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HILBERT $B(H)$-MODULES AND STATIONARY PROCESSES

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1. Introduction.

In the theory of multivariate or Hilbert space valued stationary processes, the Gram matricial structure of the time domain of processes plays an important role. The Gram matricial structure of q-variate processes forms a module over a ring of all $q\times q$-matrices with a $q\times q$-matrix valued inner product, which can be seen as a Hilbert space over a matrix ring but not over the complex number field (cf. Masani [5]). Thus it is desirable to formulate such a structure abstractly free from the underlying probabilistic structure, as Kolmogorov first emphasized in 1940 for the univariate case where unspecified Hilbert spaces are prefered to L^2-spaces (cf. [4]). The purpose of the present paper is to establish such an abstact concept of the time domain of processes.

In the next section, we shall give definitions of Hilbert $B(H)$-modules and stationary processes on them. Our definition of a Hilbert $B(H)$-module is similar to the Paschke's definition [6] of inner product modules over B^*-algebras except that we require the range of the inner product, which we call the Gramian, is contained in the trace class. Such a requirement is always satisfied for the setting of q-variate or Hilbert space valued processes, and plays an essential role in our treatment. In Sect. 3 we shall study, in a general setting, positive sesquilinear maps valued in the predual of a W^*-algebra, example of which are the Gramian and the operator valued covariance function of stationary processes, and we shall examine the relation with the $*$-representation and construct a unitary representation which is a module version of Umegaki's construction [10], which are applied to the later sections. In Sect. 4 we shall show that the structure of Hilbert $B(H)$-modules is completely determined by the power of their modular bases, and that a Fourier expansion by the modular basis and Gramian is possible in a parallel way with one on the usual Hilbert spaces. In Sect. 5 applying a general theorem obtained in Sect. 3, the equivalence of stationary processes on Hilbert $B(H)$-modules is established by their covariance functions.

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2. Hilbert $B(H)$-modules and G-stationary processes.

Let H be a Hilbert space over a complex number field C, let $B(H)$ be the algebra of all bounded operators on H, and let $T(H)$ be the ideal of $B(H)$ consisting of all trace class operators on H. For a left $B(H)$-module X we will denote the action of $B(H)$ on X by $a, x \to a \cdot x$.

DEFINITION 2.1. A *pre-Hilbert $B(H)$-module* is a left $B(H)$-module X equipped with a map $[\cdot, \cdot]$; $X \times X \to T(H)$ satisfying:

(i) $[x+y, z]=[x, z]+[y, z]$,

(ii) $[a \cdot x, y]=a[x, y]$,

(iii) $[x, y]^*=[y, x]$,

(iv) $[x, x] \geqq 0$, and $[x, x]=0$ only if $x=0$,

for all x, y, z in X, a in $B(H)$. The map $[\cdot, \cdot]$ will be called the *Gramian* on X.

It is easy to see that $1 \cdot x=x$ for the identity 1 in $B(H)$, x in X, and that $[x, a \cdot y]=[x, y]a^*$ for all a in $B(H)$, x, y in X (cf. [6]).

For a pre-Hilbert $B(H)$-module X, we define the scalar multiplication by $\alpha x=(\alpha 1) \cdot x$ for complex α, x in X, the inner product by $(x, y)=\mathrm{Tr}\,[x, y]$, and the norm by $\|x\|_2=(x, x)^{1/2}$. Then X has also a pre-Hilbert space structure. A pre-Hilbert $B(H)$-module which is complete with respect to the norm $\|\cdot\|_2$ is called a Hilbert $B(H)$-module.

When H is one-dimensional, we have $B(H)=T(H)=C$, and hence the concept of a *Hilbert $B(H)$-module* coincides with that of a Hilbert space.

The Hilbert space H itself is a simple but important example of a Hilbert $B(H)$-module. In fact, for each ξ, η in H, denote by $\xi \otimes \bar{\eta}$ the operator on H given by

$$(\xi \otimes \bar{\eta})\zeta=(\zeta, \eta)\xi$$

for all ζ in H, as in [9], then $[\xi, \eta]=\xi \otimes \bar{\eta}$ defines a Gramian on H, under the natural action of $B(H)$ on H, and the inner product given by $\mathrm{Tr}\,[x, y]$ coincides with the original one. Moreover, we can show that every Gramian on the left $B(H)$-module H is of this form if $\mathrm{Tr}\,[\xi, \eta]=(\xi, \eta)$, as follows.

PROPOSITION 2.2. *Let H be a left $B(H)$-module with natural action, and F be a $T(H)$-valued function satisfying the defining conditions of the Gramian on H. Then there is a positive λ such that $F[\xi, \eta]=\lambda \xi \otimes \bar{\eta}$ for all ξ, η in H. If $\mathrm{Tr}\, F[\xi, \xi]=(\xi, \xi)$ for some ξ then $\lambda=1$.*

Proof. Let ϕ be a unit vector in H. Then we have that $F[\phi, \phi] \geqq 0, \neq 0$ and that

$$F[\phi, \phi]=F[(\phi \otimes \bar{\phi})\phi, (\phi \otimes \bar{\phi})\phi]=\phi \otimes \bar{\phi} F[\phi, \phi] \phi \otimes \bar{\phi},$$

and hence there is a positive λ such that $F[\phi, \phi]=\lambda \phi \otimes \bar{\phi}$. For any ξ, η in H, we have that

$$F[\xi, \eta]=F[(\xi \otimes \bar{\phi})\phi, (\eta \otimes \bar{\phi})\phi]=\xi \otimes \bar{\phi} F[\phi, \phi] \phi \otimes \bar{\eta}=\lambda \xi \otimes \bar{\eta},$$

and that $\mathrm{Tr}\, F[\xi, \eta]=\lambda(\xi, \eta)$. Q. E. D.

DEFINITION 2.3. Let X be a Hilbert $B(H)$-module, and let G be a locally compact group. A family $\{x_t ; t \text{ in } G\}$ of elements of X is called a *G-stationary process* on X if the following conditions are satisfied
(i) the Gramian $[x_s, x_t]$ depends only on $t^{-1}s$,
(ii) the function $t \to [x_t, x_e]$ is weakly continuous,
(iii) $\{x_t ; t \text{ in } G\}$ spans X, that is, the smallest closed submodule containing $\{x_t ; t \text{ in } G\}$ is X. The function $\Gamma(t)=[x_t, x_e]$ is called the *covariance function* of $\{x_t\}$.

DEFINITION 2.4. Let X and Y be two Hilbert $B(H)$-modules with Gramians $[\cdot, \cdot]_X$ and $[\cdot, \cdot]_Y$, respectively. A map U from X onto Y is called an *isomorphism* if U satisfies that

(i) $U(x+y)=Ux+Uy$,

(ii) $U(a \cdot x)=a \cdot Ux$,

(iii) $[Ux, Uy]_Y=[x, y]_X$,

for all x, y in X, a in $B(H)$. We say that two Hilbert $B(H)$-modules are *equivalent* if there is an isomorphism from one onto another. Let $\{x_t\}$ and $\{y_t\}$ be two G-stationary processes on X and Y, respectively. We say that $\{x_t\}$ and $\{y_t\}$ are *equivalent* if there is an isomorphism U from X onto Y such that $Ux_t=y_t$, for all t in G.

Our formulations of Hilbert $B(H)$-modules and G-stationary processes may provide a nice setting for the Hilbert space valued stationary processes, in view of the following examples.

EXAMPLE 2.5. Let K^q be the Cartesian product of a Hilbert space K with itself n times, i.e., the set of all vectors $x=(x_1, \cdots, x_q)$ such that each x_i is in K. For x, y in K^q, the $q \times q$-matrix (a_{ij}) defined by $a_{ij}=(x_i, y_j)$ is called the Gramian of the ordered pair x, y. Then K^q is a Hilbert $B(C^q)$-module with Gramian $[x, y]=(a_{ij})$, as explained in the Masani's survey [p. 353; 5].

EXAMPLE 2.6. Let (Ω, P) be a probability measure space, let H be a separable Hilbert space, and let $L^2(H)$ the Hilbert space of all square Bochner integrable H-valued functions on (Ω, P). Then it is easy to see that $L^2(H)$ is a left $B(H)$-module in the obvious way. For any pair x, y in $L^2(H)$ there corresponds a unique trace class operator $[x, y]$ such that

$$([x, y]\xi, \eta)=\int_\Omega (\xi, y(\omega))(x(\omega), \eta)P(d\omega),$$

and that $\mathrm{Tr}\,[x, y]=\int_\Omega (x(\omega), y(\omega))P(d\omega)$. Then it is easy to see that $L^2(H)$ is a Hilbert $B(H)$-module with Gramian $[x, y]$, whose properties are investigated in Umegaki [11] in connection with the tensor product Hilbert space. Let $\{x_t; t \text{ in } R\}$ be a family of H-valued random variables in $L^2(H)$ such that $[x_s, x_t]$ depens only on $s-t$, and $\Gamma(t)=[x_t, x_0]$ is weakly continuous. Then $\{x_t\}$ is called the H-valued stationary process. In this case usually the time domain X of the process $\{x_t\}$ is defined as the closed submodule of $L^2(H)$ spanned by $\{x_t\}$. Thus the H-valued stationary process $\{x_t\}$ with time domain X is a R-stationary process on X in our sense. Further information on such a process will be found in many literatures, for instance [3].

EXAMPLE 2.7. Let $S(K, H)$ be the set of all bounded linear transformations x from a Hilbert space K to a Hilbert space H, such that xx^* is a trace class operator on H where x^* is a bounded linear transformation from H to K defined by the relation $(x^*\xi, \eta)=(\xi, x\eta)$ for all ξ in H, η in K. Then it is easy to see that $S(K, H)$ is a Hilbert $B(H)$-module with Gramian $[x, y]=xy^*$. We call this the Hilbert $B(H)$-module $S(K, H)$.

3. Positive sesquilinear maps.

In order to provide some technical results used in the later sections, we shall study positive sesquilinear maps with values in the predual of a W^*-algebra in this section.

Let P be a linear map from a C^*-algebra A into a C^*-algebra B. Let $M_n(A)$ be the C^*-algebra of all $n\times n$-matrices with entries in A, and let P_n be the linear map from $M_n(A)$ into $M_n(B)$ obtained by applying P to each entry of an element of $M_n(A)$. We say that P is *n-positive* if P_n maps positive elements in $M_n(A)$ into positive elements in $M_n(B)$, and that P is *completely positive* if P is n-positive for each positive integer n. It should be remarked [6] that P in n-positive if and only if $\sum_{i,j} b_i^*P(a_i^*a_j)b_j\geqq 0$ for all $a_1, \cdots, a_n$ in A, $b_1, \cdots, b_n$ in B.

Let M be a W^*-algebra, that is, M is a C^*-algebra which is a dual space of a Banach space M_*. We denote the norm on M and M_* by $\|\cdot\|_\infty$ and $\|\cdot\|_1$ respectively, and by $\langle\cdot, \cdot\rangle$ the dual pair on $M\times M_*$. For f in M_* and a in M, we denote by f^*, $a\cdot f$ and $f\cdot a$, the elements of M_* defined by the relations $\langle b, f^*\rangle=\overline{\langle b^*, f\rangle}$, $\langle b, a\cdot f\rangle=\langle ba, f\rangle$ and $\langle b, f\cdot a\rangle=\langle ab, f\rangle$ for all b in M. For the case in which $M=B(H)$, M_* can be regarded as $T(H)$. In this case, $\langle a, f\rangle =\mathrm{Tr}\, af$, f^* is the adjoint of f, $a\cdot f=af$ and $f\cdot a=fa$ for all a in $B(H)$ and f in $T(H)$.

Let L be a linear space. We say that F is an *M_*-sesquilinear form* on L if it is an M_*-valued sesquilinear map on $L\times L$ which is conjugate linear in the second variable. In the following we consider an M_*-sesquilinear form F on L, and write $[x, y]$ for $F(x, y)$ if no confusion may occur. For a positive integer n, F is said to be *n-positive* (or *positive*, when $n=1$) if

$$\sum_{i,j} \langle a_j^*a_i, [x_i, x_j]\rangle\geqq 0 \tag{1}$$

for all $a_1, \cdots, a_n$ in M, $x_1, \cdots, x_n$ in L. F is said to be *completely positive* if F is n-positive for all positive integer n. It is easy to see that F is n-positive if and only if $\sum_{i,j} a_i \cdot [x_i, x_j] \cdot a_j^* \geqq 0$ for all $a_1, \cdots, a_n$ in M, $x_1, \cdots, x_n$ in L.

LEMMA 3.1. *Let N be a self-adjoint subalgebra of a W^*-algebra M which is $\sigma(M, M_*)$-dense in M. Then an M_*-sesquilinear form F on L is n-positive if it satisfies the inequality* (1) *for all $a_1, \cdots, a_n$ in N, $x_1, \cdots, x_n$ in L.*

Proof. Let S be a unit sphere of M. Then by Kaplansky's density theorem, $N \cap S$ is $s(M, M_*)$-dense in S. Since the multiplication on M is jointly $s(M, M_*)$-continuous on S, it is easy to see that the inequality (1) holds for all $a_1, \cdots, a_n$ in S, and hence multiplying positive numbers it holds for all $a_1, \cdots, a_n$ in M. Q. E. D

Obviously n-positivity follows from $n+1$-positivity, and every positive M_*-sesquilinear form is symmetric in the sense that $[x, y]=[y, x]^*$ for all x, y in L. For a positive M_*-sesquilinear form F, each a in M defines a semidefinite inner product $x, y \rightarrow \langle a^*a, [x, y]\rangle$ on L, and so by the usual Schwartz inequality we have that

$$|\langle a^*a, [x, y]\rangle|^2 \leqq \langle a^*a, [x, x]\rangle \langle a^*a, [y, y]\rangle$$

for all x, y in L. But a more delicate form of a Schwartz inequality characterizes the 2-positivity of F.

THEOREM. 3.2. *Let F be a positive M_*-sesquilinear form on L. Then F is 2-positive if and only if*

$$|\langle b^*a, [x, y]\rangle|^2 \leqq \langle a^*a, [x, x]\rangle \langle b^*b, [y, y]\rangle \tag{2}$$

for all a, b in M, x, y in L.

Proof. Suppose that F is 2-positive. Let t be a real number, and α be such that $\alpha = t|\langle b^*a, [x, y]\rangle| \langle b^*a, [x, y]\rangle^{-1}$. Then we have for $a_1=a$, $a_2=b$, $x_1=\alpha x$, $x_2=y$,

$$\begin{aligned} 0 &\leqq \sum_{i,j=1}^{2} \langle a_j^* a_i, [x_i, x_j]\rangle \\ &= t^2 \langle a^*a, [x, x]\rangle + 2t|\langle b^*a, [x, y]\rangle| + \langle b^*b, [y, y]\rangle . \end{aligned}$$

Since t is arbitrary, we have the required inequality. Conversely, if the inequality (2) holds, we have

$$\begin{aligned} &\sum_{i,j=1}^{2} \langle a_j^* a_i, [x_i, x_j]\rangle \\ &= \sum_{i=1}^{2} \langle a_i^* a_i, [x_i, x_i]\rangle + 2\mathrm{Re}\langle a_2^* a_1, [x_1, x_2]\rangle \\ &\geqq 2(\langle a_1^* a_1, [x_1, x_1]\rangle \langle a_2^* a_2, [x_2, x_2]\rangle)^{1/2} + 2\mathrm{Re}\langle a_2^* a_1, [x_1, x_2]\rangle \\ &\geqq 2(|\langle a_2^* a_1, [x_1, x_2]\rangle| + \mathrm{Re}\langle a_2^* a_1, [x_1, x_2]\rangle) \geqq 0 . \end{aligned}$$

Q. E. D.

From the above inequality we can constitute an elementary proof of M. D. Choi's inequality $P(a^*a)\geqq P(a^*)P(a)$ for 2-positive unit-preserving linear maps on a C^*-algebra.

COROLLARY 3.3. *If P is a 2-positive linear map from a C^*-algebra A into a C^*-algebra B, then $\|P\|P(a^*a)\geqq P(a^*)P(a)$ for all a in A, and $\|P\|=\|P(1)\|$ if A is unital.*

Proof. Without any loss of generality, we assume B is faithfully represented on a Hilbert space L, and that P is a restriction of a normal linear map P' on A^{**} into $B(L)$. Define A^*-sesquilinear form $[\cdot, \cdot]$ on L by the relation $\langle a, [x, y]\rangle=(P'(a)x, y)$ for all a in A^{**}, x, y in L. Then it is easy to see that $[\cdot, \cdot]$ is a 2-positive A^*-sesquilinear form, from the 2-positivity of P and Lemma 3.1. Thus applying the inequality (2) for $b=1$ in A^{**}, a in A, $y=P(a)x$, the routine calculus shows that $\|P(a)x\|^2\leqq\|P'(1)\|(P(a^*a)x, x)$, so that $\|P'(1)\|\, P(a^*a)\geqq P(a^*)P(a)$. It follows that $\|P(a)\|^2=\|P(a^*)P(a)\|\leqq \|P'(1)\|\|P(a^*a)\|\leqq\|P'(1)\|\|P\|\|a\|^2$, and thus $\|P\|=\|P'(1)\|$. Q. E. D.

In the following, we denote by $(\cdot, \cdot)$ the semidefinite inner product on L given by $(x, y)=\langle 1, [x, y]\rangle$ and by $\|\cdot\|_2$ the seminorm on L given by $\|x\|_2{}^2=\|[x, x]\|_1$.

PROPOSITION 3.4. *Every 2-positive M_*-sesquilinear form F on L satisfies that*

$$\|[x, y]\|_1=\|x\|_2\|y\|_2 \tag{3}$$

for all x, y in L.

Proof. By the polar decomposition of elements of M_*[1. 14. 4; 8] there is a partial isometry u in M such that $\|[x, y]\|_1=\langle u, [x, y]\rangle$. Thus a simple application of the inequality (2) concludes the inequality (3). Q. E. D.

A positive M_*-sesquilinear form F is said to be *M_*-inner product* if $[x, x]=0$ implies $x=0$. From every positive M_*-sesquilinear form F on L, we can obtain an M_*-inner product on the factor space L/N where $N=\{x$ in L; $[x, x]=0\}$ as $[x+N, y+N]=[x, y]$. If F is a 2-positive M_*-inner product on L, then the completion $\bar{L}$ of L by the norm $\|\cdot\|_2$ is a Hilbert space, to which we can extend F uniquely by continuity shown in Proposition 3.4.

THEOREM 3.5. *Let F be an n-positive ($n\geqq 2$) M_*-sesquilinear form on L, and W be the Hilbert space obtained by factoring and completing L. Then there is a uuique unit-preserving normal n-positive linear map ρ on M into $B(W)$ such that*

$$(\rho(a)\dot{x}, \dot{y})=\langle a, [x, y]\rangle \tag{4}$$

*for all a in M, x, y in L, where $\dot{x}$, $\dot{y}$ are the corresponding elements in the factor space, which is a *-representation if and only if*

$$[\rho(a)\dot{x}, \dot{y}]=a\cdot[x, y] \tag{5}$$

for all a in M and x in L.

Proof. For each a in M, by the inequality (2) we have

$$|\langle a, [x, y]\rangle| \leqq \|a\|_\infty \|x\|_2 \|y\|_2$$

for all x, y in L. Thus the sesquilinear form $\dot{x}, \dot{y} \to \langle a, [x, y]\rangle$ on the factor space defines a unique bounded linear operator $\rho(a)$ on W such that $(\rho(a)\dot{x}, \dot{y}) = \langle a, [x, y]\rangle$ for all x, y in L. Now to check the required properties of the map $\rho: a \to \rho(a)$ is a matter of routine calculation. The last part of the assertion follows from the following relations using the fact that $[x, y]=[\dot{x}, \dot{y}]$,

$$(\rho(ab)\dot{x}, \dot{y}) = \langle ab, [x, y]\rangle, \quad \langle a, [\rho(b)\dot{x}, \dot{y}]\rangle = (\rho(a)\rho(b)\dot{x}, \dot{y})$$

for all a, b in M, x, y in L. Thus the proof is completed. Q. E. D.

The map ρ will be called the *assoiciate map* of F.

LEMMA 3.6. *Let F be a positive M_*-sesquilinear form on L, and π a map on M whose values are linear transformations on L such that $[\pi(a)x, y] = a \cdot [x, y]$ for all a in M, x, y in L. Then π induces a non-degenerate normal *-representation of M on the Hilbert space W obtained by factoring and completing L.*

Proof. First we observe that $[\pi(a)x, \pi(b)y] = a \cdot [x, y] \cdot b^*$. It follows that F is completely positive and that the set $N = \{x \text{ in } L : [x, x] = 0\}$ is invariant under $\pi(a)$. Thus we can consider that $\pi(a)$ acts on the factor space L/N. If ρ is the associate map of F, then we have that

$$(\rho(a)\dot{x}, \dot{y}) = \langle a, [x, y]\rangle = \langle 1, a \cdot [x, y]\rangle = (\pi(a)\dot{x}, \dot{y})$$

for all x, y in L. Thus $\rho = \pi$, and the conclusion follows from Theorem 3.5.
Q. E. D.

Let F be a completely positive M_*-sesquilinear form on L, and $M \otimes L$ be the algebraic tensor product of M and L. Define an M_*-sesquilinear form F' on $M \otimes L$ by

$$F'\Big[\sum_i a_i \otimes x_i, \sum_j b_j \otimes y_j\Big] = \sum_{i,j} a_i \cdot F[x_i, y_j] \cdot b_j^*$$

Then by the complete positively of F, F' is positive. Let $\pi(a)$ be the linear map on $M \otimes L$ such that

$$\pi(a) \sum_i a_i \otimes x_i = \sum_i a a_i \otimes x_i$$

Then clearly $F'[\pi(a)x, y] = a \cdot F'[x, y]$ for all x, y in $M \otimes L$, and hence by Lemma 3.6, π induces a non-degenerate normal *-representation on the Hilbert space obtained by factoring and completing $M \otimes L$. This construction of a *-representation is essentially same to that found by Stinespring [12].

Let G be a locally compact group. An M_*-valued function V on G is said to be positive definite if it satisfies that

$$\sum_{i,j} \langle a_j^* a_i, V(t_j^{-1} t_i)\rangle \geqq 0$$

for all positive integer n, $a_1, \cdots, a_n$ in M, and $t_1, \cdots, t_n$ in G, and it is said to be weakly continuous if $t \to \langle a, V(t)\rangle$ is continuous on G for all a in M.

It should be remarked that every positive definite M_*-valued function V can be extended to the completely positive M_*-sesquilinear form F on the linear space $F(G)$ of all complex-valued functions on G with finite support, which satisfies that

$$F[x, y]=\sum_s \sum_t x(s)\overline{y(t)}V(t^{-1}s)$$

for all x, y in $F(G)$.

THEOREM 3.7. *Let V be a weakly continuous positive definite M_*-valued function on G. Then there is a Hilbert space W, a non-degenerate normal *-representation π of M on W, a strongly continuous unitary representation U of G on W and a vector ξ in W satisfying:*

(i) *$U(t)$ is in $\pi(M)'$ for all t in G,*

(ii) *the linear span of the set $\{\pi(a)U(t)\xi;\ a$ in M, t in $G\}$ is dense in W,*

(iii) *$\langle a, V(t)\rangle=(\pi(a)U(t)\xi, \xi)$ for all a in M and t in G.*

Proof. Let $F(G, M)$ be the linear space of all M-valued functions on G whose value is 0 outside a finite set of G. Define an M_*-sesquilinear form $[\cdot, \cdot]$ on $F(G, M)$ by

$$[f, g]=\sum_s \sum_t f(s)\cdot V(t^{-1}s)\cdot g(t)^*$$

and put $(f, g)=\langle 1, [f, g]\rangle$ for all f, g in $F(G, M)$. Then $[\cdot, \cdot]$ is positive, and $(\cdot, \cdot)$ is a semidefinite inner product on $F(G, M)$. For a in M, t in G, define linear maps $\pi(a)$, $U(t)$ on $F(G, M)$ by the relations

$$(\pi(a)f)(s)=af(s)$$

$$(U(t)f)(s)=f(t^{-1}s)$$

for all f in $F(G, M)$, s in G. Then it is easy to see that

$$[\pi(a)f, g]=a\cdot[f, g]$$

$$[U(t)f, U(t)g]=[f, g]$$

for all f, g in $F(G, M)$. Then the subset $N=\{f$ in $F(G, M);\ [f, f]=0\}$ is invariant under $\pi(a)$ and $U(t)$. Let W be the Hilbert space obtained by completing $F(G, M)/N$. Then by Lemma 3.6, π induces a non-degenerate normal *-representation of M on W, and similarly U induces a unitary representation of G. Since it is easy to see that $\pi(a)U(t)f=U(t)\pi(a)f$ for all f in $F(G, M)$, the condition (i) is clearly satisfied. Let ξ be the vector in W induced from 1_e in $F(G, M)$ such that $1_e(e)=1$ and that $1_e(s)=0$ if $s\neq e$, where e is the unit of G. Then clearly the set $\{\pi(a)U(t)1_e;\ a$ in M, t in $G\}$ spans $F(G, M)$ and hence the condition (ii) is obvious. Observing that $[U(t)1_e, 1_e]=V(t)$, we have that $\langle a, V(t)\rangle=(\pi(a)U(t)1_e, 1_e)$, and so we obtain the condition (iii). Now we have only to show the strong continuity of U. Since the routine calculus shows that

$$(U(u)f, g)=\sum_s \sum_t \langle g(t)^*f(s), V(t^{-1}us)\rangle$$

for all u in G, f, g in $F(G, M)$, the weak continuity of U implies the weak continuity of U. Thus the strong continuity of U is concluded from the fact that the strong and weak topologies coincide on the unitary group of W.

Q. E. D.

The above construction of a unitary representation is a variation of that found by Umegaki [10].

4. The structure of Hilbert $B(H)$-modules.

In this section we shall study the structure of Hilbert $B(H)$-modules, and show that Hilbert $B(H)$-modules have a quite similar structure to that of usual Hilbert spaces except that $B(H)$ is not a field.

Let $\{X_i\}$ be a family of Hilbert $B(H)$-modules and let $\sum_i X_i$ be the Hilbert space direct sum of $\{X_i\}$ which is a left $B(H)$-module in the obvious way. Let $x=(x_i)$ and $y=(y_i)$ be in $\sum_i X_i$. By Proposition 3.4 we have $\|[x_i, y_i]\|_1 \leqq \|x_i\|_2\|y_i\|_2$, where $[\cdot, \cdot]$ is the Gramian on X_i, $\|\cdot\|_1$ is the trace norm on $T(H)$, and $\|\cdot\|_2$ is the Hilbert space norm on X_i. Then it is easy to see that the family of trace class operators $\{[x_i, y_i]\}$ is summable in the trace norm. Now we define the Gramian on $\sum_i X_i$ by $[x, y]=\sum_i [x_i, y_i]$. Then we have that $(x, y)=\sum_i (x_i, y_i)=\mathrm{Tr}\,[x, y]$, and hence $\sum_i X_i$ is a Hilbert $B(H)$-module which is called the direct sum of the family $\{X_i\}$ of Hilbert $B(H)$-modules.

LEMMA 4.1. *Let X and Y be two Hilbert $B(H)$-modules, and U be a map from X onto Y. Then the following three conditions are equivalent:*

(i) *U is an isomorphism from X to Y;*

(ii) *U is a unitary operator from X to Y such that $a\cdot Ux=U(a\cdot x)$ for all x in X;*

(iii) *U satisfies $[Ux, Uy]=[x, y]$ for all x, y in X.*

Proof. It is trivial that (i) implies (ii). The routine calculus shows that (ii) implies that $\mathrm{Tr}\,(a[Ux, Uy])=\mathrm{Tr}\,(a[x, y])$ for all a in $B(H)$, x, y in H. Thus (ii) implies (iii). It is easy to verify that (iii) implies that $[U(x+y)-Ux-Uy, U(x+y)-Ux-Uy]=0$ and that $[U(a\cdot x)-a\cdot Ux, U(a\cdot x)-a\cdot Ux]=0$ for all a in $B(H)$, x, y, z in X. Thus clearly (iii) implies (i). Q. E. D.

THEOREM 4.2. *Every Hilbert $B(H)$-module X is equivalent to a direct sum $\sum H$ of (possibly infinitely many) copy of the Hilbert $B(H)$-modules H.*

Proof. By Lemma 3.6 there is a non-degenerate normal *-representation π of $B(H)$ on the Hilbert space X such that $(\pi(a)x, y)=\langle a, [x, y]\rangle$ for all x, y in X. Since every *-representation of the C^*-algebra $C(H)$ of all compact operators on H is unitarily equivalent to a direct sum of the identity representation on H, and since $C(H)$ is weakly dense in $B(H)$, we can conclude that π is untarily equivalent to a direct sum of the identity representation of $B(H)$ on H. Thus by Lemma 4.1 this unitary equivalence induces the equivalence between two Hilbert $B(H)$-modules X and $\sum H$. Q.E.D.

COROLLARY 4.3. *Every Hilbert $B(H)$-module X is equivalent to the Hilbert $B(H)$-module $S(K, H)$ for some Hilbert space K.*

Proof. By Theorem 4.2 there is an index set I such that X is equivalent to $\sum_i \{H_i ; i \text{ in } I\}$ ($H_i=H$ for all i). Let K be a Hilbert space with basis $\{\xi_i ; i \text{ in } I\}$. For every element $x=(x_i)$ (x_i in H), define an operator $Ux=\sum_i x_i\otimes\bar{\xi}_i$ from K to H. Then it is easy to verify that Ux is in $S(K, H)$ and that the correspondence U: $x\rightarrow Ux$ is an isomorphism from X onto $S(K, H)$.
Q.E.D.

In order to proceed to the Fourier expansion of elements of a Hilbert $B(H)$-module, in which the Fourier coefficients are given by the Gramian, we shall define the basis of the Hilbert $B(H)$-module, as follows,

DEFINITION 4.4. Let $\{x_i\}$ be a family of elements of a Hilbert $B(H)$-module X. We say that $\{x_i\}$ is *modular othonormal* if

(i) $[x_i, x_j]=0$, if $i\neq j$,

(ii) $[x_i, x_i]^2=[x_i, x_i]$ and $\|x_i\|_2=1$ for each i.

A maximal modular orthonormal family is called a *modular basis.*

By Zorn's lemma, every Hilbert $B(H)$-module has a modular basis.

THEOREM 4.5. *The following conditions for a modular orthonormal family $\{x_i\}$ of elements of a Hilbert $B(H)$-module X are all equivalent.*

(i) *The family $\{x_i\}$ is a modular basis of X.*

(ii) *If x is in X and if $[x, x_i]=0$ for all i, then $x=0$.*

(iii) *If, for each i, X_i is the set $\{a\cdot x_i ; a \text{ is in } B(H)\}$, then $X=\sum_i X_i$ (the Hilbert space direct sum).*

(iv) *For all x in X, $x=\sum_i [x, x_i]\cdot x_i$.*

(v) *For all x in X, $[x, y]=\sum_i [x, x_i][x_i, y]$, where the infinite sum is defined as unconditionaly covergence in $\|\cdot\|_1$-norm.*

(vi) *For all x in X, $[x, x]=\sum_i |[x_i, x]|^2$, where $|\cdot|$ is such that $|a|=(a^*a)^{1/2}$.*

Proof. (i) implies (ii): If x is in X, if $[x, x_i]=0$ for all i, and if $x\neq 0$, then the non-zero positive trace class operator $[x, x]$ on H has a positive eigen value λ with eigen vector ϕ, $\|\phi\|_2=1$. Let a be the operator on H defined by $a\xi=\lambda^{-1/2}(\xi, \phi)\phi$ for all ξ in H, and x_0 be such an element in X that $x_0=a\cdot x$. Then the routine calculus shows that $[x_0, x_0]^2=[x_0, x_0]$ and that $\|x_0\|_2=1$. Thus we can add x_0 to the family $\{x_i\}$. This contradicts the assumed maximality of the family.

(ii) implies (iii): If $X\neq\sum_i X_i$, then there is a vector x in X such that $(x, y)=0$ for all y in $\sum_i X_i$ but that $x\neq 0$. Thus

$$\|[x, x_i]\|_1=\mathrm{Tr}\,(u[x, x_i])=(x, u^*\cdot x_i)=0$$

where u is an partial isometry on H, and hence $[x, x_i]=0$ for all i, since u^*x_i is in X_i. This contradicts (ii).

(iii) implies (iv): Assuming (iii), any x in X can be written as $x=\sum_i a_i\cdot x_i$ where a_i is in $B(H)$. Then a routine calculus using the fact that $[x_i, x_i]\cdot x_i=x_i$ shows that $a_1\cdot x_i=[x, x_i]\cdot x_i$. Thus we have the required formula.

The remaining part of the proof is now an easy matter. Q. E. D.

COROLLARY 4.6 *Any two modular basis of a Hilbert $B(H)$-module have the same power.*

Proof. Let $\{x_i;\ i \text{ in } I\}$, $\{y_j;\ j \text{ in } J\}$ be two modular basis of a Hilbert $B(H)$-module X. Then from (iii) of Theorem 4.5, we can see that X is equivalent to $\sum_i H_i$ and $\sum_j H_j$, where $H_i=H_j=H$, since $X_i=\{a\cdot x_i;\ a \text{ in } B(H)\}$ (or $=\{a\cdot y_j;\ a \text{ in } B(H)\}$) is equivalent to H. Thus the powers of I and J are same, since they are two multiplicities of the two unitarily equivalent representations of $B(H)$. Q. E. D.

The common power of all modular basis of a Hilbert $B(H)$-module X is called the *modular dimention* of X and written as $\mathrm{Dim}\,(X)$.

The following Theorem is now an immediate consequence of Theorem 4.2.

THEOREM 4.7. *Two Hilbert $B(H)$-modules are equivalent if and only if they have the same modular dimension.*

It should be remarked that $\mathrm{Dim}\,(X)\cdot\dim\,(H)=\dim\,(X)$ and that $\mathrm{Dim}\,(S(K, H))=\dim\,(K)$, where $\dim\,(\cdot)$ is the usual dimention of Hilbert spaces.

5. Equivalence of G-stationary processes.

In the following, we shall consider a fixed locally compact group G. Recall that a $T(H)$-valued function V on G is positive definite if and only if for any positive integer n, we have that $\sum_{i,j}\mathrm{Tr}\,(a_j{}^*a_i V(t_j{}^{-1}t_i))\geqq 0$ for all

$a_1, \cdots, a_n$ in $B(H)$, $t_1, \cdots, t_n$ in G. Let X be a Hilbert $B(H)$-module, and let $\{x_t\}$ be a G-stationary process on X. Then it is easy to see that the covariance function Γ of the process $\{x_t\}$ is a $T(H)$-valued positive definite function on G. Now we shall show that every positive definite $T(H)$-valued function is the covariance function of some, but unique up to equivalence, G-stationary process on a Hilbert $B(H)$-module.

THEOREM 5.1. *Let Γ be a weakly continuous positive definite $T(H)$-valued function on G. Then there exist a Hilbert $B(H)$-module X and a G-stationary process $\{x_t\}$ on X whose covariance function is Γ. In this case there is a strongly continuous unitary representation U of G on X such that $x_t=U(t)x_e$ and that $U(t)(a\cdot x)=a\cdot U(t)x$ for all t on G, a in $B(H)$, x in X.*

Proof. Applying Theorem 3.7 to the case $M=B(H)$, we have a Hilbert $B(H)$-module X, a non-degenerate normal *-representation π of $B(H)$ on X, a strongly continuous unitary representation U of G on X and a vector x_e in X which satisfy the conditions in that theorem. Put the process $\{x_t\}$ as $x_t=U(t)x_e$ for all t in G, and define the Gramian $[\cdot, \cdot]$ on X by $\mathrm{Tr}\,(a[x, y])=(\pi(a)x, y)$ for all a in $B(H)$, x, y in X. Then the routine calculus shows that

$$\mathrm{Tr}\,(a\Gamma(t^{-1}s))=(\pi(a)U(s)x_e,\ U(t)x_e)=\mathrm{Tr}\,(a[x_s,\ x_t])$$

for all a in $B(H)$, s, t in G. Thus the conclusion follows from Theorem 3.7.

Q. E. D.

THEOREM 5.2. *Let X and Y be two Hilbert $B(H)$-modules. Let $\{x_t\}$ be a G-stationary process on X with covariance function Γ_x, and let $\{y_t\}$ be a G-stationary process on Y with convariance function Γ_y. Then $\{x_t\}$ and $\{y_t\}$ are equivalent if and only if $\Gamma_x(t)=\Gamma_y(t)$ for all t in G.*

Proof. Since the "only if" part is trivial, we assume $\Gamma_x(t)=\Gamma_y(t)$ for all t in G. First we observe that

$$\left[\sum_i a_i\cdot x_{t_i},\ \sum_j b_j\cdot x_{s_j}\right]=\sum_{i,j} a_i\Gamma_x(s_j{}^{-1}t_i)b_j{}^*$$
$$=\left[\sum_i a_i\cdot y_{t_i},\ \sum_j b_j\cdot y_{s_j}\right]$$

for all $a_1, \cdots, a_n, b_1, \cdots, b_m$ in $B(H)$, $t_1, \cdots, t_n, s_1, \cdots, s_m$ in G, and given n, m. Thus putting $U'\left(\sum_i a_i\cdot x_{t_i}\right)=\sum_i a_i\cdot y_{t_i}$, we can define a map U' from X' onto Y' such that $[U'x, U'y]=[x, y]$ for all x, y in X', where X' and Y' are submodules spanned by $\{x_t\}$ and $\{y_t\}$, respectively. Then since U' is isometry, and since X' and Y' are dense in X and Y, we can extend U' to a map U on X onto Y such that $[Ux, Uy]=[x, y]$ for all x, y in X. Thus the conclusion follows from Lemma 4.1. Q. E. D.

From Theorem 4.3, every G-stationary process on a Hilbert $B(H)$-module X can be regarded as a G-stationary process on the Hilbert $B(H)$-module $S(K, H)$ for some Hilbert space K with $\dim(K)=\dim(X)$. Thus the following theorem shows that every G-stationary process on a Hilbert $B(H)$-module X is equivalent to such as given by the theorem.

THEOREM. 5.3. *Let K be a Hilbert space, and let $\{x_t\}$ be a G-stationary process on the Hilbert $B(H)$-module $S(K, H)$ with covariance function Γ. Then there is a strongly continuous unitary representation U of G on K such that $x_t=x_eU(t)^*$ for all t in G, where $x_eU(t)^*$ is the product of two operators x_e and $U(t)^*$, and that $\Gamma(t)=x_eU(t)^*x_e^*$.*

Proof. Let $H\otimes K$ be the tensor product Hilbert space of H and K, and fix a basis $\{\psi_i\}$ of K. Then we can identify $S(K, H)$ and $H\otimes K$ by the correspondence $\sum_i \phi_i\otimes\bar{\psi}_i\rightarrow\sum_i \phi_i\otimes\psi_i$. By Theorem 5.1 and 5.2, we have a strongly continuous unitary representation V of G on $H\otimes K$ such that $x_t=V(t)x_e$ and that $V(t)a\cdot x=a\cdot V(t)x$ for all t in G, x in $S(K, H)$, a in $B(H)$. Since the all actions of $B(H)$ on $H\otimes K$ constitutes the von Neumann algebra $B(H)\otimes 1$ on $H\otimes K$, and $V(G)$ is contained in $(B(H)\otimes 1)'$ which is equal to $1\otimes B(K)$, we have a strongly continuous unitary representation U of G on K such that $V(t)=1\otimes U(t)$ for all t in G. Thus the conclusion follows from the computations that

$$x_t=V(t)x_e=(1\otimes U(t))\sum_i \phi_i\otimes\bar{\psi}_i=\sum_i \phi_i\otimes\overline{U(t)\psi_i}$$

$$=\Big(\sum_i \phi_i\otimes\overline{\psi_i}\Big)U(t)^*=x_eU(t)^*,$$

where $\phi_i=x_e\psi_i$ for all i, and that

$$\Gamma(t)=[x_t, x_e]=x_tx_e^*=x_eU(t)^*x_e^*. \qquad \text{Q.E.D.}$$

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