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TOPOLOGY OF POLAR WEIGHTED HOMOGENEOUS HYPERSURFACES

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Abstract

Polar weighted homogeneous polynomials are special polynomials of real variables $x_i,\ y_i,\ i = 1, \dots, n$ with $z_i = x_i + \sqrt{-1}y_i$ which enjoy a "polar action". In many aspects, their behavior looks like that of complex weighted homogeneous polynomials. We study basic properties of hypersurfaces which are defined by polar weighted homogeneous polynomials.

1. Introduction

We consider a polynomial $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$ where $\mathbf{z} = (z_1, \dots, z_n)$, $\bar{\mathbf{z}} = (\bar{z}_1, \dots, \bar{z}_n)$, $\mathbf{z}^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$ for $\nu = (\nu_1, \dots, \nu_n)$ (respectively $\bar{\mathbf{z}}^\mu = \bar{z}_1^{\mu_1} \cdots \bar{z}_n^{\mu_n}$ for $\mu = (\mu_1, \dots, \mu_n)$) as usual. Here $\bar{z}_i$ is the complex conjugate of z_i. Writing $z_i = x_i + \sqrt{-1}y_i$, it is easy to see that f is a polynomial of $2n$-variables $x_1, y_1, \dots, x_n, y_n$. Thus f can be understood as a real analytic function $f : \mathbf{C}^n \to \mathbf{C}$. We call f *a mixed polynomial* of $z_1, \dots, z_n$.

A mixed polynomial $f(\mathbf{z}, \bar{\mathbf{z}})$ is called *polar weighted homogeneous* if there exist integers $q_1, \dots, q_n$ and $p_1, \dots, p_n$ and positive integers m_r, m_p such that

$$\gcd(q_1, \dots, q_n) = 1, \quad \gcd(p_1, \dots, p_n) = 1,$$

$$\sum_{j=1}^n q_j(\nu_j + \mu_j) = m_r, \quad \sum_{j=1}^n p_j(\nu_j - \mu_j) = m_p, \quad \text{if } c_{\nu,\mu} \neq 0$$

We say $f(\mathbf{z}, \bar{\mathbf{z}})$ is *a polar weighted homogeneous of radial weight type* $(q_1, \dots, q_n; m_r)$ *and of polar weight type* $(p_1, \dots, p_n; m_p)$. We define vectors of rational numbers $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ by $u_i = q_i/m_r$, $v_i = p_i/m_p$ and we call them *the normalized radial* (*respectively polar*) *weights*. Using a polar coordinate (r, η) of $\mathbf{C}^*$ where $r > 0$ and $\eta \in S^1$ with $S^1 = \{\eta \in \mathbf{C} \mid |\eta| = 1\}$, we define *a polar* $\mathbf{C}^*$*-action* on $\mathbf{C}^n$ by

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$$(r,\eta)\circ\mathbf{z} = (r^{q_1}\eta^{p_1}z_1,\dots,r^{q_n}\eta^{p_n}z_n),\quad (r,\eta)\in\mathbf{R}^+\times S^1$$
$$(r,\eta)\circ\bar{\mathbf{z}} = \overline{(r,\eta)\circ\mathbf{z}} = (r^{q_1}\eta^{-p_1}\bar{z}_1,\dots,r^{q_n}\eta^{-p_n}\bar{z}_n).$$

Then f satisfies the functional equality

$$f((r,\eta)\circ(\mathbf{z},\bar{\mathbf{z}})) = r^{m_r}\eta^{m_p}f(\mathbf{z},\bar{\mathbf{z}}). \tag{1}$$

This notion was introduced by Ruas-Seade-Verjovsky [12] implicitly and then by Cisneros-Molina [2].

It is easy to see that such a polynomial defines a global fibration

$$f:\mathbf{C}^n - f^{-1}(0)\to\mathbf{C}^*.$$

The purpose of this paper is to study the topology of the hypersurface $F = f^{-1}(1)$ for a given polar weighted homogeneous polynomial, which is a fiber of the above fibration. Note that F has a canonical stratification

$$F = \amalg_{I\subset\{1,2,\dots,n\}}F^{*I},\quad F^{*I} = F\cap\mathbf{C}^{*I}$$

Our main result is Theorem 10, which describes the topology of F^{*I} for a simplicial polar weighted polynomial.

2. Polar weighted homogeneous hypersurface

This section is the preparation for the later sections. Proposition 2 and Proposition 3 are added for consistency but they are essentially known from the series of works by J. Seade and coauthors [12, 13, 10, 11, 14].

2.1. Smoothness of a mixed hypersurface. Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a mixed polynomial and we consider a hypersurface $V = \{\mathbf{z}\in\mathbf{C}^n; f(\mathbf{z},\bar{\mathbf{z}}) = 0\}$. Put $z_j = x_j + iy_j$. Then $f(\mathbf{z},\bar{\mathbf{z}})$ is a real analytic function of $2n$ variables $(\mathbf{x},\mathbf{y})$ with $\mathbf{x} = (x_1,\dots,x_n)$ and $\mathbf{y} = (y_1,\dots,y_n)$. Put $f(\mathbf{z},\bar{\mathbf{z}}) = g(\mathbf{x},\mathbf{y}) + ih(\mathbf{x},\mathbf{y})$ where g, h are real analytic functions. Recall that

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right),\quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right)$$

Thus

$$\frac{\partial k}{\partial z_j} = \frac{1}{2}\left(\frac{\partial k}{\partial x_j} - i\frac{\partial k}{\partial y_j}\right),\quad \frac{\partial k}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial k}{\partial x_j} + i\frac{\partial k}{\partial y_j}\right)$$

for any analytic function $k(\mathbf{x},\mathbf{y})$. Thus for a complex valued function f, we define

$$\frac{\partial f}{\partial z_j} = \frac{\partial g}{\partial z_j} + i\frac{\partial h}{\partial z_j},\quad \frac{\partial f}{\partial \bar{z}_j} = \frac{\partial g}{\partial \bar{z}_j} + i\frac{\partial g}{\partial \bar{z}_j}$$

We assume that g, h are non-constant polynomials. Then V is a real codimension two subvariety. Put

$$d_{\mathbf{R}}g(\mathbf{x},\mathbf{y}) = \left(\frac{\partial g}{\partial x_1},\dots,\frac{\partial g}{\partial x_n},\frac{\partial g}{\partial y_1},\dots,\frac{\partial g}{\partial y_n}\right) \in \mathbf{R}^{2n}$$

$$d_{\mathbf{R}}h(\mathbf{x},\mathbf{y}) = \left(\frac{\partial h}{\partial x_1},\dots,\frac{\partial h}{\partial x_n},\frac{\partial h}{\partial y_1},\dots,\frac{\partial h}{\partial y_n}\right) \in \mathbf{R}^{2n}$$

For a complex valued mixed polynomial, we use the notation:

$$df(\mathbf{z},\bar{\mathbf{z}}) = \left(\frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_n}\right) \in \mathbf{C}^n, \quad \bar{d}f(\mathbf{z},\bar{\mathbf{z}}) = \left(\frac{\partial f}{\partial \bar{z}_1},\dots,\frac{\partial f}{\partial \bar{z}_n}\right) \in \mathbf{C}^n$$

Recall that a point $\mathbf{z} \in V$ is a singular point of V if and only if the two vectors $dg(\mathbf{x},\mathbf{y})$, $dh(\mathbf{x},\mathbf{y})$ are linearly dependent over $\mathbf{R}$ (see Milnor [4]). This condition is not so easy to be checked, as the calculation of $g(\mathbf{x},\mathbf{y})$, $h(\mathbf{x},\mathbf{y})$ from a given $f(\mathbf{z},\bar{\mathbf{z}})$ is not immediate. However we have

PROPOSITION 1. *The following two conditions are equivalent.*
(1) $\mathbf{z} \in V$ *is a singular point of* V *and* $\dim_{\mathbf{R}}(V,\mathbf{z}) = 2n-2$.
(2) *There exists a complex number* α, $|\alpha| = 1$ *such that* $\overline{df(\mathbf{z},\bar{\mathbf{z}})} = \alpha\,\bar{d}f(\mathbf{z},\bar{\mathbf{z}})$.

Proof. First assume that $d_{\mathbf{R}}g$, $d_{\mathbf{R}}h$ are linearly dependent at $\mathbf{z}$. Suppose for example that $dg(\mathbf{x},\mathbf{y}) \neq 0$ and write $dh(\mathbf{x},\mathbf{y}) = t\,dg(\mathbf{x},\mathbf{y})$ for some $t \in \mathbf{R}$. This implies that

$$\frac{\partial f}{\partial x_j} = (1+ti)\frac{\partial g}{\partial x_j}, \quad \frac{\partial f}{\partial y_j} = (1+ti)\frac{\partial g}{\partial y_j}, \quad \text{thus}$$

$$\frac{\partial f}{\partial z_j} = (1+ti)\left(\frac{\partial g}{\partial x_j} - i\frac{\partial g}{\partial y_j}\right), \quad \frac{\partial f}{\partial \bar{z}_j} = (1+ti)\left(\frac{\partial g}{\partial x_j} + i\frac{\partial g}{\partial y_j}\right).$$

Thus

$$df(\mathbf{z},\bar{\mathbf{z}}) = (1+ti)\left(\frac{\partial g}{\partial x_1} - i\frac{\partial g}{\partial y_1},\dots,\frac{\partial g}{\partial x_n} - i\frac{\partial g}{\partial y_n}\right) = 2(1+ti)\ d_{\mathbf{z}}g(\mathbf{z},\bar{\mathbf{z}})$$

$$\bar{d}f(\mathbf{z},\bar{\mathbf{z}}) = (1+ti)\left(\frac{\partial g}{\partial x_1} + i\frac{\partial g}{\partial y_1},\dots,\frac{\partial g}{\partial x_n} + i\frac{\partial g}{\partial y_n}\right) = 2(1+ti)\ d_{\bar{\mathbf{z}}}g(\mathbf{z},\bar{\mathbf{z}})$$

Here $d_{\mathbf{z}}g = \left(\frac{\partial g}{\partial z_1},\dots,\frac{\partial g}{\partial z_n}\right)$ and $d_{\bar{\mathbf{z}}}g = \left(\frac{\partial g}{\partial \bar{z}_1},\dots,\frac{\partial g}{\partial \bar{z}_n}\right)$. As g is a real valued polynomial, using the equality $\overline{d_{\mathbf{z}}g(\mathbf{x},\mathbf{y})} = d_{\bar{\mathbf{z}}}g(\mathbf{x},\mathbf{y})$ we get

$$\overline{df(\mathbf{z},\bar{\mathbf{z}})} = \frac{1-ti}{1+ti}\,\bar{d}f(\mathbf{z},\bar{\mathbf{z}}).$$

Thus it is enough to take $\alpha = \dfrac{1-ti}{1+ti}$.

Conversely assume that $\overline{df(\mathbf{z},\bar{\mathbf{z}})} = \alpha\,\bar{d}f(\mathbf{z},\bar{\mathbf{z}})$ for some $\alpha = a+bi$ with $a^2+b^2 = 1$. Using the notations

$$d_x g = \left(\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n}\right), \quad d_y g = \left(\frac{\partial g}{\partial y_1}, \dots, \frac{\partial g}{\partial y_n}\right), \text{ etc,}$$

we get

$$(1-a)\, d_x g + b\, d_y g = -b\, d_x h - (1+a)\, d_y h$$
$$-b\, d_x g + (1-a)\, d_y g = (a+1)\, d_x h - b\, d_y h.$$

Solving these equations assuming $a \neq 1$, we get

$$d_{\mathbf{R}} g = (d_x g, d_y g) = \frac{-2b}{(1-a)^2 + b^2}\, d_{\mathbf{R}} h$$

which proves the assertion. If $a = 1$, the above equations implies that $dh_{\mathbf{R}} = 0$ and the linear dependence is obvious. □

2.2. Polar weighted homogeneous hypersurfaces. Let f be a polar weighted homogeneous polynomial of radial weight type $(q_1, \dots, q_n; m_r)$ and of polar weight type $(p_1, \dots, p_n; m_p)$. By differentiating (1) in §1, we get

$$m_r f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{i=1}^{n} q_i \left(\frac{\partial f}{\partial z_i} z_i + \frac{\partial f}{\partial \bar{z}_i} \bar{z}_i\right) \tag{2}$$

$$m_p f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{i=1}^{n} p_i \left(\frac{\partial f}{\partial z_i} z_i - \frac{\partial f}{\partial \bar{z}_i} \bar{z}_i\right). \tag{3}$$

We call these equalities *Euler equalities*. Recall that $\mathbf{C}^n$ has the canonical hermitian inner product defined by

$$(\mathbf{z}, \mathbf{w}) = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n.$$

Identifying $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ by $\mathbf{z} \leftrightarrow (\mathbf{x}, \mathbf{y})$, the Euclidean inner product of $\mathbf{R}^{2n}$ is given as $(\mathbf{z}, \mathbf{w})_{\mathbf{R}} = \Re(\mathbf{z}, \mathbf{w})$. Or we can also write as

$$(\mathbf{z}, \mathbf{w})_{\mathbf{R}} = \frac{1}{2}((\mathbf{z}, \mathbf{w}) + (\bar{\mathbf{z}}, \overline{\mathbf{w}})).$$

Proposition 2. *For any $\alpha \neq 0$, the fiber $F_\alpha := f^{-1}(\alpha)$ is a smooth $2(n-1)$ real-dimensional manifold and it is canonically diffeomorphic to $F_1 = f^{-1}(1)$.*

Proof. Take a point $\mathbf{z} \in F_\alpha$. We consider two particular vectors $\mathbf{v}_r, \mathbf{v}_\theta \in T_{\mathbf{z}}\mathbf{C}^n$ which are the tangent vectors of the respective orbits of $\mathbf{R}$ and S^1:

$$\mathbf{v}_r = \left.\frac{d(r \circ \mathbf{z})}{dr}\right|_{r=1} = (q_1 z_1, \dots, q_n z_n),$$

$$\mathbf{v}_\theta = \left.\frac{d(e^{i\theta} \circ \mathbf{z})}{d\theta}\right|_{\theta=0} = (i p_1 z_1, \dots, i p_n z_n).$$

Taking the differential of the equality

$$f((r, \exp(i\theta)) \circ \mathbf{z})) = r^{m_r} \exp(m_p \theta i) f(\mathbf{z}, \bar{\mathbf{z}}),$$

we see that $df_z : T_{\mathbf{z}}\mathbf{C}^n \to T_\alpha \mathbf{C}^*$ satisfies

$$df_z(\mathbf{v}_r) = m_r|\alpha|\frac{\partial}{\partial r}, \quad df_z(\mathbf{v}_\theta) = m_p\frac{\partial}{\partial\theta}$$

where (r, θ) is the polar coordinate of $\mathbf{C}^*$. This implies that $f : \mathbf{C}^n \to \mathbf{C}$ is a submersion at $\mathbf{z}$. Thus F_α is a smooth codimension 2 submanifold. A diffeomorphism $\varphi_\alpha : F_1 \to F_\alpha$ is simply given as $\varphi(\mathbf{z}) = (r^{1/m_r}, \exp^{i\theta/m_p}) \circ \mathbf{z}$ where $\alpha = r\exp(i\theta)$. □

The above proof does not work for $\alpha = 0$. Recall that the polar $\mathbf{R}^+$-action along the radial direction is written in real coordinates as

$$r \circ (\mathbf{x}, \mathbf{y}) = (r^{q_1}x_1, \dots, r^{q_n}x_n, r^{q_1}y_1, \dots, r^{q_n}y_n), \quad r \in \mathbf{R}^+.$$

PROPOSITION 3. *Let $V = f^{-1}(0)$. Assume that $q_j > 0$ for any j. Then V is contractible to the origin O. If further O is an isolated singularity of V, $V \backslash \{O\}$ is smooth.*

Proof. A canonical deformation retract $\beta_t : V \to V$ is given as $\beta_t(\mathbf{z}) = t \circ \mathbf{z}$, $0 \le t \le 1$. (More precisely $\beta_0(\mathbf{z}) = \lim_{t\to 0} \beta_t(\mathbf{z})$.) Then $\beta_1 = \mathrm{id}_V$ and β_0 is the contraction to O. Assume that $\mathbf{z} \in V\backslash\{O\}$ is a singular point. Consider the decomposition into real analytic functions $f(z) = g(\mathbf{x}, \mathbf{y}) + ih(\mathbf{x}, \mathbf{y})$. Using the radial $\mathbf{R}^+$-action, we see that

$$g(r \circ (\mathbf{x}, \mathbf{y})) = r^{m_r} g(\mathbf{x}, \mathbf{y}), \quad h(r \circ (\mathbf{x}, \mathbf{y})) = r^{m_r} h(\mathbf{x}, \mathbf{y}). \tag{4}$$

This implies that $g(\mathbf{x}, \mathbf{y})$, $h(\mathbf{x}, \mathbf{y})$ are weighted homogeneous polynomials of $(\mathbf{x}, \mathbf{y})$ and the Euler equality can be restated as

$$m_r g(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^n p_j \left(x_j \frac{\partial g}{\partial x_j}(\mathbf{x}, \mathbf{y}) + y_j \frac{\partial g}{\partial y_j}(\mathbf{x}, \mathbf{y}) \right)$$

$$m_r h(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^n p_j \left(x_j \frac{\partial h}{\partial x_j}(\mathbf{x}, \mathbf{y}) + y_j \frac{\partial h}{\partial y_j}(\mathbf{x}, \mathbf{y}) \right).$$

Differentiating the equalities (4) in r, we get

$$\frac{\partial g}{\partial x_j}(r \circ (\mathbf{x}, \mathbf{y})) = r^{m_r - q_j} \frac{\partial g}{\partial x_j}(\mathbf{x}, \mathbf{y}), \quad \frac{\partial h}{\partial x_j}(r \circ (\mathbf{x}, \mathbf{y})) = r^{m_r - q_j} \frac{\partial h}{\partial x_j}(\mathbf{x}, \mathbf{y}).$$

This implies that these differentials are also weighted homogeneous polynomials of degree $m_r - q_j$. Thus the jacobian matrix

$$\left(\frac{\partial(g, h)}{\partial(x_i, y_i)}(r \circ (\mathbf{x}, \mathbf{y})) \right)$$

is the same with the jacobian matrix at $\mathbf{z} = (\mathbf{x}, \mathbf{y})$ up to scalar multiplications in the column vectors by $r^{m_r - q_1}, \dots, r^{m_r - q_n}$, $r^{m_r - q_1}, \dots, r^{m_r - q_n}$ respectively. Thus any points of the orbit $r \circ (\mathbf{x}, \mathbf{y})$, $r > 0$ are singular points of V. This is a contradiction to the assumption that O is an isolated singular point of V, as $\lim_{r \to 0} r \circ (\mathbf{x}, \mathbf{y}) = O$. □

Proposition 4. (*Transversality*) *Under the same assumption as in Proposition* 3, *the sphere* $S_\tau = \{\mathbf{z} \in \mathbf{C}^n; |\mathbf{z}| = \tau\}$ *intersects transversely with* V *for any* $\tau > 0$.

Proof. Let $\phi(\mathbf{x}, \mathbf{y}) = \|\mathbf{z}\|^2 = \sum_{j=1}^n (x_j^2 + y_j^2)$. Then S_τ intersects transversely with V if and only if the gradient vectors $d_{\mathbf{R}}g$, $d_{\mathbf{R}}h$, $d_{\mathbf{R}}\phi$ are linearly independent over $\mathbf{R}$. Note that $d_{\mathbf{R}}\phi(\mathbf{x}, \mathbf{y}) = 2(\mathbf{x}, \mathbf{y})$. Suppose that the sphere $S_{\|\mathbf{z}\|}$ is tangent to V at $\mathbf{z} = (\mathbf{x}, \mathbf{y}) \in V$. Then we have for example, a linear relation $dg(\mathbf{x}, \mathbf{y}) = \alpha\, dh(\mathbf{x}, \mathbf{y}) + \beta\, d\phi(\mathbf{x}, \mathbf{y})$ with some $\alpha, \beta \in \mathbf{R}$. Note that the tangent vector $\mathbf{v}_r$ to the $\mathbf{R}^+$-oribit is tangent to V and it is written $\mathbf{v}_r = (q_1 x_1, \dots, q_n x_n, q_1 y_1, \dots, q_n y_n)$ as a real vector. Then we have

$$
\begin{aligned}
0 = \left.\frac{dg(r \circ (\mathbf{x}, \mathbf{y}))}{dr}\right|_{r=1} &= \sum_{j=1}^n q_j \left(x_j \frac{\partial g}{\partial x_j}(\mathbf{x}, \mathbf{y}) + y_j \frac{\partial g}{\partial y_j}(\mathbf{x}, \mathbf{y}) \right) \\
&= (\mathbf{v}_r(\mathbf{x}, \mathbf{y}), dg(\mathbf{x}, \mathbf{y}))_{\mathbf{R}} \\
&= (\mathbf{v}_r(\mathbf{x}, \mathbf{y}), \alpha\, dh(\mathbf{x}, \mathbf{y}))_{\mathbf{R}} + (\mathbf{v}_r(\mathbf{x}, \mathbf{y}), \beta\, d\phi(\mathbf{x}, \mathbf{y}))_{\mathbf{R}} \\
&= 2\beta \sum_{j=1}^n q_j (x_j^2 + y_j^2)
\end{aligned}
$$

as $(\mathbf{v}_r(\mathbf{x}, \mathbf{y}), dh(\mathbf{x}, \mathbf{y}))_{\mathbf{R}} = 0$ by the same reason. This is the case only if $\beta = 0$ which is impossible as $V \setminus \{O\}$ is non-singular by Proposition 3. □

2.2.1. *Remark*. Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a polar weighted homogeneous polynomial with respective weights $(q_1, \dots, q_n; m_r)$ and $(p_1, \dots, p_n; m_p)$. Proposition 3 does not hold if the radial weights contain some negative q_j. Assume that $q_j \geq 0$ for any j and $I_0 := \{j \mid q_j = 0\}$ is not empty. Then it is easy to see that f does not have monomial which does not contain any z_i with $i \notin I_0$, as if such monomial exists, its radial degree is 0. This implies that $V = f^{-1}(0)$ contains the coordinate subspace $\mathbf{C}^{I_0} = \{\mathbf{z} \mid z_i = 0, i \notin I_0\}$. We call $\mathbf{C}^{I_0}$ *the canonical retract coordinate subspace*. Then Proposition 3 can be modified as $\mathbf{C}^{I_0}$ *is a deformation retract of* V. Of course, $\mathbf{C}^{I_0}$ can be contracted to O but this contraction is not through the action and not related to the geometry of V.

2.2.2. *Example.* Consider the following examples.

$$g_1(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1} \bar{z}_2 + \cdots + z_n^{a_n} \bar{z}_1, \quad a_i \geq 1,\ j = 1, \dots, n$$

and there exists j such that $a_j \geq 2$

$$g_2(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1} \bar{z}_2 + \cdots + z_{n-1}^{a_{n-1}} \bar{z}_n + z_n^{a_n}, \quad a_j \geq 1,\ j = 1, \dots, n.$$

Proposition 5. (1) *The radial weight vector* $(q_1, \dots, q_n)$ *of* $g_1(\mathbf{z}, \bar{\mathbf{z}})$ *is semi-positive, i.e.* $q_j \geq 0$ *for any* j *if* $a_i \geq 1$ *for any* i. ($\exists j$, $a_j \geq 2$ *by the existence of polar action.*) *It is not strictly positive if and only if* $n = 2m$ *is even and either* (a) $a_1 = a_3 = \cdots = a_{2m-1} = 1$ *or* (b) $a_2 = a_4 = \cdots = a_{2m} = 1$.

In case (a) (*respectively* (b)), *we have* $q_2 = q_4 = \cdots = q_{2m} = 0$ *and* $q_{2j+1} \geq 1$, $0 \leq j \leq m-1$ (*resp.* $q_1 = q_3 = \cdots = q_{2m-1} = 0$ *and* $q_{2j} \geq 1$, $1 \leq j \leq m$).

(2) *The radial weight vector* $(q_1, \dots, q_n)$ *of* $g_2(\mathbf{z}, \bar{\mathbf{z}})$ *is semi-positive. It is not strictly positive if and only if* $a_n = 1$. *Let* s *be the integer such that* $a_n = a_{n-2} = \cdots = a_{n-2s} = 1$ *and* $a_{n-2s-2} \geq 2$. *Then* $q_{n-1} = \cdots = q_{n-2s+1} = 0$ *and* $q_j \geq 1$ *otherwise.*

Proof. We first consider $g_1(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1} \bar{z}_2 + \cdots + z_n^{a_n} \bar{z}_1$. By an easy calculation, using the notation $a_{i+n} = a_i$ the normalized radial weigts $(u_1, \dots, u_n)$ are given as

$$u_j = \frac{1}{a_1 \cdots a_n - 1} \sum_{i=0}^{m-1} (a_{j+2i+1} - 1) a_{j+2i+2} \cdots a_{j+n-1}, \quad \text{if } n = 2m$$

$$u_j = \frac{1}{a_1 \cdots a_n + 1} \left(1 + \sum_{i=0}^{m-1} (a_{j+2i+1} - 1) a_{j+2i+2} \cdots a_{j+n-1} \right), \quad \text{if } n = 2m+1$$

and the assertion follows immediately from this expression.

Next we consider $g_2(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1} \bar{z}_2 + \cdots + z_n^{a_{n-1}} \bar{z}_n + z_n^{a_n}$. Then the normalized radial weigts $(u_1, \dots, u_n)$ are given as

$$\begin{aligned} u_j &= \frac{1}{a_j} - \frac{1}{a_j a_{j+1}} + \cdots + (-1)^{n-j} \frac{1}{a_j a_{j+1} \cdots a_n} \\ &= \begin{cases} \dfrac{a_{j+1} - 1}{a_j a_{j+1}} + \cdots + \dfrac{a_n - 1}{a_j a_{j+1} \cdots a_n}, & n - j : \text{odd} \\ \dfrac{a_{j+1} - 1}{a_j a_{j+1}} + \cdots + \dfrac{a_{n-1} - 1}{a_j a_{j+1} \cdots a_{n-1}} + \dfrac{1}{a_j a_{j+1} \cdots a_n} & n - j : \text{even} \end{cases} \end{aligned}$$

As $a_i \geq 1$, the assertion follows from the above expression. □

2.3. Simplicial mixed polynomial. Let $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{j=1}^{s} c_j \mathbf{z}^{\mathbf{n}_j} \bar{\mathbf{z}}^{\mathbf{m}_j}$ be a mixed polynomial. Here we assume that $c_1, \dots, c_s \neq 0$. Put

$$\hat{f}(\mathbf{w}) := \sum_{j=1}^{s} c_j \mathbf{w}^{\mathbf{n}_j - \mathbf{m}_j}.$$

We call $\hat{f}$ the *the associated Laurent polynomial.* This polynomial plays an important role for the determination of the topology of the hypersurface $F = f^{-1}(1)$. Note that

Proposition 6. *If $f(\mathbf{z},\bar{\mathbf{z}})$ is a polar weighted homogeneous polynomial of polar weight type $(p_1,\dots,p_n;m_p)$, $\hat{f}(\mathbf{w})$ is also a weighted homogeneous Laurent polynomial of type $(p_1,\dots,p_n;m_p)$ in the complex variables $w_1,\dots,w_n$.*

A mixed polynomial $f(\mathbf{z},\bar{\mathbf{z}})$ is called *simplicial* if the exponent vectors $\{\mathbf{n}_j \pm \mathbf{m}_j \mid j=1,\dots,s\}$ are linearly independent in $\mathbf{Z}^n$ respectively. In particular, simplicity implies that $s \le n$. When $s=n$, we say that f is *full*. Put $\mathbf{n}_j = (n_{j,1},\dots,n_{j,n})$, $\mathbf{m}_j = (m_{j,1},\dots,m_{j,n})$ in $\mathbf{N}^n$. Assume that $s \le n$. Consider two integral matrix $N=(n_{i,j})$ and $M=(m_{i,j})$ where the k-th row vectors are $\mathbf{n}_k$, $\mathbf{m}_k$ respectively.

Lemma 7. *Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a mixed polynomial as above. If $f(\mathbf{z},\bar{\mathbf{z}})$ is simplicial, then $f(\mathbf{z},\bar{\mathbf{z}})$ is a polar weighted homogeneous polynomial. In the case $s=n$, $f(\mathbf{z},\bar{\mathbf{z}})$ is simplicial if and only if* $\det(N \pm M) \neq 0$.

Proof. First we assume that $s=n$ and consider the system of linear equations

$$(5)\qquad \begin{cases} (n_{1,1}+m_{1,1})u_1+\cdots+(n_{1,n}+m_{1,n})u_n = 1 \\ \quad\cdots \\ (n_{n,1}+m_{n,1})u_1+\cdots+(n_{n,n}+m_{n,n})u_n = 1 \end{cases}$$

$$(6)\qquad \begin{cases} (n_{1,1}-m_{1,1})v_1+\cdots+(n_{1,n}-m_{1,n})v_n = 1 \\ \quad\cdots \\ (n_{n,1}-m_{n,1})v_1+\cdots+(n_{n,n}-m_{n,n})v_n = 1 \end{cases}$$

It is easy to see that equations (5) and (6) have solutions if $\det N \pm M \neq 0$ which is equivalent for f to be simplicial by definition. Note that the solutions $(u_1,\dots,u_n)$ and $(v_1,\dots,v_n)$ are rational numbers. We call them *the normalized radial* (*respectively polar*) *weights*. Now let m_r, m_p be the least common multiple of the denominators of $u_1,\dots,u_n$ and $v_1,\dots,v_n$ respectively. Then the weights are given as $q_j = u_j m_r$, $p_j = v_j m_p$, $j=1,\dots,n$ respectively.

Now suppose that $s<n$. It is easy to choose positive integral vectors $\mathbf{n}_j$, $j=s+1,\dots,n$ (and put $\mathbf{m}_j=0$, $j=s+1,\dots,n$) such that $\det(\tilde{N}\pm\tilde{M}) \neq 0$, where $\tilde{N}$ and $\tilde{M}$ are $n\times n$-matrices adding $(n-s)$ row vectors $\mathbf{n}_{s+1},\dots,\mathbf{n}_n$. Then the assertion follows from the case $s=n$. □

This corresponds to considering the mixed polynomial:

$$f(\mathbf{z},\bar{\mathbf{z}}) = \sum_{j=1}^{s} c_j \mathbf{z}^{\mathbf{n}_j}\bar{\mathbf{z}}_j^{\mathbf{m}_j} + 0 \times \sum_{j=s+1}^{n} \mathbf{z}^{\mathbf{n}_j}.$$

2.3.1. *Example.* Let

$$f_{\mathbf{a},\mathbf{b}}(\mathbf{z},\bar{\mathbf{z}}) = z_1^{a_1}\bar{z}_2^{b_1}+\cdots+z_n^{a_n}\bar{z}_1^{b_n},\quad a_i,b_i \ge 1,\ i=1,\dots,n$$

$$k(\mathbf{z},\bar{\mathbf{z}}) = z_1^d(\bar{z}_1+\bar{z}_2)+\cdots+z_n^d(\bar{z}_n+\bar{z}_1),\quad d \ge 2.$$

The associated Laurent polynomials are

$$\widehat{f_{\mathbf{a},\mathbf{b}}}(\mathbf{w}) = w_1^{a_1} w_2^{-b_1} + \cdots + w_n^{a_n} w_1^{-b_n}$$
$$\hat{k}(\mathbf{w}) = w_1^d(1/w_1 + 1/w_2) + \cdots + w_n^d(1/w_n + 1/w_1).$$

Corollary 8. *For the polynomial $f_{\mathbf{a},\mathbf{b}}$, the following conditions are equivalent.*

(1) *$f_{\mathbf{a},\mathbf{b}}$ is simplicial.*
(2) *$f_{\mathbf{a},\mathbf{b}}$ is a polar weighted homogeneous polynomial.*
(3) (SC) $a_1 \cdots a_n \neq b_1 \cdots b_n$.

Proof. The assertion follows from the equality:

$$\det(\mathbf{n} \pm \mathbf{m}) = \det\begin{pmatrix} a_1 & 0 & \cdots & \pm b_n \\ \pm b_1 & a_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \pm b_{n-1} & a_n \end{pmatrix}$$
$$= \begin{cases} a_1 a_2 \cdots a_n + (-1)^{n-1} b_1 b_2 \cdots b_n & \text{for } \mathbf{n}+\mathbf{m} \\ a_1 a_2 \cdots a_n - b_1 b_2 \cdots b_n & \text{for } \mathbf{n}-\mathbf{m}. \end{cases}$$ □

The polynomial $k(\mathbf{z}, \bar{\mathbf{z}})$ is a polar weighted homogeneous polynomial with respective weight types $(1, \dots, 1; d+1)$ and $(1, \dots, 1; d-1)$. However it is not simplicial.

Now we consider an example which does not satisfy the simplicial condition (SC) of Corollary 8: $\phi_a := z_1^a \bar{z}_1^a + \cdots + z_n^a \bar{z}_n^a$. This does not have any polar action as they are polynomials of $|z_1|^2, \dots, |z_n|^2$ and it takes only non-negative values. Note also that $\phi_a^{-1}(1)$ is real codimension 1 as $\phi_a(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^n (x_j^2 + y^2)^a$.

As typical simplicial polar weighted polynomials, we consider again the following two polar weighted polynomials.

$$g_1(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1} \bar{z}_2 + \cdots + z_n^{a_n} \bar{z}_1, \quad a_i \geq 1,\ j = 1, \dots, n$$
and there exists j such that $a_j \geq 2$
$$g_2(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1} \bar{z}_2 + \cdots + z_{n-1}^{a_{n-1}} \bar{z}_n + z_n^{a_n}, \quad a_i \geq 1,\ j = 1, \dots, n.$$

The polynomial $g_1(\mathbf{z}, \bar{\mathbf{z}})$ with $a_i \geq 2$, $(i = 1, \dots, n)$ is a special case of σ-twisted Brieskorn polynomial and has been studied intensively ([12]). In our case, we only assume $a_i \geq 2$ for some i. The existence of i with $a_i \geq 2$ is the condition for the existence of polar action. We consider the two hypersurfaces defined by $V_i = g_i^{-1}(0)$ for $i = 1, 2$. The condition for a hypersurface defined by a polar weighted homogeneous polynomial to have an isolated singularity is more complicated than that of the singularity defined by a complex anaytic hypersurface. For the above examples, we assert the following.

Proposition 9. *For V_1, V_2, we have the following criterion.*

(1) $V_i \cap \mathbf{C}^{*n}$, $i = 1, 2$ *are non-singular.*

(2) $V_1 = g_1^{-1}(0)$ *has no singularity outside of the origin if and only if one of the following conditions is satisfied.*

  (a) *n is odd.*

  (b) *n is even and there are (at least) two indices i, j $(i < j)$ such that $a_i, a_j \geq 2$ and $j - i$ is odd.*

(3) $V_2 = g_2^{-1}(0)$ *has no singularity outside of the origin if and only if one of the following conditions is satisfied.*

  (a) $a_n \geq 2$.

  (b) $a_n = 1$, *$n = 2m + 1$ is odd and $a_{2j-1} = 1$ for any $1 \leq j \leq m + 1$.*

Proof. We use Proposition 1. So assume that

$$(\#):\quad \overline{df(\mathbf{z}, \bar{\mathbf{z}})} = \alpha\, \bar{d}f(\mathbf{z}, \bar{\mathbf{z}}), \quad |\alpha| = 1.$$

(1) We consider V_1. Suppose $\mathbf{z} \in V_1 \cap \mathbf{C}^{*n}$ is a singular point. Note that

$$df(\mathbf{z}, \bar{\mathbf{z}}) = (a_1 z_1^{a_1 - 1} \bar{z}_2, \dots, a_n z_n^{a_n - 1} \bar{z}_1), \quad \bar{d}f(\mathbf{z}, \bar{\mathbf{z}}) = (z_n^{a_n}, z_1^{a_1}, \dots, z_{n-1}^{a_{n-1}})$$

$(\#)$ implies that

$$a_j \bar{z}_j^{a_j - 1} z_{j+1} = \alpha z_{j-1}^{a_{j-1}}, \quad j = 1, \dots, n,\ |\alpha| = 1. \tag{7}$$

In this case, indices should be understood to be integers modulo n. So $z_{n+1} = z_1$, and so on. If $\mathbf{z} \in \mathbf{C}^{*n}$, the multiplication of the absolute values of the both sides gives a contradiction: $\prod_{i=1}^n a_i |z_i|^{a_i} = \prod_{i=1}^n |z_i|^{a_i}$.

Now we consider the smoothness on $V_1 \setminus \{O\}$. Assume that $\mathbf{z}$ is a singular point of $V_1 \setminus \{O\}$. For simplicity, we may assume that $a_n \geq 2$ as g_1 is symmetric with the permutation $i \to i + 1$.

Assume that $z_\iota \neq 0$. Then the $(\iota + 1)$-th component of $\bar{d}f(\mathbf{z}, \bar{\mathbf{z}})$ is non-zero. Thus by $(\#)$, $(\iota + 1)$-th component of $df(\mathbf{z}, \bar{\mathbf{z}})$ is also non-zero. That is, $z_{\iota+1}^{a_\iota - 1} \bar{z}_{\iota+2} \neq 0$. In particular, $z_{\iota+2} \neq 0$. We repeat the same argument and get a sequence of non-zero components $z_\iota, z_{\iota+2}, \dots$. Thus we arrive to the conclusion that either $z_{n-1} \neq 0$ (if $n - \iota$ is odd) or $z_n \neq 0$ (if $n - \iota$ is even).

– If $n - \iota$ is odd and $z_{n-1} \neq 0$, the last component of $df(\mathbf{z}, \bar{\mathbf{z}})$ is non-zero and we have $z_n, z_1 \neq 0$ as we have assumed that $a_n \geq 2$. This creates two non-zero sequences $z_n, z_2, z_4, \dots$ and $z_1, z_3, \dots$. Thus we conclude that $\mathbf{z} \in \mathbf{C}^{*n}$, which is impossible by the first argument.

– If $n - \iota$ is even, $z_\iota, z_{\iota+2}, \dots, z_n \neq 0$. Thus we see that the first component of $\bar{d}f(\mathbf{z}, \bar{\mathbf{z}})$ is non-zero. By the same argument, we get a non-zero sequence $z_2, z_4, \dots$.

Thus to show that $\mathbf{z} \in \mathbf{C}^{*n}$, it is enough to show that $z_{n-1} \neq 0$.

(a) Assume first n is odd. If ι is even, then we see that $z_\iota, z_{\iota+2}, \dots, z_{n-1} \neq 0$ and we are done.

If ι is odd, we get $z_n \neq 0$, which implies the first component of $\bar{d}f(\mathbf{z}, \bar{\mathbf{z}})$ is non-zero. Thus as the second round, we have non-zero a sequence $z_2, z_4, \dots$ which contains z_{n-1}. Thus we are done.

(b) Now we assume that n is even but there is another integer $1 \le i < n$ such that $a_i \ge 2$ and $a_n \ge 2$ and i is odd. If $\imath$ is odd, we have shown that $\mathbf{z} \in \mathbf{C}^{*n}$. If $\imath$ is even, we get $z_n \ne 0$ and thus $z_2 \ne 0$. Then the sequence $z_2, z_4, \dots$ contains z_{i-1}. As $a_i \ge 2$, looking at the i-th component of $df(\mathbf{z}, \bar{\mathbf{z}})$, we get $z_i \cdot z_{i+1} \ne 0$. Thus we get a non-zero sequence $z_i, z_{i+2}, \dots$ which contains z_{n-1}, and we are done.

Now to show that one of the conditions (a) or (b) is necessary, we assume that n is even and $a_\nu = 1$ for any odd ν and $a_n \ge 2$. Thus putting $n = 2m$,

$$f = (z_1\bar{z}_2 + z_2^{a_2}\bar{z}_3) + \cdots + (z_{2m-1}\bar{z}_{2m} + z_{2m}^{a_{2m}}\bar{z}_1).$$

Consider the subvariety $z_1 = z_3 = \cdots = z_{n-1} = 0$. Then

$$df(\mathbf{z}, \bar{\mathbf{z}}) = (\bar{z}_2, 0, \bar{z}_4, 0, \dots, \bar{z}_{2m}, 0), \quad \bar{d}f(\mathbf{z}, \bar{\mathbf{z}}) = (z_n^{a_n}, 0, \dots, z_{2m-2}^{a_{2m-2}}, 0)$$

the condition (#) is written as

$$(\#) \quad z_2 = \alpha z_n^{a_n}, \quad z_4 = \alpha z_2^{a_2}, \dots, z_{2m} = \alpha z_{2m-2}^{a_{2m-2}}$$

which has real one-dimensional solution

$$z_{2j} = \alpha^{\beta_j} u^{\gamma_j} \quad (j = 1, \dots, m), \quad \alpha^{\beta_m} u^{\gamma_m a_{2m} - 1} = 1$$

$$\beta_j = 1 + \sum_{i=1}^{j-1} a_{2(j-1)} a_{2(j-2)} \cdots a_{2(j-i)}, \quad \gamma_j = a_2 a_4 \cdots a_{2(j-1)}$$

(2) We consider the case V_2. We will see first $V_2 \cap \mathbf{C}^{*n}$ is non-singular. Take a singular point of V_2. Then we have some $\alpha \in S^1$ so that

$$(\#): \quad \overline{df}(\mathbf{z}, \bar{\mathbf{z}}) = \alpha\, \bar{d}f(\mathbf{z}, \bar{\mathbf{z}}).$$

As we have

$$df(\mathbf{z}, \bar{\mathbf{z}}) = (a_1 z_1^{a_1 - 1}\bar{z}_2, \dots, a_{n-1} z_{n-1}^{a_{n-1}-1}\bar{z}_n, a_n z_n^{a_n - 1}),$$
$$\bar{d}f(\mathbf{z}, \bar{\mathbf{z}}) = (0, z_1^{a_1}, \dots, z_{n-1}^{a_{n-1}})$$

we see that (#) implies that $z_1^{a_1-1}\bar{z}_2 = 0$. Thus there are no singularities on $V_2 \cap \mathbf{C}^{*n}$. Suppose that $z_\imath \ne 0$ for some $\imath$. If $\imath < n-1$, this implies $(\imath+1)$-th component of $\bar{d}f(\mathbf{z}, \bar{\mathbf{z}})$ is non-zero. Thus (#) implies that $(\imath+1)$-th component of df is non-zero. In particular, $z_{\imath+2}$ is non-zero. (Of course, $z_{\imath+1} \ne 0$ if $a_{\imath+1} > 1$.) Repeating this argument, we arrive to the conclusion: either z_{n-1} or z_n is non zero.

First assume that $a_n \ge 2$. Comparing the last components of $df(\mathbf{z}, \bar{\mathbf{z}})$ and $\bar{d}f(\mathbf{z}, \bar{\mathbf{z}})$, we observe that z_{n-1} and z_n are both non-zero. Now we go in the reverse direction. As the $(n-1)$-th component of $df(\mathbf{z}, \bar{\mathbf{z}})$ is non-zero, the corresponding $(n-1)$-th component $z_{n-2}^{a_{n-2}}$ of $\bar{d}f(\mathbf{z}, \bar{\mathbf{z}})$ is non-zero. Then the $(n-2)$-th component of $df(\mathbf{z}, \bar{\mathbf{z}})$ is non-zero. Going downwords, we see that $z \in \mathbf{C}^{*n}$. However this is impossible, as we have already seen above.

Next we assume that $a_n = 1$ and n is odd and $a_{2j-1} = 1$ for any j. Note that the last component of $df(\mathbf{z}, \bar{\mathbf{z}})$ is 1. Thus $z_{n-1} \ne 0$. If $z_n \ne 0$, we get a contradiction as above $\mathbf{z} \in \mathbf{C}^{*n}$. Thus we may assume that $z_n = 0$. Comparing $(2j)$-components of $df(\mathbf{z}, \bar{\mathbf{z}})$ and $\alpha\, \bar{d}f(\mathbf{z}, \bar{\mathbf{z}})$, we get

$$z_2 = 0, \quad z_4 = \alpha z_2^{a_2}, \dots, z_{n-1} = z_{n-3}^{a_{n-3}}$$

which has no solution with $z_{n-1} \neq 0$.

Now we show that the condition (a) or (b) in (3) is necessary.

(i) Assume that $a_n = 1$ and n is even ans put $n = 2m$. Let s be the maximal integer such that $a_{2s} \geq 2$. If there does not exists such s, we put $s = 0$. Non-isolated singularities are given by the solutions of

$$z_2 = z_4 = \cdots = z_{2m} = 0, \quad z_{2j-1} = 0,\ j \leq s$$
$$z_{2s+3} = \alpha z_{2s+1}^{a_{2s+1}}, \dots, z_{2m-1} = \alpha z_{2m-3}^{a_{2m-3}}, \quad 1 = \alpha z_{2m-1}^{a_{2m-1}}.$$

(ii) Assume that $a_n = 1$, $n = 2m+1$ is odd, and there exists odd index such that $a_{2j+1} \geq 2$. Put s be the maximum integer of such j. Non-isolated singularities are given by the solutions of

$$z_1 = z_3 = \cdots = z_{2m+1} = 0, \quad z_{2j} = 0,\ j \leq s$$
$$z_{2s+4} = \alpha z_{2s+2}^{a_{2s+2}}, \dots, z_{2m} = \alpha z_{2m-2}^{a_{2m}}, \quad 1 = \alpha z_{2m}^{a_{2m}}. \qquad \square$$

2.3.2. *Remark.* 1. The polynomial $g_1(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1} \bar{z}_2 + \cdots + z_n^{a_n} \bar{z}_1$ is an example of so-called *σ-twisted Brieskorn polynomial* if $a_i \geq 2$, $i = 1, \dots, n$. Let σ be a permutation of $\{1, 2, \dots, n\}$. Then σ-twisted Brieskorn polynomial is defined as

$$f_\sigma(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1} \bar{z}_{\sigma(1)} + \cdots + z_n^{a_n} \bar{z}_{\sigma(n)}, \quad a_1, \dots, a_n \geq 2.$$

and the corresponding assertions in Proposition 3 and 4 are proved in [13]. See also [14] for more systematical treatment for real analytic polynomials which define Milnor fibrations. In [3], similar conditions for the isolatedness condition as Proposition 9 are considered. For our purpose, we call $f_\sigma(\mathbf{z}, \bar{\mathbf{z}})$ a *weak σ-twisted Brieskorn polynomial* if $\sigma \in \mathcal{S}_n$ and $a_i \geq 1$ for any $i = 1, \dots, n$.

2. Consider a product $\mathbf{C}^n = \mathbf{C}^s \times \mathbf{C}^{n-s}$ and use variables $\mathbf{v} \in \mathbf{C}^s$ and $\mathbf{w} \in \mathbf{C}^{n-s}$. Assume that there exist mixed polynomials $h(\mathbf{v}, \bar{\mathbf{v}})$ and $k(\mathbf{w}, \bar{\mathbf{w}})$ so that $f(\mathbf{z}, \bar{\mathbf{z}}) = h(\mathbf{v}, \bar{\mathbf{v}}) + k(\mathbf{w}, \bar{\mathbf{w}})$. $f(\mathbf{z}, \bar{\mathbf{z}})$ is a polar weighted polynomial if and only if $h(\mathbf{v}, \bar{\mathbf{v}})$, $k(\mathbf{w}, \bar{\mathbf{w}})$ are polar weighted polynomial and it is known that $f^{-1}(1)$ is homotopic to the join $h^{-1}(1) \star k^{-1}(1)$ if f is polar weighted. Such a polynomial is called a *polynomial of join type* ([2], see also [6]).

Now consider a weak σ-twisted Brieskorn polynomial $f_\sigma(\mathbf{z}, \bar{\mathbf{z}})$. If σ has order n, it is (up to a change of ordering) equal to the cyclic permutation $\sigma = (1, 2, \dots, n)$ and $f_\sigma = g_1$. In general, σ can be written as a product of mutually commuting cyclic permutations $\sigma = \tau_1 \tau_2 \cdots \tau_\nu$. Put $|\tau_i| = \{j \mid \tau_i(j) \neq j\}$ and put f_{τ_i} be the partial sum of monomials in $f(\mathbf{z}, \bar{\mathbf{z}})$ written in variables $\{z_j \mid j \in |\tau_i|\}$. Thus f_σ is a join type polynomial of ν weak τ_i-twisted Brieskorn polynomial f_{τ_i}. Thus $f_\sigma(\mathbf{z}, \bar{\mathbf{z}})$ has an isolated singularity if and only if each polynomial f_{τ_i} has an isolated singularity. A similar assertion is also proved in [3].

3. Observe that the singularities of V_1, V_2 are on the canonical retract coordinate subspaces $\mathbf{C}^{I_0}$. Note also that the polar action is trivial on $\mathbf{C}^{I_0}$.

2.4. Milnor fibration. Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a polar weighted homogeneous polynomial of radial weight type $(q_1, \dots, q_n; m_r)$ and of polar weight type $(p_1, \dots, p_n; m_p)$. Then

$$f : \mathbf{C}^n - f^{-1}(0) \to \mathbf{C}^*$$

is a locally trivial fibration. The local triviality is given by the action. In particular, the monodromy map $h : F \to F$ is given by $h(\mathbf{z}) = \exp(2\pi i/m_p) \circ \mathbf{z} = (z_1 \exp(2p_1\pi i/m_p), \dots, z_n \exp(2p_n\pi i/m_p))$ where $F = f^{-1}(1)$ ([12, 2]).

3. Topology of simplicial polar weighted homogeneous hypersurfaces

Let $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{j=1}^{s} c_j \mathbf{z}^{\mathbf{n}_j} \bar{\mathbf{z}}^{\mathbf{m}_j}$ be a polar weighted homogeneous polynomial of radial weight type $(q_1, \dots, q_n; m_r)$ and of polar weight type $(p_1, \dots, p_n; m_p)$. Let $F = f^{-1}(1)$ be the fiber.

3.1. Canonical stratification of F and the topology of each stratum. For any subset $I \subset \{1, 2, \dots, n\}$, we define

$$\mathbf{C}^I = \{\mathbf{z} \mid z_j = 0, j \notin I\}, \quad \mathbf{C}^{*I} = \{\mathbf{z} \mid z_i \neq 0 \text{ iff } i \in I\}, \quad \mathbf{C}^{*n} = \mathbf{C}^{*\{1,\dots,n\}}$$

and we define mixed polynomials f^I by the restriction: $f^I = f|_{\mathbf{C}^I}$. For simplicity, we write a point of $\mathbf{C}^I$ as $\mathbf{z}_I$. Put $F^{*I} = \mathbf{C}^{*I} \cap F$. Note that F^{*I} is a non-empty subset of $\mathbf{C}^{*I}$ if and only if $f^I(\mathbf{z}_I, \bar{\mathbf{z}}_I)$ is not constantly zero. Now we observe that the hypersurface $F = f^{-1}(1)$ has the canonical stratification

$$F = \amalg_I F^{*I}.$$

Thus it is essential to determine the topology of each stratum F^{*I}. Put $F^* := F \cap \mathbf{C}^{*n}$, the open dense stratum and put $\hat{F}^* := \hat{f}^{-1}(1) \cap \mathbf{C}^{*n}$ where $\hat{f}(\mathbf{w})$ is the associated Laurent weighted homogeneous polynomial.

THEOREM 10. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a simplicial polar weighted homogeneous polynomial and let $\hat{f}(\mathbf{w})$ be the associated Laurent weighted homogeneous polynomial. Then there exists a canonical diffeomorphism $\varphi : \mathbf{C}^{*n} \to \mathbf{C}^{*n}$ which gives an isomorphism of the two Milnor fibrations defined by $f(\mathbf{z}, \bar{\mathbf{z}})$ and $\hat{f}(\mathbf{w})$:*

$$\begin{array}{ccc} \mathbf{C}^{*n} - f^{-1}(0) & \xrightarrow{f} & \mathbf{C}^* \\ \downarrow \varphi & & \downarrow \mathrm{id} \\ \mathbf{C}^{*n} - \hat{f}^{-1}(0) & \xrightarrow{\hat{f}} & \mathbf{C}^* \end{array}$$

and it satisfies $\varphi(F^{*n}) = \hat{F}^{*n}$ and φ is compatible with the respective canonical monodromy maps.

Proof. Assume first that $s = n$ for simplicity. Recall that

$$\hat{f}(\mathbf{w}) = \sum_{j=1}^{n} c_j \mathbf{w}^{\mathbf{n}_j - \mathbf{m}_j}.$$

Let $\mathbf{w} = (w_1, \dots, w_n)$ be the complex coordinates of $\mathbf{C}^n$ which is the ambient space of $\hat{F}$. We construct $\varphi : \mathbf{C}^{*n} \to \mathbf{C}^{*n}$ so that $\varphi(\mathbf{z}) = \mathbf{w}$ satisfies

$$\mathbf{w}(\varphi(\mathbf{z}))^{\mathbf{n}_j - \mathbf{m}_j} = \mathbf{z}^{\mathbf{n}_j} \bar{\mathbf{z}}^{\mathbf{m}_j}, \quad \text{thus} \quad \hat{f}(\varphi(\mathbf{z})) = f(\mathbf{z}).$$

For the construction of φ, we use the polar coordinates (ρ_j, θ_j) for $z_j \in \mathbf{C}^*$ and the polar coordinates (ξ_j, η_j) for $\mathbf{w}_j$. Thus $\mathbf{z}_j = \rho_j \exp(i\theta_j)$ and $\mathbf{w}_j = \xi_j \exp(i\eta_j)$. First we take $\eta_j = \theta_j$. Put $\mathbf{n}_j = (n_{j,1}, \dots, n_{j,n})$, $\mathbf{m}_j = (m_{j,1}, \dots, m_{j,n})$ in $\mathbf{N}^n$. Consider two integral matrix $N = (n_{i,j})$ and $M = (m_{i,j})$ where the k-th row vector are $\mathbf{n}_k$, $\mathbf{m}_k$ respectively. Now taking the logarithm of the equality $\mathbf{z}^{\mathbf{n}_j} \bar{\mathbf{z}}^{\mathbf{m}_j} = \mathbf{w}^{\mathbf{n}_j - \mathbf{m}_j}$, we get an equivalent equality:

$$\begin{aligned}(n_{j1} + m_{j1}) \log \rho_1 + \cdots + (n_{jn} + m_{jn}) \log \rho_n \\ = (n_{j1} - m_{j1}) \log \xi_1 + \cdots + (n_{jn} - m_{jn}) \log \xi_n, \quad j = 1, \dots, n.\end{aligned}$$

This can be written as

$$(N + M) \begin{pmatrix} \log \rho_1 \\ \vdots \\ \log \rho_n \end{pmatrix} = (N - M) \begin{pmatrix} \log \xi_1 \\ \vdots \\ \log \xi_n \end{pmatrix} \tag{8}$$

Put $(N - M)^{-1}(N + M) = (\lambda_{ij}) \in \mathrm{GL}(n, \mathbf{Q})$. Now we define φ as follows.

$$\begin{aligned}\varphi : \mathbf{C}^{*n} \to \mathbf{C}^{*n}, \quad \mathbf{z} = (\rho_1 \exp(i\theta_1), \dots, \rho_n \exp(i\theta_n)) \\ \mapsto \mathbf{w} = (\xi_1 \exp(i\theta_1), \dots, \xi_n \exp(i\theta_n))\end{aligned}$$

where ξ_j is given by $\xi_j = \exp(\sum_{i=1}^{n} \lambda_{ji} \log \rho_i)$ for $j = 1, \dots, n$. It is obvious that φ is a real analytic isomorphism of $\mathbf{C}^{*n}$ to $\mathbf{C}^{*n}$. Let us consider the Milnor fibrations of $f(\mathbf{z}, \bar{\mathbf{z}})$ and $\hat{f}(\mathbf{w})$ in the respective ambient tori $\mathbf{C}^{*n}$.

$$f : \mathbf{C}^{*n} \backslash f^{-1}(0) \to \mathbf{C}^*, \quad \hat{f} : \mathbf{C}^{*n} \backslash \hat{f}^{-1}(0) \to \mathbf{C}^*.$$

Recall that the monodromy maps h^*, $\hat{h}^*$ are given as

$$\begin{aligned}h^* : F^* \to F^*, \quad \mathbf{z} \mapsto \exp(2\pi i/m_p) \circ \mathbf{z} \\ \hat{h}^* : \hat{F}^* \to \hat{F}^*, \quad \mathbf{w} \mapsto \exp(2\pi i/m_p) \circ \mathbf{w}.\end{aligned}$$

Note that the $\mathbf{C}^*$-action associated with $\hat{f}(\mathbf{w})$ is the polar action of $f(\mathbf{z}, \bar{\mathbf{z}})$. Namely $\exp i\theta \circ \mathbf{w} = (\exp(ip_1\theta)w_1, \dots, \exp(ip_n\theta)w_n)$. Thus we have the commutative diagram:

$$\begin{array}{ccc} F_\alpha^* & \xrightarrow{h^*} & F_\alpha^* \\ \downarrow \varphi & & \downarrow \varphi \\ \hat{F}_\alpha^* & \xrightarrow{\hat{h}^*} & \hat{F}_\alpha^* \end{array}$$

where $F_\alpha^* = f^{-1}(\alpha) \cap \mathbf{C}^{*n}$ and $\hat{F}_\alpha^* = \hat{f}^{-1}(\alpha) \cap \mathbf{C}^{*n}$ for $\alpha \in \mathbf{C}^*$. □

3.1.1. *Remark.* The case $f(\mathbf{z}, \bar{\mathbf{z}}) = z_1^{a_1}\bar{z}_1 + \cdots + z_n^{a_n}\bar{z}_n$ is studied in [12].

3.2. Zeta-functions. Now we know that by [7, 8], the inclusion map $\hat{F}^* \hookrightarrow \mathbf{C}^{*n}$ is $(s-1)$-equivalence and $\chi(\hat{F}^*) = (-1)^{n-1}\det(N - M)$ for $s = n$ and 0 otherwise. Note also the monodromy map $\hat{h} : \hat{F}^* \to \hat{F}^*$ has a period m_p. The fixed point locus of $(\hat{h})^k$ is F^* if $m_p | k$ and $\emptyset$ otherwise. Thus using the formula of the zeta function (see, for example [4]),

$$\zeta_{\hat{h}^*}(t) = \exp\left(\sum_{j=0}^{\infty}(-1)^{n-1}\, dt^{jm_p}/(jm_p)\right) = (1 - t^{m_p})^{(-1)^n d/m_p}$$

where $d = \det(N - M)$ if $s = n$ and $d = 0$ for $s < n$. Translating this in the monodromy $h^* : F^* \to F^*$, we obtain

Corollary 11. *F^* has a homotopy type of CW-complex of dimension $n-1$ and the inclusion map $F^* \hookrightarrow \mathbf{C}^{*n}$ is an $(s-1)$-equivalence. The zeta function $\zeta_{h^*}(t)$ of $h^* : F^* \to F^*$ is given as $(1 - t^{m_p})^{(-1)^n d/m_p}$ with $d = \det(N - M)$ if $s = n$ and $\zeta_{h^*}(t) = 1$ for $s < n$.*

3.2.1. *Remark.* In general, the restriction of the polar action on $\mathbf{C}^n$ to $\mathbf{C}^{*I}$ may not effective and to make the action effective, we need to define polar weights as $p_{I,i} = p_i/r_I$ and $m_{I,P} = m_p/r_I$ where r_I is the gratest common divisor of $\{p_i \mid i \in I\}$. However the monodromy map $h_I : F^{*I} \to F^{*I}$ is equal to the restriction of $h : F \to F$.

4. Connectivity of F

Now we are ready to patch together the information of the strata F^{*I} for the topology of F. First we introduce the notion of *k-convenience* which is introduced for holomorphic functions ([8]). We say $f(\mathbf{z}, \bar{\mathbf{z}})$ is *k-convenient* if $f^I \not\equiv 0$ for any $I \subset \{1, 2, \dots, n\}$ with $|I| \geq n - k$. The following is obvious by the definition.

Proposition 12. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a simplicial polar weighted homogeneous polynomial with s monomials and assume that f is k-convenient. Then $k \leq s - 1$.*

Now we have the following result about the connectivity of F.

THEOREM 13. *Assume that $f(\mathbf{z}, \bar{\mathbf{z}})$ is a simplicial polar weighted homogeneous polynomial with s monomials and assume that f is k-convenient. Then F is* $\min(k, n-2)$*-connected.*

For the proof, we show the following stronger assertion. Let $I \subset \{1, 2, \dots, n\}$ and put

$$\mathbf{C}^n(*I) = \{\mathbf{z} = (z_1, \dots, z_n) \in \mathbf{C}^n \mid z_j \neq 0, j \in I\}, \quad F(*I) = F \cap \mathbf{C}^n(*I).$$
$$\mathbf{C}^{*I} = \{\mathbf{z} \in \mathbf{C}^n \mid z_j \neq 0 \text{ iff } j \in I\}, \quad F^{*I} = F \cap \mathbf{C}^{*I}.$$

LEMMA 14. *Under the assumption as in Theorem* 13, *the inclusion* $F(*I) \hookrightarrow \mathbf{C}^n(*I)$ *is* $\min(k+1, n-1)$*-equivalence.*

We prove the assertion by double induction on (n, k). Put

$$I_j = \{j, \dots, n\}, \quad K_j = \{1, \dots, \check{j}, \dots, n\}$$
$$\mathbf{C}_j^{n-1} = \mathbf{C}^{K_j} = \mathbf{C}^n \cap \{z_j = 0\}, \quad F_j = F \cap \mathbf{C}_j^{n-1}.$$

Note that F_j is the Milnor fiber of f^{K_j}. Theorem 13 follows from Lemma 14 by taking $I = \emptyset$. Changing the ordering if necessary, we may assume that $I = I_t$ for some t. We consider the filtration of F:

$$F^* = F(*I_1) \subset F(*I_2) \subset F(*I_3) \subset \cdots \subset F(*I_n) \subset F = F(*\emptyset).$$

A key lemma is

LEMMA 15. *The inclusion map* $(F(*I_j), F(*I_{j-1})) \hookrightarrow (\mathbf{C}^n(*I_j), \mathbf{C}^n(*I_{j-1}))$ *is* $\min(k+1, n-1)$*-equivalence.*

Proof. Let T_j be a tubular neighborhood of $\{z_j = 0\}$ in $\mathbf{C}^n(*I_{j+1})$ such that $T_j \cap F(*I_{j+1})$ is a tubular neighborhood of $F_j(*I_{j+1}) = \{z_j = 0\} \cap F(*I_{j+1})$ in $F(*I_{j+1})$. Consider the following diagrams by the excision isomorphisms and by the Thom isomorphisms ψ for D^2-bundle:

$$\begin{array}{ccc}
H_{\ell+1}(F(*I_{j+1}), F(*I_j)) & \xrightarrow{\cong} & H_{\ell+1}(F(*I_{j+1}) \cap T_j, F(*I_j) \cap T_j) \\
\downarrow \tau_j & & \downarrow \tau_j' \\
H_{\ell+1}(\mathbf{C}^n(*I_{j+1}), \mathbf{C}^n(*I_j)) & \xrightarrow{\cong} & H_{\ell+1}(T_j, \mathbf{C}^n(*I_j) \cap T_j) \\
& \xrightarrow{\psi} & H_{\ell-1}(F_j(*I_{j+1})) \\
& & \downarrow \tau_j'' \\
& \xrightarrow{\psi} & H_{\ell-1}(\mathbf{C}_j^{n-1}(*I_{j+1}))
\end{array}$$

Now note that f^{K_j} is $(k-1)$-convenient. Thus by the induction assumption on Lemma 15, τ_j'' is isomorphism for $\ell - 1 \leq k - 1$. This implies that τ_j', τ_j is isomorphism for $\ell + 1 \leq k + 1$. □

Proof of Lemma 14. Now we can prove Lemma 14 by the induction on j and Five Lemma, assuming $I = I_j$ for some j, applied to two exact sequences for the pairs $(F(*I_{j+1}), F(*I_j))$ and $(\mathbf{C}^n(*I_{j+1}), F(*I_j))$ and commutative diagrams:

$$\begin{array}{ccccc}
H_{\ell+1}(F(*I_{j+1}), F(*I_j)) & \longrightarrow & H_\ell(F(*I_j)) & \longrightarrow & H_\ell(F(*I_{j+1})) \\
\Big\downarrow \tau_j & & \Big\downarrow \iota_j & & \Big\downarrow \iota_j \\
H_{\ell+1}(\mathbf{C}^n(*I_{j+1}), \mathbf{C}^n(*I_j)) & \longrightarrow & H_\ell(\mathbf{C}^n(*I_j)) & \longrightarrow & H_\ell(\mathbf{C}^n(*I_{j+1}))
\end{array}$$

Induction starts for $j = 1$: ι_1 is $\min(k+1, n-1)$-equivalence by Corollary 11. This completes the proof of Lemma 14. □

4.1. Euler numbers and zeta functions. Let $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{j=1}^{s} c_j \mathbf{z}^{\mathbf{n}_j} \bar{\mathbf{z}}^{\mathbf{m}_j}$ be a simplicial polar weighted homogeneous. Let

$$\mathscr{S} = \{I \subset \{1, \dots, n\}; f^I \text{ is } \mathit{full}\}$$

and put $r_I = \gcd_{i \in I}\{p_i\}$ and $m_{p,I} = m_p / r_I$ and put $d_I = |\det_{i \in I}(\mathbf{n}_i - \mathbf{m}_i)|$. Thus for $I \in \mathscr{S}$, f^I is a simplicial full polar weighted homogeneous polynomial of polar weight type $(p_i/r_I)_{i \in I}$ with degree $m_{p,I}$. We observed in Remark 3.2.1 that the monodromy map $h^{*I} : F^{*I} \to F^{*I}$ is equal to the restriction of the monodromy map $h : F \to F$. We denote the zeta function of the monodromy map

$$h : F \to F, \quad h^{*I} = h|_{F^{*I}} : F^{*I} \to F^{*I}$$

by $\zeta(t)$, $\zeta^{*I}(t)$ respectively. Recall that $\zeta(t)$ is an alternating product of characteristic polynomials ([4]). Namely

$$\zeta(t) = \prod_{j=0}^{n-1} P_j(t)^{(-1)^{j+1}}$$

where P_j is the characteristic polynomial of the monodromy action on $h_* : H_j(F, \mathbf{Q}) \to H_j(F, \mathbf{Q})$. By Theorem 10 and the additive formula for the Euler characteristics, using a similar argument as that of Proposition 2.8, [8], we have:

THEOREM 16. (1) $\chi(F) = \sum_{I \in \mathscr{S}} (-1)^{|I|-1} d_I$.
(2) $\zeta(t) = \prod_{I \in \mathscr{S}} \zeta^{*I}(t)$, $\zeta^{*I}(t) = (1 - t^{m_{p,I}})^{(-1)^{|I|} d_I / m_{p,I}}$.

4.2. Examples. 1. Assume that $f_1(\mathbf{z})$ is a homogeneous polynomial defined by

$$f_1(\mathbf{z}) = z_1^{a_1} + z_2^{a_2} + \dots + z_n^{a_n}, \quad a_1, \dots, a_n \geq 2.$$

Then $F = f_1^{-1}(1)$ is $(n-2)$-connected and

$$\chi(F) = \sum_{j=1}^{n} \sum_{|I|=j} \chi(F^{*I}) = (a_1 - 1)(a_2 - 1) \cdots (a_n - 1) - (-1)^n$$

and

$$\operatorname{div}(\zeta_h) = (\Lambda_{a_1} - 1)\cdots(\Lambda_{a_n} - 1) - (-1)^n$$

as is well-known by [9, 1, 5]. Here $\operatorname{div}((t-\lambda_1)\cdots(t-\lambda_k)) = \sum_{i=1}^k \lambda_i \in \mathbf{Z}\cdot\mathbf{C}^*$ and $\Lambda_m = \operatorname{div}(t^m - 1)$.

2. Consider

$$f_2(\mathbf{z},\bar{\mathbf{z}}) = z_1^{a_1}\bar{z}_2 + \cdots + z_{n-1}^{a_{n-1}}\bar{z}_n + z_n^{a_n}$$

Then f_2 is a simplicial polar weighted polynomial and put

$$\mathscr{S} = \{I_j = \{1,\dots,j\} \mid j = 0,\dots,n-1\}.$$

Thus we have

$$\chi(F) = (-1)^{n-1}(a_1a_2\cdots a_n - a_2\cdots a_n + \cdots + (-1)^{n-1}a_n)$$

$$\log\zeta(t) = (-1)^n\left(\frac{1}{(1-t^{a_1\cdots a_n})} - \frac{1}{(1-t^{a_2\cdots a_n})} + \cdots + (-1)^{n-1}\frac{1}{(1-t^{a_n})}\right)$$

Proof. The polar weight of f_2 is given by $(p_1,\dots,p_n;m_p)$ where

$$m_p = a_1\cdots a_n, \quad p_1 = m_p\left(\frac{1}{a_1} + \cdots + \frac{1}{a_1\cdots a_n}\right),$$

$$p_2 = m_p\left(\frac{1}{a_2} + \cdots + \frac{1}{a_2\cdots a_n}\right)$$

$$\vdots$$

$$p_{n-1} = m_p\left(\frac{1}{a_{n-1}} + \frac{1}{a_{n-1}a_n}\right), \quad p_n = \frac{m_p}{a_n}$$

Thus the assertion follows from Corollary 11. □

4.3. Surface cases. Consider the case $n = 3$. We consider two simplicial polar weighted homogeneous polynomials.

$$f_1(\mathbf{z},\bar{\mathbf{z}}) = z_1^{a_1}\bar{z}_2^{b_1} + z_2^{a_2}\bar{z}_3^{b_2} + z_3^{a_3}, \quad a_1,a_2,b_1,b_2 > 0$$

$$f_2(\mathbf{z},\bar{\mathbf{z}}) = z_1^{a_1}\bar{z}_2^{b_1} + z_2^{a_2}\bar{z}_3^{b_2} + z_3^{a_3}\bar{z}_1^{b_3}, \quad a_1a_2a_3 > b_1b_2b_3 > 0.$$

They are 1-convenient. Let $F_1 = f_1^{-1}(1)$ and $F_2 = f_2^{-1}(1)$. By Theorem 13, F_1, F_2 are simply connected. Their Betti numbers $b_2(F_i)$ are given as

$$b_2(F_1) = a_1a_2a_3 - a_2a_3 + a_3 - 1, \quad b_2(F_2) = a_1a_2a_3 - b_1b_2b_3 - 1.$$

(I) First we consider f_1. The normalized polar weight for f_1 is given as

$$v_1 = \frac{b_1b_2}{a_1a_2a_3} + \frac{b_1}{a_1a_2} + \frac{1}{a_1}, \quad v_2 = \frac{b_2}{a_2a_3} + \frac{1}{a_2}, \quad v_3 = \frac{1}{a_3}$$

Let $r = \gcd(b_1b_2, a_1a_2a_3)$, $r_1 = \gcd(b_2, a_2a_3)$. Then m_p is given as $a_1a_2a_3/r$ and the zeta function of $h_1 : F_1 \to F_1$ is given as

$$\zeta_{h_1}(t) = P_0(t)^{-1}P_2(t)^{-1} = \frac{(1 - t^{a_2a_3/r_1})^{r_1}}{(1 - t^{a_1a_2a_3/r})^r(1 - t^{a_3})}$$

where $P_2(t)$ is the characteristic polynomial of the monodromy action $h_{1*} : H_2(F_1; \mathbf{Q}) \to H_2(F_1; \mathbf{Q})$. Note that $P_0(t) = 1 - t$. For example,

$$\zeta_{h_1}(t) = \frac{(1 - t^{a_2a_3})}{(1 - t^{a_1a_2a_3})(1 - t^{a_3})}, \quad b_1 = b_2 = 1$$

$$\zeta_{h_1}(t) = \frac{(1 - t^{a_2'a_3})^2}{(1 - t^{a_1'a_2'a_3})^4(1 - t^{a_3})}, \quad a_1 = 2a_1', a_2 = 2a_2', b_1 = b_2 = 2.$$

(II) We consider f_2. The normalized polar weight for f_2 is given as:

$$v_1 = \frac{a_2a_3 + b_1a_3 + b_1b_2}{a_1a_2a_3 - b_1b_2b_3}, \quad v_2 = \frac{a_1a_3 + a_1b_2 + b_2b_3}{a_1a_2a_3 - b_1b_2b_3}, \quad v_3 = \frac{a_1a_2 + a_2b_3 + b_1b_3}{a_1a_2a_3 - b_1b_2b_3}.$$

Put $d = a_1a_2a_3 - b_1b_2b_3$. The least common multiple m_p of the denominators of v_1, v_2, v_3 depends on $\gcd(d, a_2a_3 + b_1a_3 + b_1b_2)$ and so on. We only gives two examples.

(1) Assume that $a_1 = a_2 = a_3 = a$, $b_1 = b_2 = b_3 = b$. Then $v_1 = v_2 = v_3 = \frac{1}{a - b}$. Thus

$$\zeta_{h_2}(t) = (1 - t^{a-b})^{a^2+ab+b^2}.$$

(2) Assume that $\gcd(d, a_2a_3 + b_1a_3 + b_1b_2) = \gcd(d, a_1a_3 + a_1b_2 + b_2b_3) = \gcd(d, a_1a_2 + a_2b_3 + b_1b_3) = 1$. Then $m_p = d$ and $\zeta_{h_2}(t) = (1 - t^d)$.

For example, if $a_1 = 2$, $a_2 = 3$, $a_3 = 5$ and $b_1 = b_2 = b_3 = 1$, we get $\zeta_{h_2}(t) = (1 - t^{29})$.

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