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HOLOMORPHIC MAPS OF RIEMANN SURFACES AND WEIERSTRASS POINTS

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To the memory of Professor Nobuyuki Suita

Abstract

We investigate holomorphic maps between hyperelliptic Riemann surfaces and prove rigidity theorems given by correspondences of Weierstrass points.

1. Introduction

In this paper, all of the Riemann surfaces will be compact and of genera greater than 1 if not specified. Concerning automorphisms of Riemann surfaces, the induced permutations of Weierstrass points determine the automorphisms. Namely, if an automorphism T on a Riemann surface X fixes all of the Weierstrass points, then $T = \mathrm{id.}$ or X is hyperelliptic and T is the hyperelliptic involution. This fact implies that the number of automorphisms is finite (Schwarz). It is natural to ask what happens when the target Riemann surface is different from the source. Let X and Y be compact Riemann surfaces and let $h : X \to Y$ be a nonconstant holomorphic map. Martens [Mr] pointed out the following (for Riemann surfaces of positive genera). *If $D = \sum m_i Q_i$ is a positive divisor of degree n and* (*projective*) *dimension r on X, then $h(D) = \sum m_i h(Q_i)$ is a positive divisor of degree n and dimension $\geq r$ on Y. If the complete linear system determined by D is without fixed points, then so is that determined by $h(D)$.* Hence it follows that when X is hyperelliptic, Y is also hyperelliptic and every Weierstrass point on X maps on a Weierstrass point on Y. We denote by W the set of Weierstrass points on X. We will show

Theorem 1. *Let X be a hyperelliptic Riemann surface and let Y be a Riemann surface. Let $h_1, h_2 : X \to Y$ be nonconstant holomorphic maps* (*then Y is also hyperelliptic*). *If $h_1|_W = h_2|_W$, then $h_1 = h_2$ or $h_1 = \tau \circ h_2$, where τ is the hyperelliptic involution.*

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This assertion gives an upper bound for the number of nonconstant holomorphic maps between hyperelliptic Riemann surfaces depending only on the number of Weierstrass points (or, we can say, on the genus).

We may put another interpretation on Theorem 1 in terms of homology groups. A holomorphic map $h : X \to Y$ induces a homomorphism

$$h_{*n} : H_1(X, \mathbf{Z}_n) \to H_1(Y, \mathbf{Z}_n)$$

between the first homology groups with coefficients in the integers mod n. For automorphisms, denote by $\phi : \mathrm{Aut}(X) \to \mathrm{Sp}(g, \mathbf{Z}_n)$ the natural homomorphism, where $\mathrm{Sp}(g, \mathbf{Z}_n)$ is the symplectic group of genus g over $\mathbf{Z}_n$. Then, it is shown that for $n > 2$, ϕ is injective, and for $n = 2$, only automorphisms of order 2 is in the kernel of ϕ (for the proof see [F-K] Chapter 5). For holomorphic maps, if two holomorphic maps $h_i : X \to Y$ $(i = 1, 2)$ induce the same homomorphism $H_1(X, \mathbf{Z}_n) \to H_1(Y, \mathbf{Z}_n)$ for some $n > \sqrt{8(g-1)}$ where g is the genus of X, then $h_1 = h_2$ (see [T]). For hyperelliptic case, we may take $n = 2$.

Theorem 2. *Let X be a hyperelliptic Riemann surface and let Y be a Riemann surface. Let $h_1, h_2 : X \to Y$ be nonconstant holomorphic maps* (*then Y is also hyperelliptic*). *If h_1 and h_2 induce the same homomorphism $H_1(X, \mathbf{Z}_2) \to H_1(Y, \mathbf{Z}_2)$, then $h_1 = h_2$ or $h_1 = \tau \circ h_2$.*

Further, we will investigate the case where the target surface is not fixed.

For hyperelliptic Riemann surfaces, the case where the covering is normal has been studied (see [F], [H], [K] and [Mc]).

2. Preliminaries

Let X be a compact Riemann surface of genus $g > 1$. Recall that for each point $p \in X$, there are g integers called gaps

$$1 = n_1 < n_2 < \cdots < n_g < 2g$$

such that there does not exist a meromorphic function on X holomorphic on $X - \{p\}$ with a pole of order m at p if and only if $m = n_i$. Except only a finite number of points, $\{n_1, \dots n_g\} = \{1, \dots, g\}$ and the excepted points are called Weierstrass points. We denote by W the set of Weierstrass points. Then, the number of Weierstrass points $\#W$ satisfies

$$2g + 2 \le \#W \le g^3 - g.$$

A Riemann surface with a meromorphic function of degree 2 is called to be hyperelliptic. X is hyperelliptic if and only if $\#W = 2g + 2$. Each Weierstrass point on a hyperelliptic Riemann surface of genus g has the gap sequence $\{1, 3, \dots, 2g - 1\}$.

Let $h : X \to Y$ denotes a nonconstant holomorphic map between Riemann surfaces. The Martens' assertion in the introduction (or a little modification of his proof in [Mr]) gives that *if D is the polar divsor of some meromorphic function* *on X, then $h(D)$ is the polar divisor of some meromorphic function on Y.* Thus, we see that if n is a non-gap for $p \in X$, then n is also a non-gap for $h(p) \in Y$. Especially,

LEMMA 1 (Martens [Mr]). *When X is hyperelliptic, Y is also hyperelliptic and every Weierstrass point of X maps on a Weierstrass point of Y. Furthermore, the involution of X followed by h is equivalent to h followed by the involution of Y.*

The first assertion makes the assumption of Theorem 1 meaningful.

A preimage of a Weierstrass point is not always a Weierstrass point even if X is hyperelliptic. We see this by the following example.

Example 1. let X be a hyperelliptic Riemann surface of genus 3 which has an fixed-point-free automorphism of order 2. Then the quotient surface Y is of genus 2 and there are 6 Weierstrass points on Y. Thus, the number of preimages of them is 12, while the number of Weierstrass points on X is 8.

To prove Theorem 2, we use theory of Jacobian varieties. By definition, $T := \mathbf{C}^g/\Lambda$ is a complex torus, where Λ is a lattice in $\mathbf{C}^g$. Choose bases $e_1, \dots, e_g$ of $\mathbf{C}^g$ and $\chi_1, \dots, \chi_{2g}$ of the lattice Λ. Write χ_j in terms of the basis $e_1, \dots, e_g$: $\chi_j = \sum_{k=1}^{g} \chi_{kj} e_k$. We denote by $M(n, m; K)$ the set of $n \times m$ matrices with K-coefficients. The matrix $\Pi = (\chi_{kj}) \in M(g, 2g; \mathbf{C})$ is called a period matrix for T. By an *underlying real structure* for T, we mean the real torus $\mathbf{R}^{2g}/\mathbf{Z}^{2g}$ together with a map $\mathbf{R}^{2g}/\mathbf{Z}^{2g} \to T$ induced by a linear map $\mathbf{R}^{2g} \ni x \mapsto \Pi x \in \mathbf{C}^g$, where Π is a period matrix. Note that $\Pi : \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to T$ is homeomorphic. We denote by T_n $(n \in \mathbf{N})$ the group of n-division points of T, that is, the kernel of the homomorphism $n_T : T \to T$ defined by $\mathbf{z} \mapsto n \times \mathbf{z}$ $(\mathbf{z} \in T)$. On the real torus, n-division points are vectors of the form ${}^t(m_1/n, \dots, m_{2g}/n), m_1, \dots, m_{2g} \in \mathbf{Z}$, and we also denote by $(\mathbf{R}^{2g}/\mathbf{Z}^{2g})_n$ the set of n-division points on the underlying real torus.

Let T and T_1 be complex tori of dimension g and γ, respectively. Denote by Π and Π_1 period matrices for T and T_1, respectively. It is known that for any homomorphism (i.e., a holomorphic map compatible with the group action) $f : T \to T_1$, there are $A \in M(\gamma, g; \mathbf{C})$ and $M \in M(2\gamma, 2g; \mathbf{Z})$ such that the following diagram is commutative (cf. [L-B], Chapter 1).

$$\begin{array}{ccccc} \mathbf{R}^{2g} & \xrightarrow{\Pi} & \mathbf{C}^g & \longrightarrow & T \\ \big\downarrow M & & \big\downarrow A & & \big\downarrow f \\ \mathbf{R}^{2\gamma} & \xrightarrow{\Pi_1} & \mathbf{C}^\gamma & \longrightarrow & T_1 \end{array}$$

We will call a map $F : \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to \mathbf{R}^{2\gamma}/\mathbf{Z}^{2\gamma}$ a linear map if F is induced by a linear map $\tilde{F} : \mathbf{R}^{2g} \to \mathbf{R}^{2\gamma}$ with $\tilde{F}|_{\mathbf{Z}^{2g}} \subset \mathbf{Z}^{2\gamma}$. By the diagram above, we see that there exists a linear map $F : \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to \mathbf{R}^{2\gamma}/\mathbf{Z}^{2\gamma}$ which satisfies $\Pi_1 \circ F = f \circ \Pi$.

Let $\{\chi_1, \dots, \chi_{2g}\}$ be a basis for $H_1(X, \mathbf{Z})$ and let $\{\omega_1, \dots, \omega_g\}$ be a basis for the space of holomorphic differentials on X. By definition, the Jacobian variety $J(X) = \mathbf{C}^g/\Lambda$ is the complex torus where Λ is the lattice generated by the period matrix with respect to the bases. Let $p_0 \in X$. Define $\phi_X : X \to J(X)$ by $p \mapsto \{\int_{p_0}^p \omega_j\}_{j=1}^g$. It is known that ϕ_X is an embedding. Let $h : X \to Y$ be a holomorphic map of Riemann surfaces. Then, there exists a homomorphism $f : J(X) \to J(Y)$ which satisfies $f \circ \phi_X = \phi_Y \circ h$ where ϕ_X (resp. ϕ_Y) is the embedding with the base point p_0 (resp. $h(p_0)$).

Lemma 2. *Let* $F_i : \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to \mathbf{R}^{2\gamma}/\mathbf{Z}^{2\gamma}$ *be linear maps* $(i = 1, 2)$. *Then,* $F_1|_{(\mathbf{R}^{2g}/\mathbf{Z}^{2g})_n} = F_2|_{(\mathbf{R}^{2g}/\mathbf{Z}^{2g})_n}$ *if and only if* $F_{1*n} = F_{2*n}$, *where* F_{i*n} *are homomorphisms between homology groups* $H_1(\mathbf{R}^{2g}/\mathbf{Z}^{2g}, \mathbf{Z}_n) \to H_1(\mathbf{R}^{2\gamma}/\mathbf{Z}^{2\gamma}, \mathbf{Z}_n)$ *induced by* F_i $(i = 1, 2)$.

Proof. Suppose $F_{1*n} = F_{2*n}$ holds. Let $\mathbf{x} \in (\mathbf{R}^{2g}/\mathbf{Z}^{2g})_n$. Then, it can be written in the form $\mathbf{x} = {}^t(m_1/n, \ldots, m_{2g}/n), m_1, \ldots, m_{2g} \in \mathbf{Z}$. Let $F_{1*} - F_{2*} : H_1(\mathbf{R}^{2g}/\mathbf{Z}^{2g}, \mathbf{Z}) \to H_1(\mathbf{R}^{2\gamma}/\mathbf{Z}^{2\gamma}, \mathbf{Z})$ denote the difference of two induced homomorphisms between homology groups with $\mathbf{Z}$-coefficients. Then, the assumption $F_{1*n} - F_{2*n} = 0$ means that every coefficient of $F_{1*} - F_{2*}$ is a multiple of n. Recalling that $H_1(\mathbf{R}^{2g}/\mathbf{Z}^{2g}, \mathbf{Z}) \simeq \mathbf{Z}^{2g}$ and that we can identify it with the lattice of the real torus, we see $(\tilde{F}_1 - \tilde{F}_2)(\mathbf{x}) \in \mathbf{Z}^{2\gamma}$, equivalently $F_1 - F_2|_{(\mathbf{R}^{2g}/\mathbf{Z}^{2g})_n} = 0$. Conversely, if $F_1|_{(\mathbf{R}^{2g}/\mathbf{Z}^{2g})_n} = F_2|_{(\mathbf{R}^{2g}/\mathbf{Z}^{2g})_n}$ holds, then denoting by $\mathbf{e}_j$ the vector whose j-th entry is 1 and others are 0, $\tilde{F}_1 - \tilde{F}_2$ maps $\mathbf{e}_j/n$ to a integral vector for every $j = 1, 2, \ldots, 2g$. It implies that every coefficient of $F_{1*} - F_{2*}$ is a multiple of n and we see $F_{1*n} = F_{2*n}$. □

3. Rigidity theorems

Proof of Theorem 1. Let π be a meromorphic function on Y of degree 2. Consider 2 meromorphic functions $\pi \circ h_1$ and $\pi \circ h_2$. Put $F = \pi \circ h_1 - \pi \circ h_2$. Suppose $F \not\equiv 0$. By the Riemann-Hurwitz relation, we see that the degree of $\pi \circ h_1$ and $\pi \circ h_2$ is $\leq 2(g-1)/(\gamma-1)$. Thus, the degree of the polar divisor of F is $\leq 4(g-1)/(\gamma-1)$. On the other hand, the zero divisor of F is $\geq 4(g+1)$ because each of the Weierstrass points is one of zero points of F and the ramification index of it is at least 2 for a Weierstrass point on X maps on a Weierstrass point on Y and the Weierstrass points on Y is the ramification points of π. But $4(g+1) > 4(g-1)/(\gamma-1)$, contradiction. Thus, we must have $F \equiv 0$, equivalently $\pi \circ h_1 = \pi \circ h_2$. Suppose $h_1 \neq h_2$. Then, for a small neighborhood U_p of an arbitrary point $p \in X$, we have $h_1|_{U_p} = \tau \circ h_2|_{U_p}$ and it implies $h_1 = \tau \circ h_2$ by the theorem of identity. □

Proof of Theorem 2. Let $\{\chi_1, \ldots, \chi_{2g}\}$ be a basis for $H_1(X, \mathbf{Z})$ and let $\{\omega_1, \ldots, \omega_g\}$ be a basis for the space of holomorphic differentials on X. Let $J(X) = \mathbf{C}^g/\Lambda$ be the Jacobian variety of X where Λ is the lattice generated by the period matrix with respect to the bases. Let $p_0 \in X$. Define $\phi : X \to J(X)$

by $p \mapsto \{\int_{p_0}^{p} \omega_j\}_{j=1}^{g}$. Then, if p_0 is a Weierstrass point, any Weierstrass point $p \in W$ satisfies $\phi(p) \in J(X)_2$ and $\{\phi(p)\}_{p \in W}$ generate $J(X)_2$ (cf. [F-K] Chapter 7). We denote by $f_i : J(X) \to J(Y)$ the homomorphisms induced by h_i and by F_i the homomorphism of underlying real tori induced by h_i $(i = 1, 2)$. Then each induced homomorphism between homology groups of underlying real tori (which is the same as the homomorphism between the homology groups of Jacobians) is the same as that of Riemann surfaces. Thus, by the assumption and Lemma 2, $F_1|_{(\mathbf{R}^{2g}/\mathbf{Z}^{2g})_2} = F_2|_{(\mathbf{R}^{2g}/\mathbf{Z}^{2g})_2}$. Recalling the diagram in Preliminaries, we see $f_1|_{J(X)_2} = f_2|_{J(X)_2}$. It implies $h_1|_W = h_2|_W$, and by Theorem 1, we get the conclusion. $\square$

Now, we turn to the case where the target is not fixed. Let X be a hyperelliptic Riemann surface of genus g and let Y_1 and Y_2 be Riemann surfaces of the same genus γ. Let $h_i : X \to Y_i$ $(i = 1, 2)$ be nonconstant holomorphic maps. In this situation, the condition $h_1|_W = h_2|_W$ is meaningless. Thus, we consider the following combinatorial condition.

Ordering Weierstrass points p_j^i $(j = 1, 2, \dots 2\gamma + 2)$ on Y_i $(i = 1, 2)$ properly, if

$$W \cap h_1^{-1}(p_j^1) = W \cap h_2^{-1}(p_j^2)$$

holds set-theoretically for every $j = 1, 2, \dots 2\gamma + 2$, then we will say that h_1 and h_2 satisfy *the W-condition*.

PROPOSITION. *Let $h_i : X \to Y_i$ $(i = 1, 2)$ be holomorphic maps, where Y_1 and Y_2 are of the same genus $\gamma > 4$. If h_1 and h_2 satisfy the W-condition, then there exist double-covers $\pi_i : Y_i \to \hat{\mathbf{C}}$ $(i = 1, 2)$ which satisfies $\pi_1 \circ h_1 = \pi_2 \circ h_2$.*

Proof. Without loss of generality, we may assume that the number of elements of sets satisfy $\#\{W \cap h_1^{-1}(p_1^1)\} \geq \#\{W \cap h_1^{-1}(p_2^1)\} \geq \cdots \geq \#\{W \cap h_1^{-1}(p_{2\gamma+2}^1)\}$ and double-covers satisfy $\pi_i(p_1^i) = 1$, $\pi_i(p_2^i) = 2$ and $\pi_i(p_3^i) = 3$ $(i = 1, 2)$. Put $F = \pi_1 \circ h_1 - \pi_2 \circ h_2$. Then, $W \cap h_1^{-1}(p_j^1)$ $(j = 1, 2, 3)$ are zeros of F and so the degree of zero divisor of F is $\geq 2 \times 3 \times (g + 1)/(\gamma + 1)$. On the other hand, if $F \not\equiv 0$, the degree of polar divisor of F is $\leq 2 \times 2 \times (g - 1)/(\gamma - 1)$ where $(g - 1)/(\gamma - 1)$ comes from the maximum degree for h_i $(i = 1, 2)$. Assumption $\gamma > 4$ leads us to the conclusion that $F \equiv 0$. $\square$

If there is a conformal map $S : Y_1 \to Y_2$ with $S \circ h_1 = h_2$, then we will write $h_1 \simeq h_2$. Even if $\pi_1 \circ h_1 = \pi_2 \circ h_2$, it does not always mean $h_1 \simeq h_2$. But for unramified cases, we have

THEOREM 3. *Let X be a hyperelliptic Riemann surface. Let $h_i : X \to Y_i$ $(i = 1, 2)$ be nonconstant holomorphic maps where Y_i $(i = 1, 2)$ are of the same genus $\gamma > 4$. If h_i $(i = 1, 2)$ are unramified and satisfy the W-condition, then $h_1 \simeq h_2$.*

Proof. By Proposition, we have $\pi_1 \circ h_1 = \pi_2 \circ h_2$. All of the branch points on $\hat{\mathbf{C}}$ are image of Weierstrass points on Y_i since h_i are unramified $(i = 1, 2)$. Thus, Y_1 and Y_2 are conformally equivalent since they are expressed by the same algebraic function as a double-cover over $\hat{\mathbf{C}}$. Applying Theorem 1, we see $h_1 \simeq h_2$. □

In this case, we also have a theorem in terms of homology groups. Let G, G_1 and G_2 be groups and let $H_1 : G \to G_1$ and $H_2 : G \to G_2$ be homomorphisms. If there exists a isomorphism $\psi : G_1 \to G_2$ such that $\psi \circ H_1 = H_2$, we will say H_1 and H_2 are isomorphic.

THEOREM 4. *Let X be a hyperelliptic Riemann surface. Let $h_i : X \to Y_i$ $(i = 1, 2)$ be nonconstant holomorphic maps where Y_i $(i = 1, 2)$ are of the same genus $\gamma > 4$. If h_i $(i = 1, 2)$ are unramified and induced homomorphisms $h_{i*2} : H_1(X, \mathbf{Z}_2) \to H_1(Y_i, \mathbf{Z}_2)$ $(i = 1, 2)$ are isomorphic, then $h_1 \simeq h_2$.*

Proof. We use underlying real structures. On underlying real tori, choosing homology bases properly, linear maps $F_i : \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to \mathbf{R}^{2\gamma}/\mathbf{Z}^{2\gamma}$ induced by h_i $(i = 1, 2)$ satisfy $F_{1*2} = F_{2*2}$. By Lemma 2, it is equivalent to $F_1|_{(\mathbf{R}^{2g}/\mathbf{Z}^{2g})_2} = F_2|_{(\mathbf{R}^{2g}/\mathbf{Z}^{2g})_2}$. Recall the diagram in Preliminaries and that any Weierstrass point $p \in W$ satisfies $\phi(p) \in J(X)_2$ if we choose a Weierstrass point as the base point of the embedding. Then we see that h_1 and h_2 satisfy the W-condition. Applying Theorem 3, we get the conclusion. □

When h_i $(i = 1, 2)$ are ramified, Theorem 3, 4 do not hold. We see this by the following example.

Example 2. We picture two copies of the sphere $\hat{\mathbf{C}}$. We label these two copies sheet 1 and sheet 2. Each sheet, we cut along segments $[-1, 1], [2, 3], [-2, -3], [4, 5], [-4, -5], [6, 7], [-6, -7], \ldots, [10, 11], [-10, -11]$. Each cut is considered to have two banks; a $+$bank and a $-$bank. We construct a Riemann surface R by joining every $+$bank on sheet 1 to a $-$bank of the corresponding cut on sheet 2, and then joining the corresponding $+$bank on sheet 2 to the $-$bank of the corresponding cut on sheet 1. Then R is a hyperelliptic Riemann surface of genus 10 and z is a double-cover of $\hat{\mathbf{C}}$. Rotation of angle π of center 0 on $\hat{\mathbf{C}}$, that is, $z \mapsto -z$, can be extended to R and we denote the extended map by T. We put $Y_1 := R/\langle T \rangle$ and $Y_2 := R/\langle \tau \circ T \rangle$, where τ is the involution. Then, Y_1 and Y_2 are both of genus 5. Let $h_i : R \to Y_i$ be the projections for $i = 1, 2$. Then, h_1 and h_2 satisfy the W-condition. Y_1 is the double-cover of $\hat{\mathbf{C}}$ branched over $0, 1, 2, \ldots, 11$. On the other hand, Y_2 is the double-cover of $\hat{\mathbf{C}}$ branched over $1, 2, \ldots, 11, \infty$. We see that Y_1 and Y_2 are not conformally equivalent (e.g., comparing cross ratios of branch points).

With slight and obvious modification, we can construct such examples for higher (and also lower) genera.

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