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ON A SOLUTION OF $w''+e^{-z}w'+(az+b)w=0$

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§ 1. Introduction.

It is well-known that the general solution of the following differential equation

$$(1)\qquad w''+P(z)w'+Q(z)w=0$$

with a transcendental entire function P and a polynomial Q is an entire function of infinite order. In spite of this fact a particular solution may be an entire function of finite order. This is shown by $w''+e^{-z}w'-w=0$. Our main interest lies in the following problem: When does (1) have an entire solution of finite order? This problem seems to be very important but very hard. So far as we know there are only few results concerning the above problem.

In § 2 we shall give a general negative criterion. In § 3 we shall prove a theorem which guarantees the existence of a non-zero asymptotic value of an entire solution of finite order of the given differential equation (1). In § 4 by making use of the above theorem we shall give two applications. One is a negative result and the other is a positive result. In § 5 we shall prove a theorem concerning the boundedness of the solution of

$$y''+F(z)y=0$$

along a ray. Applying these results in § 6 we shall consider the differential equation

$$w''+e^{-z}w'+(az+b)w=0\,.$$

§ 2. We shall prove the following.

THEOREM 1. *Every entire solution of* (1) *is of infinite order, if* $P(z)$ *is of order less than* 1/2.

Proof. In order to prove this theorem we need the following Besicovitch theorem: Let $f(z)$ be an entire function of order ρ less than 1/2. Let $m(r)$ and $M(r)$ be the minimum and maximum modulus of $f(z)$ on $|z|=r$ respectively. Let X be the set of r for which $\log m(r)>(\cos \pi\rho')\log M(r)$, where ρ' is any number satisfying $\rho<\rho'<1/2$. Then the upper density of X is greater than $1-\rho/\rho'$.

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See [1].

Further we need the following fact: Let $n(r)$ be the central index of an entire transcendental function $w(z)$. Then the order of w is equal to

$$\overline{\lim_{r\to\infty}}\frac{\log n(r)}{\log r}.$$

Further let ζ be the point at which $|w(\zeta)|=M(r, w)=\max_{|z|=|\zeta|=r}|w(z)|$. Then for $r\to\infty$, $r\notin\Delta$

$$w^{(p)}(\zeta)=\left(\frac{n(r)}{\zeta}\right)^{p}(1+\eta_p)w(\zeta),$$

$$\eta_p=\frac{h_p(\zeta)}{n(r)^{\gamma}},\quad |h_p(\zeta)|<K,\quad 0<\gamma<1/2.$$

Here Δ has finite logarithmic measure. See [3], [7], [8].

By the above fact the given differential equation (1) gives

$$\left(\frac{n(r)}{\zeta}\right)^{2}(1+\eta_2)+P(\zeta)\left(\frac{n(r)}{\zeta}\right)(1+\eta_1)+Q(\zeta)=0$$

for $r=|\zeta|\notin\Delta$. Hence with $m(r)=\min_{|z|=r}|P(z)|$, $M(r)=\max_{|z|=r}|P(z)|$

$$\left(\frac{n(r)}{r}\right)^{2}(1+|\eta_2|)\geqq|P(\zeta)|\frac{n(r)}{r}(1-|\eta_1|)-|Q(\zeta)|$$

$$\geqq m(r)\frac{n(r)}{r}(1-|\eta_1|)-Ar^{K}$$

$$\geqq M(r)^{\cos\pi\rho'}\frac{n(r)}{r}(1-|\eta_1|)-Ar^{K}$$

for $r\in X\cap\Delta^{c}$. Since P is transcendental entire, $M(r)\geqq r^{S}$ for any arbitrarily large number S. Hence

$$\frac{n(r)}{r}\geqq r^{S\cos\pi\rho'}\frac{1-|\eta_1|}{1+|\eta_2|}-A\frac{r^{k+1}}{n(r)(1+|\eta_2|)}$$

$$\geqq r^{S'}$$

for $r\in X\cap\Delta^{c}$ with an arbitrarily large number S'. This implies that

$$\overline{\lim_{r\to\infty}}\frac{\log n(r)}{\log r}\geqq S'+1.$$

Thus every solution of (1) is of infinite order. Thus we have the desired result.

§ 3. Existence of an asymptotic value in a sector.

THEOREM 2. *Suppose that $w(z)$ is an entire solution of* (1) *and of finite order. Let $P(z)$ be an entire function such that $|P(z)|>Ae^{r^{\rho}}$, $\rho>0$ in $0\leqq\arg z\leqq\alpha$*

with a positive constant A and $|z|=r$. Then

(i) *$w'(z)$ is bounded in D: $0\leqq\arg z\leqq\alpha$,*

(ii) *$w'(z)\to 0$, $w''\to 0$ as $z\to\infty$ in D_ε: $\varepsilon\leqq\arg z\leqq\alpha-\varepsilon$ for an arbitrary positive ε,*

(iii) *$w(z)\to B$ as $z\to\infty$ in D_ε,*

(iv) *$B\neq 0$.*

Proof. We need the following Lemma: Let $f(z)$ be a meromorphic function of order $\rho<\infty$. Then

$$\left|\frac{f'(z)}{f(z)}\right|\leqq 18|z|^{\gamma},\quad \gamma>1+3\rho$$

holds excepting a set of $|z|$ of finite measure. See Hille's excellent book [4] p. 123. We shall denote the above exceptional set by Δ for simplicity's sake.

(i) By (1) and by the above Lemma for $|z|\notin\Delta$

$$|P(z)||w'(z)|\leqq 18|z|^{\gamma}|w'(z)|+\hat{Q}(|z|)|w(z)|,$$

where $\hat{Q}(z)=\sum|a_j|z^j$ with $Q(z)=\sum a_j z^j$. On the other hand

$$|w(z)|\leqq|w(0)|+\left|\int_0^z w'(t)dt\right|$$

$$\leqq|w(0)|+rM_1^{\theta}(r),\quad |z|=r\,,$$

where $M_1^{\theta}(r)=\max\limits_{0\leq|z|\leq r}|w'(|z|e^{i\theta})|$. Hence for $|z|\in\Delta^c$, $\arg z=\theta$,

$$(|P(z)|-18|z|^{\gamma})|w'(z)|\leqq\hat{Q}(|z|)(|w(0)|+|z|M_1^{\theta}(|z|))\,.$$

If θ satisfies $0\leqq\theta\leqq\alpha$, then

$$(Ae^{r^{\rho}}-18r^{\gamma})|w'(z)|\leqq\hat{Q}(r)(|w(0)|+rM_1^{\theta}(r))\,.$$

Assume that $M_1^{\theta}(r)$ is unbounded for $r\to\infty$. Since $M_1^{\theta}(r)$ is evidently monotone non-decreasing, there is a sequence $\{r_m\}$, $r_m\in\Delta^c$ such that $|w'(z_m)|=M_1^{\theta}(r_m)$, $r_m=|z_m|$, $\arg z_m=\theta$. Therefore

$$(Ae^{r_m^{\rho}}-18r_m^{\gamma})M_1^{\theta}(r_m)-\hat{Q}(r_m)r_m M_1^{\theta}(r_m)\leqq\hat{Q}(r_m)|w(0)|\,.$$

This is a contradiction. Hence $M_1^{\theta}(r)$ is bounded for $r\to\infty$. In this case $M_1^{\theta}(r)$ is continuous for θ. Therefore $w'(z)$ is uniformly bounded in $0\leqq\arg z\leqq\alpha$.

(ii) We may assume that $|w'(z)|\leqq K$ for $0\leqq\arg z\leqq\alpha$. Then

$$|w''(re^{i\theta})|\leqq\frac{1}{2\pi}\int_0^{2\pi}\frac{|w'(re^{i\theta}+se^{i\phi})|}{s^2|e^{2i\phi}|}s\,d\phi$$

$$\leqq\frac{K}{s}\,.$$

Here $\zeta=re^{i\theta}+se^{i\phi}\in D$. Hence the above estimation holds for $0<\varepsilon\leqq\theta\leqq\alpha-\varepsilon$. Thus

$$|P(z)||w'|\leqq \hat{Q}(|z|)|w|+\frac{K}{s}$$

$$\leqq \hat{Q}(|z|)(|w(0)|+rK)+\frac{K}{s}.$$

This implies that

$$|w'(z)|\leqq \frac{\hat{Q}(r)(rK+|w(0)|)+K/s}{A\exp r^{\rho}}$$

tends to zero as $z\to\infty$, $z\in D_\varepsilon$. Hence making use of this estimation for $w'(z)$ we have $w''(z)\to 0$ in D_ε by the Cauchy integral formula.

(iii) Let a_0 be

$$\int_0^\infty w'(te^{i\theta})e^{i\theta}dt$$

for $\varepsilon\leqq\theta\leqq\alpha-\varepsilon$, $\varepsilon>0$. It is very easy to prove the existence of a_0 and the independence of θ. For $z=|z|e^{i\phi}$, $\varepsilon\leqq\phi\leqq\alpha-\varepsilon$, $\varepsilon>0$

$$w(z)-w(0)-a_0=\int_0^z w'(t)dt-\int_0^\infty w'(se^{i\theta})e^{i\theta}ds$$

$$=\int_0^{|z|} w'(se^{i\varphi})e^{i\varphi}ds-\int_0^\infty w'(se^{i\theta})e^{i\theta}ds$$

$$=\int_\varphi^\theta w'(|z|e^{i\eta})|z|e^{i\eta}i\,d\eta-\int_{|z|}^\infty w'(te^{i\theta})e^{i\theta}dt.$$

Since

$$|w'(|z|e^{i\eta})|\leqq\frac{1}{A\exp r^{\rho}}\{\hat{Q}(r)(rK+|w(0)|)+K\}$$

for $|z|e^{i\eta}\in D_\varepsilon$,

$$\left|\int_\varphi^\theta w'(|z|e^{i\eta})|z|e^{i\eta}i\,d\eta\right|\to 0$$

as $|z|\to\infty$. Similarly

$$\int_{|z|}^\infty w'(te^{i\theta})e^{i\theta}dt\to 0$$

as $|z|\to\infty$. Hence $w(z)\to w(0)+a_0$ as $z\to\infty$, $z\in D_\varepsilon$. Thus $B=w(0)+a_0$.

(iv) Suppose $B=0$. Then for $z\in D_\varepsilon$

$$w(z)=w(0)+\int_0^z w'(t)dt$$

$$=B-\int_z^\infty w'(t)dt$$

Hence we may start from

$$w(z)=-\int_{|z|}^\infty w'(se^{i\theta})e^{i\theta}ds.$$

Let $|w(0)|$ be less than K. Then

$$|w'(z)| \leqq \frac{\hat{Q}(r)(r+1)+1}{A \exp r^{\rho}} K, \quad |z|=r$$

for $z \in D_{\varepsilon}$. Therefore

$$|w(z)| \leqq K\int_r^{\infty} \frac{\hat{Q}(t)(t+1)+1}{A \exp t^{\rho}} dt$$

$$\leqq K \frac{\hat{Q}(r)(r+1)+1}{\frac{\rho}{2} r^{\rho-1} A \exp \frac{r^{\rho}}{2}} \int_r^{\infty} \frac{\rho}{2} \frac{t^{\rho-1}}{e^{t^{\rho}/2}} dt$$

$$= \frac{2K}{\rho A} \frac{r^{1-\rho}\{\hat{Q}(r)(r+1)+1\}}{\exp \frac{r^{\rho}}{2}} \exp\left(-\frac{r^{\rho}}{2}\right)$$

$$= \frac{2K}{\rho A} \frac{r^{1-\rho}\{\hat{Q}(r)(r+1)+1\}}{\exp r^{\rho}} .$$

Further by the Cauchy integral formula

$$|w''(z)| \leqq \frac{1}{2\pi} \int_0^{2\pi} \frac{|w'(z+e^{i\phi})|}{|e^{2i\phi}|} |e^{i\phi}| d\phi$$

$$\leqq \frac{\hat{Q}(r_{\zeta})(r_{\zeta}+1)+1}{A \exp r_{\zeta}^{\rho}} K$$

with $r_{\zeta}=|\zeta|=|z+e^{i\phi}|$. Therefore for a sufficiently large r and for $re^{i\theta} \in D_{\varepsilon}$

$$|w(r e^{i\theta})| < \varepsilon , \quad |w''(r e^{i\theta})| < \varepsilon .$$

Then by (1)

$$|P(z)|\,|w'(z)| \leqq \hat{Q}(r)|w(r e^{i\theta})| + |w''(r e^{i\theta})|$$

and

$$|w'(r e^{i\theta})| \leqq \varepsilon \frac{\hat{Q}(r)+1}{A \exp r^{\rho}} < \varepsilon^2.$$

In this case

$$|w(r e^{i\theta})| \leqq \frac{\varepsilon}{A} \int_r^{\infty} \frac{\hat{Q}(t)+1}{\exp t^{\rho}} dt$$

$$\leqq \frac{2\varepsilon}{\rho A} \frac{r^{1-\rho}}{\exp r^{\rho}} (\hat{Q}(r)+1) < \varepsilon^2 .$$

and

$$|w''(r e^{i\theta})| \leqq \frac{\varepsilon}{A} \frac{\hat{Q}(r_{\zeta})+1}{\exp r_{\zeta}^{\rho}} < \varepsilon^2 .$$

Repeating this process we have

$$w'(r e^{i\theta}) \equiv 0$$

for $r \geqq r_0$. This is a contradiction. Hence $B \neq 0$.

In this theorem we may replace $|P(z)|>A\exp r^{\rho}$ by $|P(z)|>A\exp(C r^{\rho})$ with positive C.

§ 4. Applications of Theorem 2.

THEOREM 3. *Let* $\alpha_1<\beta_1<\alpha_2<\beta_2<\cdots<\alpha_p<\beta_p$, $0\leqq\alpha_1$, $\beta_p<\alpha_1+2\pi$. *Let* D_j *be the sector* $\alpha_j\leqq\arg z\leqq\beta_j$. *Suppose that* $\sum(\alpha_{j+1}-\beta_j)<\varepsilon$ *for an arbitrary positive number* ε. *(Here* $\alpha_{p+1}=\alpha_1+2\pi$*). Suppose that* $|P(z)|>A_j\exp(C_j r^{\rho_j})$ *with positive constants* A_j, C_j, ρ_j *in* D_j. *Then every solution of* (1) *is of infinite order.*

Proof. Assume that $w(z)$ is an entire solution of (1) and of finite order. By Theorem 2 $w'(z)$ is uniformly bounded in each D_j. Let $|w'(z)|$ be less than K_j in D_j. Let K be $\max K_j$. Let D be an unbounded domain in which $|w'(z)|>K$. Then D lies in one of the remaining sectors $(D_1\cup D_2\cup\cdots\cup D_p)^c$, say the sector $\beta_1<\arg z<\alpha_2$. Let $r\theta(r)$ be the arc length of $\{|z|=r\}\cap D$. Let $M(r)$ be $\max|w'(z)/K|$ on $\{|z|=r\}\cap D$. Then there is a constant $K(0<K<1)$ such that

$$\log\log M(r)\geqq\pi\int_{r_0}^{Kr}\frac{dt}{\theta(t)t}-\text{const.}$$

See Tsuji's book [6] p. 117. (The formulation in [6] is more complicated than the above.) Here $\theta(t)<\varepsilon$. Hence

$$\log\log M(r)\geqq\frac{\pi}{\varepsilon}\log\frac{Kr}{r_0}-\text{const.}$$

This implies that

$$\varlimsup_{r\to\infty}\frac{\log\log M(r)}{\log r}\geqq\frac{\pi}{\varepsilon}$$

Here ε is arbitrary. Thus the order of $w(z)$ is infinite. This is absurd.

By this theorem (1) does not admit any entire solution of finite order if $P(z)=\Pi\left(1-\frac{z}{n^2}\right)$ or if $P(z)=\frac{R}{z^{p/2}}\sin z^{n/2}$, R : polynomial, $0\leqq p\leqq n$. There are lots of such examples. Expecially we can construct such an example for which P has the given order by making use of the Mittag-Leffler function.

Another application is a positive result. Frei [2] had proved the following

THEOREM 4. *If there is an entire function* $w(z)$ *of finite order satisfying the differential equation*

$$w''+e^{-z}w'+cw=0\,,\quad c=\text{const.},$$

then $c=-n^2$.

Wittich [9] had also given another proof. The following proof is due to T. Kobayashi.

Proof. Let w_1 and w_2 be two independent solutions of the given equation.

We may assume that $w_1(z)$ is of finite order. Evidently $w_1(z+2\pi i)$ satisfies the equation. Hence

$$w_1(z+2\pi i)=\alpha_1 w_1(z)+\alpha_2 w_2(z).$$

Since $w_1(z+2\pi i)$ is of finite order too, α_2 should be equal to zero. Further by Theorem 2 $w_1(z)\to B\neq 0$, $w_1(z+2\pi i)\to B\neq 0$ if $z\to\infty$ in $\pi/2+\varepsilon\leqq\arg z\leqq 3\pi/2-\varepsilon$, $\varepsilon>0$. Hence $\alpha_1=1$, that is, $w_1(z+2\pi i)=w_1(z)$. This implies that there is a one-valued regular function $f(x)$ in $0<|x|<\infty$ such that $w_1(z)=f(e^z)$. If $f(x)$ has an essential singularity at $x=0$, then $f(e^z)$ does not have any finite asymptotic value in $\pi/2+\varepsilon\leqq\arg z\leqq 3\pi/2-\varepsilon$. If $f(x)$ has a pole at $x=0$, then $f(e^z)$ tends to ∞ as $z\to\infty$ in $\pi/2+\varepsilon\leqq\arg z\leqq 3\pi/2-\varepsilon$. Hence

$$f(x)=\sum_{j=0}^{\infty} a_j x^j.$$

Substituting this into the given equation, we have

$$\Big(\sum_{j=0}^{\infty} a_j e^{jz}\Big)''+\Big(\sum_{j=0}^{\infty} a_j e^{jz}\Big)' e^{-z}+c\Big(\sum_{j=0}^{\infty} a_j e^{jz}\Big)=0.$$

This gives

$$(n^2+c)a_n=-(n+1)a_{n+1}\quad (n\geqq 1)$$

$$a_1=c\,a_0.$$

Firstly $a_0=B\neq 0$. If $c\neq -n^2$ for all integers n, then

$$\frac{|a_n|}{|a_{n+1}|}\to 0$$

as $n\to\infty$. This shows that the radius of convergence of $f(x)=\sum a_j x^j$ is equal to zero. This is absurd. Hence $c=-n^2$ for some integer $n>0$. Then $a_{n+k}=0$, $k=1, 2, \cdots$. Thus $f(x)=\sum_0^n a_j x^j$. It is very easy to determine all the a_j. This gives the desired result.

§5. **Boundedness criterion along a ray.** In the real case there are lots of boundedness criteria. We know very few such criteria in the complex case.

THEOREM 5. *Suppose that* $F(z)=g(r)e^{i\gamma(r)}$ *along the ray* $re^{i\theta}$ (θ: *fixed*) *such that*

$$g(r)\cos(\gamma(r)+2\theta)=S(r)+O\Big(\frac{1}{\exp r^{\rho}}\Big)$$

is monotone increasing for $r\geqq r_0$, *where* S *is a non-constant polynomial and* $\rho>0$, *and*

$$g(r)\sin(\gamma(r)+2\theta)=\frac{A}{\exp r^{\rho}}+O\Big(\frac{B}{\exp r^{\rho'}}\Big)$$

with $\rho'>\rho$ *for* $r\geqq r_0$, *where* A, B *are polynomials of* r *or functions of* r *admitting* *polynomial majorants. Suppose further that $F(z)$ is transcendental entire. Then every solution of $y''+F(z)y=0$ is bounded along the ray $re^{i\theta}$.*

Proof. Let us put the solution $y=R(r)e^{i\Theta(r)}$ along the ray $re^{i\theta}$. Then the differential equation gives(*)

(2) $$R''(r)+\{g(r)\cos(\gamma(r)+2\theta)-\Theta'(r)^2\}R(r)=0\,,$$
$$\{\Theta'(r)R(r)^2\}'+g(r)\sin(\gamma(r)+2\theta)R(r)^2=0\,.$$

For simplicity's sake we put

$$X(r)=g(r)\cos(\gamma(r)+2\theta)\,,\quad Y(r)=g(r)\sin(\gamma(r)+2\theta)\,.$$

Let U be

$$\int_{r_1}^{r}\Theta'^2RR'dt\,.$$

Then

$$U=\frac{1}{2}R(r)^2\Theta'(r)^2-\frac{1}{2}R(r_1)^2\Theta'(r_1)^2-\int_{r_1}^{r}R^2\Theta'\Theta''dt\,.$$

By the second equation of (2)

$$\Theta''R+2\Theta'R'+YR=0\,.$$

Hence

$$-\int_{r_1}^{r}R^2\Theta'\Theta''dt=\int_{r_1}^{r}R\Theta'(2\Theta'R'+YR)dt=2U+\int_{r_1}^{r}YR^2\Theta'dt\,.$$

Thus

$$U=\frac{1}{2}R(r)^2\Theta'(r)^2-\frac{1}{2}R(r_1)^2\Theta'(r_1)^2+2U+\int_{r_1}^{r}YR^2\Theta'dt\,.$$

Therefore

$$U=-\frac{1}{2}R(r)^2\Theta'(r)^2+\frac{1}{2}R(r_1)^2\Theta'(r_1)^2-\int_{r_1}^{r}Y\Theta'R^2dt\,.$$

On the other hand by the first equation of (2)

$$\begin{aligned}U&=\int_{r_1}^{r}\Theta'^2RR'dt=\int_{r_1}^{r}R''R'dt+\int_{r_1}^{r}XRR'dt\\&=\frac{1}{2}R'(r)^2-\frac{1}{2}R'(r_1)^2+\frac{1}{2}X(r)R(r)^2\\&\quad-\frac{1}{2}X(r_1)R(r_1)^2-\frac{1}{2}\int_{r_1}^{r}R^2dX(t)\,.\end{aligned}$$

Thus we have

(*) With respect to the equation (2) it is necessary to pay attention to its meaning at each zero of y. See C.-T. Taam. Oscillation theorems. Amer. J. Math. 74 (1952), 317-324. See also Hille [4].

$$\frac{1}{2}R'(r)^2+\frac{1}{2}R(r)^2\Theta'(r)^2+\frac{1}{2}X(r)R(r)^2$$

$$=\frac{1}{2}R'(r_1)^2+\frac{1}{2}R(r_1)^2\Theta'(r_1)^2+\frac{1}{2}X(r_1)R(r_1)^2$$

$$+\frac{1}{2}\int_{r_1}^{r}R^2dX(t)-\int_{r_1}^{r}Y\Theta'R^2dt\,.$$

Now we shall estimate the last integral. By the second equation of (2)

$$-\int_{r_1}^{r}YR^2dt=\Theta'(r)R^2(r)-\Theta'(r_1)R^2(r_1)\,.$$

Hence

$$-\int_{r_1}^{r}Y(t)\Theta'(t)R(t)^2dt=-\int_{r_1}^{r}\Theta'(r_1)R(r_1)^2Y(t)dt$$

$$+\int_{r_1}^{r}Y(t)\int_{r_1}^{t}Y(s)R(s)^2dsdt\,.$$

Therefore,

$$\left|-\int_{r_1}^{r}Y\Theta'R^2dt\right|\leqq|\Theta'(r_1)R(r_1)^2|\int_{r_1}^{r}|Y(t)|dt$$

$$+\int_{r_1}^{r}|Y(t)|R(t)^2dt\int_{r_1}^{r}|Y(t)|dt\,.$$

On the other hand

$$\int_{r_1}^{r}|Y(t)|dt\leqq\int_{r_1}^{\infty}|Y(t)|dt=C_0<\infty\,.$$

Hence

$$\left|-\int_{r_1}^{r}Y\Theta'R^2dt\right|\leqq C_0|\Theta'(r_1)|R(r_1)^2+C_0\int_{r_1}^{r}|Y(t)|R(t)^2dt\,.$$

Now we take r_1 sufficiently large such that

$$\frac{|Y(t)|R(t)^2}{\frac{1}{2}X'(t)R(t)^2}\leqq\frac{|B_1|\exp(-t^{\rho})}{S'(t)\left(1+O\left(\frac{1}{t}\right)\right)}<\varepsilon$$

for $t\geqq r_1$. Therefore

$$C_0\int_{r_1}^{r}|Y(t)|R(t)^2dt\leqq\frac{\varepsilon}{2}\int_{r_1}^{r}R(t)^2dX(t)\,.$$

Thus we have

$$\frac{1}{2}X(r)R(r)^2\leqq C_1(r_1)+\frac{1+\varepsilon}{2}\int_{r_1}^{r}R^2(t)dX(t)\,.$$

By the similar process of proof of the Gronwall inequality we have

$$\frac{1}{2}X(r)R(r)^2 \leqq C_1 + \frac{1+\varepsilon}{2}\int_{r_1}^{r} R^2(t)X(t)\frac{X'(t)}{X(t)}dt$$

$$\leqq C_1 \frac{X(r)^{1+\varepsilon}}{X(r_1)^{1+\varepsilon}}.$$

Therefore

$$R(r)^2 \leqq 2C_1(r_1)\frac{X(r)^{\varepsilon}}{X(r_1)^{1+\varepsilon}}.$$

Using this intermediate estimation we again estimate

$$\int_{r_1}^{r} Y\theta' R^2 dt .$$

Similarly

$$|\Theta'(r)R(r)^2 - \Theta'(r_1)R(r_1)^2| = \left| -\int_{r_1}^{r} YR^2 dt \right|$$

$$\leqq 2C_1 \frac{1}{X(r_1)^{1+\varepsilon}}\int_{r_1}^{\infty} |Y(t)|\,|X(t)|^{\varepsilon} dt = C_2 .$$

Thus

$$|\Theta'(r)R(r)^2| \leqq C_2 + |\Theta'(r_1)|\,R(r_1)^2 = C_3 .$$

This implies that

$$\left|\int_{r_1}^{r} Y\Theta' R^2 dt\right| \leqq C_3 \int_{r_1}^{\infty} |Y|\,dt = C_4 .$$

Therefore

$$\frac{1}{2}X(r)R(r)^2 \leqq C + \frac{1}{2}\int_{r_1}^{r} R(t)^2 X(t)\frac{X'(t)}{X(t)}dt .$$

By Gronwall's inequality

$$R(r)^2 \leqq 2C\frac{1}{X(r_1)},$$

which gives the desired result.

§ 6. The differential equation $w'' + e^{-z}w' + (az+b)w = 0$.

Our problem is whether this equation admits an entire solution of finite order. By Frei's theorem we may assume that $a \neq 0$. By a suitable translation we may consider

(3) $$w'' + e^{-z+c}w' + azw = 0 .$$

By the well-known transformation

$$w = y\frac{\exp\left(\frac{1}{2}e^{-z+c}\right)}{\exp\left(\frac{1}{2}e^{c}\right)},$$

we have

$$y''+\left(az+\frac{1}{2}e^{-z+c}-\frac{1}{4}e^{-2z+2c}\right)y=0\,.$$

We denote this by $y''+F(z)y=0$. Let a be $|a|e^{i\alpha}$ and $z=re^{i\theta}$ (θ: fixed). Then we put $F(z)=g(r)e^{i\gamma(r)}$. In this case

$$g(r)=|a|r+\frac{1}{2}e^{-r\cos\theta+c_1}\cos(r\sin\theta-C_2+\theta+\alpha)+\text{higher order terms},$$

$$\gamma(r)=\theta+\alpha-\frac{1}{8}\frac{e^{-r\cos\theta+c_1}}{|a|r\cos^2(\theta+\alpha)}\sin(r\sin\theta-C_2+\theta+\alpha)+\text{higher order terms},$$

where $C=C_1+iC_2$. In order to apply Theorem 5 we need to examine the assumptions. $g(r)\cos(\gamma(r)+2\theta)$ should be monotone increasing for $r\geqq r_0$ and hence $\cos(\gamma(r)+2\theta)>0$. Further $g(r)\sin(\gamma(r)+2\theta)$ tends to zero very rapidly as $r\to\infty$. Hence $\sin(\gamma(r)+2\theta)$ tends to zero very rapidly as $r\to\infty$. This gives that $\alpha+3\theta=2p\pi$. Thus

$$\theta=\frac{2p\pi}{3}-\frac{\alpha}{3}\,.$$

Further $\cos\theta$ should be positive. Hence $-\pi/2<\theta<\pi/2$. One of three rays $re^{-i\alpha/3}$, $re^{-i\alpha/3+i2\pi/3}$, $re^{-i\alpha/3+i4\pi/3}$ lies in the right half-plane. We can apply Theorem 5 along this ray. In the first place we assume that there is only one ray $re^{i\theta}$ along which the assumptions of Theorem 5 are satisfied. Then

$$-\frac{\pi}{6}\leqq\theta\leqq\frac{\pi}{6}$$

In this case we may assume that $0\leqq\theta\leqq\pi/6$. By Theorem 5 y is bounded along this ray $re^{i\theta}$. Now we assume that w is an entire solution, being of finite order, of (3). Then by Theorem 2 w tends to a non-zero constant B when $z\to\infty$ in $\pi/2+\varepsilon\leqq\arg z\leqq 3\pi/2-\varepsilon$. w is also bounded along the ray $re^{i\theta}$. Suppose that w is unbounded in the sector $S: \theta<\phi<\pi/2+\varepsilon$. Then there is an unbounded domain, contained in S, in which w is unbounded. In this case

$$\log\log M(r)\geqq\pi\int_{r_0}^{Kr}\frac{dt}{t\theta(t)}-\text{const.}$$

$$\geqq\frac{\pi}{\frac{\pi}{2}+\varepsilon-\theta}\log\frac{Kr}{r_0}-\text{const.}$$

Hence

$$\varliminf_{r\to\infty}\frac{\log\log M(r)}{\log r}\geqq\frac{\pi}{\frac{\pi}{2}+\varepsilon-\theta}\,.$$

ε is arbitrary. Thus the lower order of w is greater than $2\pi/(\pi-2\theta)\geqq 2$. Let ζ

be a point such that $|\zeta|=r$, $|w(\zeta)|=\max_{|z|=r}|w(z)|$. By the Wiman-Valiron method

$$\left(\frac{n(r)}{\zeta}\right)^2(1+\eta_2)+e^{-\zeta+c}\frac{n(r)}{\zeta}(1+\eta_1)+a\zeta=0$$

holds excepting a set Δ of finite logarithmic measure. Assume that there is a sequence $\{\zeta_m\}$ such that $\mathcal{R}\zeta_m\geqq 0$, $|\zeta_m|\in\Delta^c$. Then along $|\zeta_m|=r_m$

$$\frac{n(r)}{r}=|a|^{1/2}r^{1/2}\left(1+O\left(\frac{1}{r}\right)\right),$$

since $e^{-\zeta_m+c}$ is bounded. This implies that

$$\varliminf_{r\to\infty}\frac{\log n(r)}{\log r}\leqq\frac{3}{2}.$$

It is known that

$$\varliminf_{r\to\infty}\frac{\log\log M(r, w)}{\log r}=\varliminf_{r\to\infty}\frac{\log n(r)}{\log r}.$$

Hence we have a contradiction. Therefore there is a sequence $\{\zeta_m\}$ such that $|\zeta_m|\in\Delta^c$, $R\zeta_m<0$. Let us put $\zeta_m=r_m e^{i\phi_m}$. We may assume that $\pi/2<\phi_m<\pi/2+\varepsilon$. $\phi_m\to\pi/2$ as $r_m\to\infty$. Now we shall omit the index m, since this does not make any confusion. Then

$$\left(\frac{n(r)}{\zeta}\right)^2(1+\eta_2)+e^{-\zeta+c}\frac{n(r)}{\zeta}(1+\eta_1)+a\zeta=0.$$

Let us put $c=c_1+ic_2$, $a=|a|e^{i\alpha}$. Then

$$(1+\eta_2)e^{-i\phi}+\frac{r}{n(r)}e^{-r\cos\phi+c_1}e^{-ir\sin\phi+ic_2}(1+\eta_1)+\frac{|a|r^3}{n(r)^2}e^{2i\phi+i\alpha}=0.$$

Since $n(r)\geqq r^{2-\delta'}$, $\delta'>0$,

$$\frac{r}{n(r)}e^{-r\cos\phi+c_1}=1+O\left(\frac{1}{r^{1-\delta'}}\right).$$

Thus

$$\left(\frac{n(r)}{r}\right)^2\{1+(1+\varepsilon_1)\cos(-r\sin\phi+c_2+\phi)-i(1+\varepsilon_2)\sin(-r\sin\phi+c_2+\phi)\}$$

$$=-|a|r\cos(3\phi+\alpha)-i|a|r\sin(3\phi+\alpha),\quad \varepsilon_1,\ \varepsilon_2\to 0\ (r\to\infty).$$

Therefore by $n(r)\geqq r^{2-\delta'}$

$$r^{1-2\delta'}\{1+(1+\varepsilon_1)\cos(-r\sin\phi+c_2+\phi)\}\leqq-|a|\cos(3\phi+\alpha),$$

$$r^{1-2\delta'}(1+\varepsilon_2)|\sin(-r\sin\phi+C_2+\phi)|\leqq|a|\,|\sin(3\phi+\alpha)|.$$

Let us put $\phi=\pi/2+\psi$. Then

$$r\cos\psi-\psi-c_2=2p\pi-\frac{\pi}{2}+o(1).$$

Which implies that

$$r=2p\pi-\frac{\pi}{2}+c_2+o(1).$$

Let Δ_p be $[r_p-o(1)<r<r_p+o(1)]$, $r_p=2p\pi+c_2-\pi/2$. Then

$$\log\frac{r_{p+1}-o(1)}{r_p+o(1)}\geqq\log\frac{2(p+1)\pi+c_2-\pi/2-o(1)}{2p\pi+c_2-\pi/2+o(1)}$$

$$\geqq\frac{1}{2}\cdot\frac{1}{p-\frac{1}{4}+\frac{c_2}{2\pi}}.$$

Hence

$$\sum\log\frac{r_{p+1}-o(1)}{r_p+o(1)}\geqq\frac{1}{2}\sum\frac{1}{p-\frac{1}{4}+\frac{c_2}{2\pi}}=\infty.$$

This shows that the logarithmic measure of $(\cup\Delta_p)^c$ is infinite. Thus $(\cup\Delta_p)^c$ is not contained in Δ. Therefore there is a set F of r of infinite logarithmic measure such that for $r\in F$, $|\zeta|=r$, $\mathcal{R}\zeta\geqq 0$, $M(r, w)=\max|w(re^{i\phi})|=|w(\zeta)|$. We have, then, that the lower order of w is not greater than 3/2. This is a contradiction. Hence $w(z)$ should be bounded in S. This shows that $w(z)$ is bounded in the sector $T: \theta\leqq\phi\leqq 3\pi/2-\varepsilon$. We now make use of the classical Lindelöf-Iversen-Gross theorem [5] and have the existence of the asymptotic value $B\neq 0$ of w in

$$T_1: \theta+\varepsilon\leqq\phi\leqq 3\pi/2-2\varepsilon.$$

Therefore $y\to B\exp(e^c/2)$ as $z\to\infty$ in $S': \theta+\varepsilon\leqq\arg z\leqq\pi/2-\varepsilon$. Then by the Cauchy integral formula $y'\to 0$ in the same sector S'. Now the so-called Green's transform [4] is useful. Let us consider $y''+F(z)y=0$ with $F(z)=az+e^{-z+c}/2-e^{-2z+2c}/4$. Green's transform gives

$$\bar{y}(re^{i\phi})y'(re^{i\phi})-\bar{y}(r_0e^{i\phi})y'(r_0e^{i\phi})$$

$$=\int_{r_0}^{r}|y'(te^{i\phi})|^2dte^{-i\phi}-\int_{r_0}^{r}F(te^{i\phi})|y(te^{i\phi})|^2dte^{i\phi}.$$

Hence

$$e^{i\phi}\{\bar{y}(re^{i\phi})y'(re^{i\phi})-\bar{y}(r_0e^{i\phi})y'(r_0e^{i\phi})\}$$

$$=\int_{r_0}^{r}|y'(te^{i\phi})|^2dt-e^{3i\phi+i\alpha}\int_{r_0}^{r}(X+iY)|y(te^{i\phi})|^2dt$$

$$=I_1-e^{3i\phi+i\alpha}(I_2+iI_3).$$

Thus

$$|\bar{y}(re^{i\phi})y'(re^{i\phi})-\bar{y}(r_0e^{i\phi})y'(r_0e^{i\phi})|^2$$

$$=(I_1\cos(3\phi+\alpha)-I_2)^2+(I_1\sin(3\phi+\alpha)+I_3)^2.$$

Here $\theta+\varepsilon\leqq\phi\leqq\pi/2-\varepsilon$,

$$X=|a|t+\frac{1}{2}e^{-t\cos\phi+c_1}\cos(-t\sin\phi+c_2-\alpha-\phi)$$
$$-\frac{1}{4}e^{-2t\cos\phi+2c_1}\cos(-2t\sin\phi+2c_2-\alpha-\phi)$$

and

$$Y=\frac{1}{2}e^{-t\cos\phi+c_1}\sin(-t\sin\phi+c_2-\alpha-\phi)$$
$$-\frac{1}{4}e^{-2t\cos\phi+2c_1}\sin(-2t\sin\phi+2c_2-\alpha-\phi)\,.$$

Hence

$$I_3=\int_{r_0}^{r}Y(t)|y(te^{i\phi})|^2dt$$

is bounded as $r\to\infty$, since $y\to B\exp(e^c/2)$ as $r\to\infty$ and $|Y(t)|\leqq\exp(c_1-t\cos\phi)$ for any sufficiently large t. We now rewrite the above formula

$$(R(r)R'(r)-R(r_0)R'(r_0))^2+(\Theta'(r)R(r)^2-\Theta'(r_0)R(r_0)^2)^2$$
$$=(I_1\cos(3\phi+\alpha)-I_2)^2+(I_1\sin(3\phi+\alpha)+I_3)^2\,.$$

By the above observation

$$R(r)\to|B|\exp\frac{\Re e^c}{2}\neq0\,,$$
$$R'(r)\to0\,,\quad \Theta'(r)R(r)\to0$$

as $r\to\infty$. Thus $(I_1\cos(3\phi+\alpha)-I_2)^2+(I_1\sin(3\phi+\alpha)+I_3)^2$ is bounded for $r\to\infty$. However

$$I_2=\int_{r_0}^{r}(t|a|+O(e^{-t}))R(t)^2dt$$

tends to infinity as $r\to\infty$. We can take ϕ such that $\cos(3\phi+\alpha)\neq0$, $\sin(3\phi+\alpha)\neq0$. Hence the boundedness of $(I_1\cos(3\phi+\alpha)-I_2)^2$ implies that I_1 is unbounded and hence $(I_1\sin(3\phi+\alpha)+I_3)^2$ is unbounded. This is a contradiction.

In the second place we assume that there are two rays $re^{i\theta_1}$, $re^{i\theta_2}$ along which the assumptions of Theorem 5 are satisfied. Then

$$-\frac{\pi}{2}<\theta_2=\theta_1-\frac{2\pi}{3}<\theta_1<\frac{\pi}{2}\,.$$

The same reasoning does work in this case too. We finally arrive at the same contradiction.

We have the following

THEOREM 6. *There is no entire function of finite order satisfying the differential equation*

$$w''+e^{-z}w'+(az+b)w=0,$$

if $a\neq 0$.

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