks like the distance between level sets. Therefore, L and the geodesics (integral curves) defined by ∇f are enough to construct Fermi coordinates of N. □

Let us recall the next result, see our previous work [6].

Lemma 5.4. *Let M be a differentiable manifold. Then M can be immersed as an helix submanifold with angle $\theta \neq 0$ w.r.t. to a direction d of some Euclidean* *space if and only if M admits a Riemannian metric g and an eikonal function $f : M \to \mathbf{R}$ w.r.t. to g such that*:

$$\cos(\theta) = \frac{-1}{\sqrt{1 + \|\nabla f\|^2}}$$

Half of this last result is based on the next Theorem, also from [6].

THEOREM 5.5. *Let $i : M \to \mathbf{R}^n$ be a submanifold and let $f : M \to \mathbf{R}$ be an eikonal function, where M has the induced metric by $\mathbf{R}^n$, i.e. the metric of the image $i(M) \subset \mathbf{R}^n$. Then $\phi(M)$ is a helix, where $\phi : M \to \mathbf{R}^n \times \mathbf{R}$ is the immersion given by*

$$\phi(p) := (i(p), f(p)).$$

The direction is $d = (0, 1)$ and the angle θ between d and $\nu(M)$ (normal space) is determined by the equality

$$\cos(\theta) = \frac{-1}{\sqrt{1 + \|\nabla f\|^2}}.$$

Example 5.6. Here we construct a non ruled helix. Let N^2 be the following surface in $\mathbf{R}^3$:

$$N^2 = \{(x, y) \in \mathbf{R}^2 \mid (x, y, y(x^2 + y^2))\}.$$

Let us take L as the curve $L := \{(0, y, y^3) : y \in \mathbf{R}\} \subset N^2$. By using the constructions of Theorems 5.2 and 5.3 to these N and L, we get a helix surface in $\mathbf{R}^4$ with only one straight line segment helix line. Because N itself contains only one straight line: $\{(x, 0, 0) : x \in \mathbf{R}\}$. Thus, $N^2 \subset \mathbf{R}^4$ is a non ruled helix submanifold.

Remark 5.7. We can also construct ruled helix of codimension k in $\mathbf{R}^n$: Take as M the Euclidean space $\mathbf{R}^{n-k}$ and as L any immersed hypersurface in M. So, the geodesics of M which are orthogonal to L are straight line segments.

Example 5.8. Here is another example of a non ruled helix. First we will construct an eikonal function in the upper half-space model of the hyperbolic space $\mathbf{H}^m = \{(x_1, \ldots, x_m) \in \mathbf{R}^m \mid x_m > 0\}$. Let $\partial_j := \partial_{x_j}$ the canonical vector fields of $\mathbf{R}^m$ restricted to $\mathbf{H}^m$. Then the Riemannian metric in $\mathbf{H}^m$ at the point $y = (y_1, \ldots, y_m)$ is determined by $g(\partial_i, \partial_j) = \dfrac{1}{y_m^2}$. So, the basis $y_m\partial_1, \ldots, y_m\partial_m$ is a orthonormal basis of $T_y\mathbf{H}^n$. We need also the distance between two points $x, y \in \mathbf{H}^n$. It is given by

$$d(x, y) = \cosh^{-1}\left(1 + \frac{|x - y|}{2x_m y_m}\right).$$

For example the function $f(y) = \ln y_m$ is eikonal on $\mathbf{H}^m$. In this case the solution can be constructed by taking global Fermi coordinates of $\mathbf{H}^m$ around the hypersurface $L = \{(x_1, \dots, x_m) \mid x_m = 1\}$, which is a horosphere. In order to find such Fermi coordinates, it is necessary to find the distance from any point $y \in \mathbf{H}^m$ to L: it is given by $\ln y_m$. Let us observe that L is isometric to the standard $m-1$-dimensional euclidean space. The level sets of this function are parallel horospheres (with the same infinity point). The horospheres are flat with respect to the induced metric.

But let us find another eikonal function, where L will be isometric to an $m-1$-dimensional upper half-space hyperbolic space. More exactly, let us consider $L = \{x \in \mathbf{H}^m \mid x_1 = 0\}$, which is a totally geodesic hypersurface. Let $y = (y_1, \dots, y_m) \in \mathbf{H}^m$, then the geodesic orthogonal to L through y intersects L at the point $p = (0, y_2, \dots, y_{m-1}, y_1^2 + y_m^2)$. So the distance between this point p and y is given by

$$f(y_1, \dots, y_m) = \cosh^{-1}\left(\frac{(y_1^2 + y_m^2)^{1/2}}{y_m}\right).$$

Let us verify that this function $f : \mathbf{H}^m \to \mathbf{R}$ is eikonal. A direct calculus shows that

$$\partial_1 f = \frac{1}{(y_1^2 + y_m^2)^{1/2}} \quad \text{and} \quad \partial_m f = \frac{x}{y_m(y_1^2 + y_m^2)^{1/2}}.$$

The gradient of f at the point y in the upper half-space model of the hyperbolic space is $\nabla f = (y_m \partial_1 f) y_m \partial_1 + (y_m \partial_m f) y_m \partial_m$. So

$$\|\nabla f\|^2 = (y_m \partial_1 f)^2 + (y_m \partial_m f)^2 = 1.$$

The level set $f^{-1}(a) \subset \mathbf{H}^m$, with the induced metric, is isometric to a $(m-1)$-dimensional hyperbolic space $\mathbf{H}^{m-1}$ (up to multiplication by a constant of the standard metric of $\mathbf{H}^{m-1}$). These level sets, $f^{-1}(a)$, are called *equidistant hypersurfaces* (See Spivak [13], page 22).

Let $i : \mathbf{H}^m \to \mathbf{R}^n$ be an isometric immersion of the upper half-space model of the hyperbolic space into $\mathbf{R}^n$ with the standard metric (for some n). Let us consider the immersion $\phi : \mathbf{H}^m \to \mathbf{R}^{n+1}$, $\phi(x) = (i(x), f(x))$. Then by Theorem 5.5, the immersed $M = \phi(\mathbf{H}^m)$ is a helix submanifold with the induced metric and the helix direction is $d := e_{n+1} = (0, \dots, 0, 1)$. Let us observe that the set levels $f^{-1}(a) \subset \mathbf{H}^m$ of f are isometrically immersed in M. In others words, if H is an orthogonal hyperplane to d in $\mathbf{R}^{n+1}$, then the intersection $H \cap M$ is isometric to the set levels $f^{-1}(a) \subset \mathbf{H}^m$ for every $a \in \mathbf{R}$. So, the helix M intersects any such hyperplane H in a submanifold isometric (up to a constant) to a $(m-1)$-dimensional hyperbolic space. The helix so constructed can not be ruled. This is so since an isometric immersion $i : \mathbf{H}^m \to \mathbf{R}^n$ of the hyperbolic space can not be ruled. Indeed, a ruled immersion $i : \mathbf{H}^m \to \mathbf{R}^n$ of the hyperbolic space produce a Jacobi vector field Y along the rule whose length $|Y|$ contradicts a Theorem of Rauch [2, p. 86, Theorem II.6.4].

5.1. Constant angle surfaces in $\mathbf{S}^2 \times \mathbf{R}$

As an application of the above Theorem 5.5, and Theorem 3.4 of [6] we get the following theorem of [8].

Theorem 5.9 (Dillen et al, [8]). *A surface M immersed in $\mathbf{S}^2 \times \mathbf{R}$ is a constant angle surface if and only if the immersion $F : M \to \mathbf{S}^2 \times \mathbf{R}$ is (up to isometries of $\mathbf{S}^2 \times \mathbf{R}$) locally given by*

$$F(u,v) = (\cos(u\cos(\theta))f(v) + \sin(u\cos(\theta))f(v) \times f'(v), u\sin(\theta)),$$

$f : I \to \mathbf{S}^2$ is a unit speed curve in $\mathbf{S}^2$ and $\theta \in [0,\pi]$ is the constant angle.

Explanation. By definition (see [8, pag. 91]) a constant angle surface $M \subset \mathbf{S}^2 \times \mathbf{R} \subset \mathbf{R}^3 \times \mathbf{R}$ is a helix with respect to $d = (0,0,0,1)$.

Notice that the map $i : M \to \mathbf{S}^2 \subset \mathbf{R}^3$ locally given by

$$i(u,v) = \cos(u\cos(\theta))f(v) + \sin(u\cos(\theta))f(v) \times f'(v)$$

is an immersion. Then the coordinate u regarded as a function $u : M \to \mathbf{R}$ is an eikonal function with respect to the metric induced by i. Actually, u is constructed as in Theorems 5.2 and 5.3 by using the unit speed curve $f : I \to \mathbf{S}^2$ as the submanifold of codimension one i.e. as the zero level set of u. $\square$

It is interesting to notice that equation (28) in [8, pag. 93] show that the above helix $M^2 \subset \mathbf{R}^4$ is ruled if and only if $\theta = \dfrac{\pi}{2}$ and in this case the helix is a Riemannian product of a curve in $\mathbf{S}^2$ and $\mathbf{R}$.

6. Helix with parallel normal direction

Let us recall that if $M \subset \mathbf{R}^n$ is a helix of angle θ with respect to the unitary direction d, then the normal direction of M is the unitary direction $\xi : M \to TM^\perp$, where $\sin(\theta)\xi$ is the orthogonal projection of d onto $TM^\perp$.

We say that M is a helix with parallel normal direction if $\nabla^\perp\xi = 0$, where $\nabla^\perp$ is the normal connection of the isometric immersion $M \subset \mathbf{R}^n$.

6.1. Strong r-helix submanifolds

It can happens that a given submanifold $M \subset \mathbf{R}^n$ is a helix w.r. to two or more independent directions. For an hypersurface M notice that if M is an helix w.r. to d and d' then M is also a helix w.r. to any direction in the linear span of d and d'. This gives a motivation for the following definition.

Definition 6.1. A submanifold $M \subset \mathbf{R}^n$ is a weak r-helix if there exist r linearly independent directions $d_1, \dots, d_r$, such that M is a helix with respect to every d_j. We say that is a strong r-helix if there exist a linear subspace $\mathscr{H} \subset \mathbf{R}^n$ of dimension $r = \dim(\mathscr{H})$ such that M is a helix w.r. to any direction $d \in \mathscr{H}$. The subspace $\mathscr{H}$ will be called *the subspace of helix directions*.

Remark 6.2. A characterization of a strong r-helix in terms of the orthogonal projections $\pi_p : \mathbf{R}^n \to T_p M$.

The submanifold $M \subset \mathbf{R}^n$ is a strong r-helix if and only if there exist a r-dimensional lineal subspace $\mathscr{H} \subset \mathbf{R}^n$, such that: For all $v \in \mathscr{H}$, $\|\pi_p(v)\|$ does not depends of $p \in M$.

Remark 6.3. Notice that if a submanifold $M \subset \mathbf{R}^n$, of any codimension, is a weak helix w.r. to the directions d_1 and d_2 then M is not necessarily a strong 2-helix.

Let now M be a weak r-helix with respect to the linearly independent directions $(d_1, d_2, \dots, d_r)$. We can decompose each vector d_j in its tangent and normal components:

$$d_j = \cos(\theta_j)\mathrm{T}_j(p) + \sin(\theta_j)\xi_j,$$

where each θ_j is constant. Taking derivative with respect to $X \in TM$ we obtain (See Lemma 2.7):

$$0 = \cos(\theta_j)\nabla_X \mathrm{T}_j(p) - \sin(\theta_j)A^{\xi_j}(X) \quad \text{and} \tag{2}$$

$$0 = \cos(\theta_j)\alpha(X, \mathrm{T}_j(p)) + \sin(\theta_j)\nabla_X^{\perp}\xi_j. \tag{3}$$

Definition 6.4. The *second normal space* of $M \subset \mathbf{R}^n$ consist of the normal vectors, $\xi \in \nu(M)$, such that the shape operator in its direction is zero, i.e. $A^{\xi} = 0$.

We will consider the following problem raised in [6]:

Problem. Classify weak r-helices so that the normal components, ξ_j, of the directions d_j satisfy:

$$\nabla^{\perp}\xi_j = 0,$$

i.e. every ξ_j is parallel with respect to the normal connection.

Lemma 6.5. *Let M be a weak r-helix with respect to the directions d_j, $j = 1, \dots, r$. If the normal component ξ_j of d_j is normal parallel then M is a strong r-helix with respect to the subspace generated by $d_1, \dots, d_r$.*

Proof. Since $\nabla^{\perp}\xi_j = 0$, the functions $\langle \xi_j, \xi_k \rangle$ are constants. The first author proved (Proposition 2.4 in [5]), that if such functions are constant, then M is a strong r-helix. □

Theorem 6.6. *Let M be a weak r-helix with respect to the directions d_j, $j = 1, \dots, r$. Let us assume that the normal component ξ_j of d_j is normal parallel. Then M is locally immersed at $x \in M$ as $y + s_1 T_1 + \cdots + s_r T_r$, where T_j is the normalized projection of d_j onto TM, $y \in M \cap (\bigcap_j H_{x,d_j})$ and $s_j \in (-\varepsilon_j, \varepsilon_j) \subset \mathbf{R}$.*

Proof. By Lemma 6.5, M is a strong helix submanifold. It means that M is a helix with respect to any direction in the vector space generated by

$d_1, \dots, d_r$. Moreover, by Theorem 4.3, each *helix line* with respect to any helix direction is a straight line segment in the ambient.

Let us consider any $x \in M$ and the direction $d = a_1 d_1 + \cdots + a_r d_r$, with $|d| = 1$. The orthogonal projection T of d on TM is given by: $T = a_1 T_1 + \cdots + a_r T_r$. So, if s is small, M contains the straight line segment defined by $x + sT = x + sa_1 T_1 + \cdots + sa_r T_r$. This implies that M contains a relative open piece of an affine subspace of dimension r, which is generated by the T_j's. Finally, let us observe that $x \in M \cap (\bigcap_j H_{x,d_j})$ and $y \in M \cap (\bigcap_j H_{x,d_j})$ if and only if $M \cap (\bigcap_j H_{y,d_j}) = M \cap (\bigcap_j H_{x,d_j})$.

So we can parametrize M with the data given by $M \cap (\bigcap_j H_{x,d_j})$ and the T_j's as this Theorem implies. In fact, we can parametrize M with the alternative data $M \cap (\bigcap_j H_{x,d_j})$, d_j's and the η_j's, where each η_j is the orthogonal projection of T_j in the hyperplane Hx, d_j. With these conditions T_j is a linear combination of d_j and η_j. $\square$

7. Minimal ruled helices

Now, we are going to characterize minimal ruled helix submanifolds of arbitrary dimension and codimension of Euclidean spaces.

Let M as before, i.e. a helix submanifold w.r. to d and with angle θ, such that the helix lines are straight lines.

Given $p \in M$, H_p will denote the hyperplane through p and orthogonal to d. As before we can assume that $\theta \neq \left\{0, \frac{\pi}{2}\right\}$ otherwise we are done. Hence, d is transversal and nonorthogonal to M. So, $L := H_p \cap M$ is a submanifold.

THEOREM 7.1. *A ruled helix submanifold M is minimal if and only if $L := H_p \cap M$ is also minimal in $\mathbf{R}^n$, for every $p \in M$.*

Proof. Let T and ξ be the tangent and normal helix directions of M. The Weingarten's formula for $L \subset \mathbf{R}^n$ implies that $D_X Z = -A_L^Z(X) + \nabla^{L^\perp}{}_X Z$, for every tangent vector field X on L and $Z \in TL^\perp$. Here, A_L^Z is the shape operator of L (as submanifold of $\mathbf{R}^n$) in the direction Z. The corresponding formula for $M \subset \mathbf{R}^n$ is $D_X Z = -A^Z(X) + \nabla_X^\perp Z$, for every tangent vector field X on L and $Z \in TM^\perp$.

We deduce from these formulas that for every $X \in TL$ and $Z \in TM^\perp$,

$$\langle A^Z(X), X \rangle = -\langle D_X Z, X \rangle = \langle A_L^Z(X), X \rangle. \tag{4}$$

Let $k = \dim M$ and let $X_1, X_2, \dots, X_k = T$ be a local orthonormal basis of M around p, where $X_1, X_2, \dots, X_{k-1}$ is a local orthonormal basis of $L = H_p \cap M$.

For every $Z \in TM^\perp$, we have the relation.

$$\sum_{j=1}^{k-1} \langle \alpha(X_j, X_j), Z \rangle = \sum_{j=1}^{k-1} \langle A^Z(X_j), X_j \rangle = \sum_{j=1}^{k-1} \langle A_L^Z(X_j), X_j \rangle = \sum_{j=1}^{k-1} \langle \alpha_L(X_j, X_j), Z \rangle,$$

where α, α_L are the second fundamental forms of $M \subset \mathbf{R}^n$ and $L \subset \mathbf{R}^n$ respectively.

So, we have, for every $Z \in TM^{\perp}$, the relation.

$$\sum_{j=1}^{k-1} \langle \alpha_L(X_j, X_j), Z \rangle = \sum_{j=1}^{k-1} \langle \alpha(X_j, X_j), Z \rangle. \tag{5}$$

Let us observe that $TL^{\perp} = TM^{\perp} \oplus \langle T \rangle$.

Also, we have the relation

$$\sum_{j=1}^{k-1} \langle \alpha_L(X_j, X_j), T \rangle = c \sum_{j=1}^{k-1} \langle \alpha(X_j, X_j), \xi \rangle, \tag{6}$$

where $c \neq 0$ is a constant. It follows from the next calculus:

Since $d = \cos(\theta)T + \sin(\theta)\xi$ and $\alpha_L \in TL^{\perp} \cap TH_p$ (H_p is a hyperplane), we obtain

$$\langle \alpha_L(X_j, X_j), T \rangle = c\langle \alpha_L(X_j, X_j), \xi \rangle = c\langle A_L^{\xi}(X_j), X_j \rangle.$$

Now applying (4), we get

$$\langle \alpha_L(X_j, X_j), T \rangle = c\langle A^{\xi}(X_j), X_j \rangle = c\langle \alpha(X_j, X_j), \xi \rangle.$$

Now, if M is minimal and since $\alpha(T, T) = 0$, the right hand side of (5) and (6) are equal to zero. Which implies that L is minimal in $\mathbf{R}^n$.

Finally, in the case that L is minimal in $\mathbf{R}^n$, it is enough to use that the left hand side of (5) is zero to deduce that M is minimal. $\square$

Let H_p as before, where p is any point of a helix submanifold M.

The next result completes the result that given a helix submanifold M, if we assume any two conditions in $\{M$ *ruled*, M *minimal*, L_p *minimal in* $\mathbf{R}^n\}$, then we can deduce the third condition.

Proposition 7.2. *Let M be a minimal helix submanifold such that, for every $p \in M$, $L := H_p \cap M$ is minimal in $\mathbf{R}^n$. Then M is a ruled helix.*

Proof. Let us observe that the equality (5) holds for every immersed hypersurface L in any Euclidean submanifold M. In particular it is true for $L := H_p \cap M$. Since M and L_p are minimal in $\mathbf{R}^n$, then

$$\sum_{j=1}^{k-1} \alpha(X_j, X_j) + \alpha(T, T) = 0, \quad \sum_{j=1}^{k-1} \alpha_L(X_j, X_j) = 0.$$

Using (5), we conclude that $\alpha(T, T) = 0$. Which is equivalent for M to be a ruled helix. $\square$

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