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# CONFORMAL SLIT MAPPINGS FROM PERIODIC DOMAINS

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*To the memory of Professor Nobuyuki Suita*

## 1. Introduction

K. Amano et al. proposed a numerical conformal mapping from a multiply-connected domain to some typical slit domains by charge simulation method and gave the effective result. Further H. Ogata, D. Okano, K. Amano try a numerical conformal mapping from an infinitely-connected domain called periodic structure domain to a periodic parallel slit domain. The periodic structure domain is a domain in the complex plane $\mathbf{C}$ as follow

$$D = \mathbf{C} - \bigcup_{m \in Z} \{z + ma; z \in D_0\},$$

where $D_0$ is a simply connected closed domain surrounded by a closed Jordan curve, a is a positive constant such that $D_0 \cap \{z + ma; z \in D_0\} = \emptyset$ for every positive integer $m$. The periodic parallel slit domain is a domain in the complex plane $\mathbf{C}$ as follow

$$S = \mathbf{C} - \bigcup_{m \in Z} S_m(\varphi, d, z_0),$$

where $S_m(\varphi, d, z_0) = \{z_0 + ma + tde^{i\varphi}; 0 \leq t \leq 1\}(z_0 \in D_0)$ is a rectilinear slit from $z_0 + ma$ with inclination $\varphi(-\pi/2 < \varphi \leq \pi/2)$ and length $d$. They proved the existence of such a conformal mapping $f$ from a periodic structure domain $D$ to a periodic parallel slit domain $S$ with slits given inclination $\varphi$ such that

$$f(z + a) = f(z) + a(z \in D), \quad f(z) \sim z \pm c \ (\Re z \text{ is fixed}, \ \Im z \Rightarrow \pm\infty),$$

where $c$ is a complex constant. We note the uniqueness of the conformal mapping $f$, because it is important for getting required mapping numerically. Further, we will note that there exists uniquely a normalized conformal mapping from more general periodic domain to a periodic parallel slit domain. We don't know, generally, the value of function theoretic quantities for a given domain. Conformal mappings by numerical method may be able to give them approximately. We wish those data of quantities sublimate to quality and gives theoretical meaning. Conversely it seems that function theoretic quantities play a role of getting good approximation of required conformal mapping.

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## 2. Periodic domains

Let $c$ be a complex number whose real part is positive and consider a parallel displacement $g_c(z) = z + c$. We call a domain $G \subset \mathbf{C}$ periodic domain of period $c$ if $g_c$ gives a conformal mapping from $G$ onto $G$ and there exists a positive constant $M$ such that $G \cap B = B$, where $B = \{z = x + iy; 0 \le x \le \Re\, c, |y| \ge M - |\Im\, c| > 0\}$. Further, when the boundary $\partial G$ of $G$ in the extended complex plane $\hat{\mathbf{C}}$ consists of vertical (horizontal) slit segments and $\{\infty\}$, we call $G$ vertical (horizontal) slit periodic domain.

A periodic domain $G$ of period $c$ is mapped to a domain $G_1 \subset \mathbf{C}$ by $g_1(z) = \exp\left(\frac{2\pi i z}{c}\right)$. Note that $g_1(z+c) = g_1(z)$ and $G_1$ has punctures $\{0\}$ and $\{\infty\}$. By a classical theorem there exists a conformal mapping $g_2$ from $G_1 \cup \{0\} \cup \{\infty\}$ to a radial slit domain whose each boundary component lies on a radial line, where $g_2(0) = 0$, $g_2(\infty) = \infty$. If the number of boundary components of $G_1$ is finite and $g_2(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}$ at a neighborhood of $\infty$, then $g_2(z)$ is unique. Here we get a composite function $F(z) = \log g_2 \circ g_1(z)$ which is a conformal mapping from $G$ to a horizontal slit periodic domain $G_H$ of period $2\pi$. As a boundary behavior it may be $F(z) = \frac{2\pi i}{c} z + O(1)$ at $\infty$. For $c > 0$ $f_v(z) = \frac{c}{2\pi i} \log g_2 \circ g_1(z)$ is a conformal mapping from $G$ to a vertical slit periodic domain $G_V$ of period $2\pi$ and $f_v(z) = z + O(1)$ at $\infty$. Similarly, using a conformal mapping $\tilde{g}_2$, $(\tilde{g}_2(z) = z + \sum_{n=0}^{\infty} b_n z^{-n})$ from $G_1 \cup \{0\} \cup \{\infty\}$ to a circular slit domain, we can get a conformal mapping $f_h = \frac{c}{2\pi i} \log \tilde{g}_2 \circ g_1(z)$ from $G$ to a horizontal slit periodic domain such that $f_h(z) = z + O(1)$ at $\infty$.

## 3. Periodic slit conformal mapping

We would like to show a slight extension of previous assertion.

Proposition. *For a periodic domain $G(\ni 0)$ of period $c$, there exists a vertical slit periodic domain $G_V$ of period $2\pi$ and a conformal mapping $f$ from $G$ to $G_V$ such that*

$$f(0) = 0, \quad f : G \to G_V \ \textit{conformal}, \quad f \circ g_c = g_{2\pi} \circ f.$$

*Further, if $G$ has a countable number of the boundary components, then $G_V$ and $f$ are uniquely determined.*

*Proof.* Let two points $z_1, z_2 \in G$ be equivalent if there exists $m \in \mathbf{Z}$ such that $z_2 = z_1 + mc$ and denote $z_1 \sim z_2$. By this equivalence relation the quotient space $R = G/\sim$ becomes a punctured Riemann surface and $\Pi$ denotes the projection from $G$ to $R$. Let $+\infty$ $(-\infty)$ be the puncture whose neighborhood corresponds to $\{z = x + iy; 0 \le x \le \Re\, c, y \ge M\}$ $(\{z = x + iy; 0 \le x \le \Re\, c, y \le -M\})$.

When we take local variables $\left\{w = \exp\left(\frac{2\pi iz}{c}\right)\right\}$ and $\left\{w = \exp\left(\frac{-2\pi iz}{c}\right)\right\}$ at the puncture $\{+\infty\}$ and $\{-\infty\}$, $\hat{R} = R \cup \{+\infty\} \cup \{-\infty\}$ becomes a Riemann surface.

For the function $f$ in Proposition, $df$ is regarded as a holomorphic differential on $R$. Let $C_+$ be a cycle which is realized as a segment from $iM$ to $iM + c$. Then

$$\int_{C_+} df = 2\pi.$$

The residue of $df$ is $-i$ at $\{+\infty\}$ and $i$ at $\{-\infty\}$.

Let $\Gamma$ be a real Hilbert space which consists of square integrable real differentials on $\hat{R}$ and has the Dirichlet's inner product:

$$(\omega, \sigma) = \int \omega \wedge *\sigma \quad for \ \omega, \sigma \in \Gamma,$$

where $*\sigma$ is a conjugate differential of $\sigma$. We use the following subspaces of $\Gamma$:

$$\Gamma_h = \{\omega \in \Gamma; \omega \text{ is harmonic}\},$$

$$\Gamma_{eo} = \{\omega \in \Gamma; (\omega, \sigma) = 0 \text{ for any } \sigma \in \Gamma_h\}.$$

$$\Gamma_{hse} = \left\{\omega \in \Gamma_h; \int_\gamma \omega = 0 \text{ for dividing regular closed curve } \gamma\right\},$$

$$\Gamma_{hm} = \{\omega \in \Gamma_h; (\omega, *\sigma) = 0 \text{ for any } \sigma \in \Gamma_{hse}\}.$$

Note that the differential in $\Gamma_{hse}$ is exact on a planar domain $\hat{R}$ and $\Gamma_{hse}$ may be denoted by $\Gamma_{he}$ on $\hat{R}$. Set

$$\Lambda_{eo} = \{\omega + i\sigma; \omega, \sigma \in \Gamma_{eo}\},$$

$$\Lambda_{hm} = \{\omega + i\sigma; \omega \in \Gamma_{hm}, \sigma \in \Gamma_{hse}\}, \quad *\Lambda_{hm} = \{*\omega; \omega \in \Lambda_{hm}\},$$

$$\Lambda_{hse} = \{\omega + i\sigma; \omega \in \Gamma_{hse}, \sigma \in \Gamma_{hm}\}, \quad *\Lambda_{hse} = \{*\omega; \omega \in \Lambda_{hse}\}.$$

A meromorphic differential $\tau$ on $\hat{R}$ has $\Lambda_{hm}$-behavior ($\Lambda_{hse}$-behavior) if there exists an $\omega \in \Lambda_{hm} + \Lambda_{eo}$ ($\omega \in \Lambda_{hse} + \Lambda_{eo}$) such that $\tau = \omega$ on a neighborhood of the boundary of $\hat{R}$.

If $f$ and $f_1$ satisfy the conditions stated in Proposition and the boundary components of $G$ is countable, by Proposition 2 of [4], [5] we know there exists $\omega \in \Lambda_{hm} + \Lambda_{eo}$ such that $d(f - f_1) - \omega$ has a compact support. Then $d(f - f_1) - \omega \in \Lambda_{hm} + \Lambda_{eo}$. Above all $d(f - f_1) \in \Lambda_{hm}$. On the other hand, since $d(f - f_1)$ is holomorphic on $\hat{R}$,

$$d(f - f_1) = -i * d(f - f_1) \in \Lambda_{hm} \cap *\Lambda_{hse} = \{0\}.$$

This show the uniqueness of required mapping.

For the sake of showing the existence of required mapping, we note that there exists a third kind of meromorphic differential $\Psi_{+\infty,-\infty}$ with $\Lambda_{hm}$-behavior such that $\Psi_{+\infty,-\infty}$ has the residue $-i$ on $\{+\infty\}$ and $i$ on $\{-\infty\}$ (cf. [3]). Since $\Psi_{+\infty,-\infty}$ is semiexact in a neighborhood of the boundary, the function

$$f(z) = \int_\gamma \Psi_{+\infty,-\infty} \circ \Pi$$

is determined independently of the choice of regular curve $\gamma$ which start from 0 to $z$. The $f$ gives the mapping to a vertical slit domain and satisfies that $f(0) = 0$,

$$\begin{aligned} f(z+c) &= \int_0^{z+c} \Psi_{+\infty,-\infty} \circ \Pi \\ &= \int_0^z \Psi_{+\infty,-\infty} \circ \Pi + \int_z^{iM} \Psi_{+\infty,-\infty} \circ \Pi \\ &\quad + \int_{C_+} \Psi_{+\infty,-\infty} \circ \Pi + \int_{iM+c}^{z+c} \Psi_{+\infty,-\infty} \circ \Pi \\ &= f(z) + 2\pi. \end{aligned}$$

Therefore $f \circ g_c = g_{2\pi} \circ f$ and we have the assertion of Proposition.

Using a third kind of meromorphic differential $\Psi^d_{+\infty,-\infty}$ on $\hat{R}$ with $\Lambda_{hm}$-behavior, which has residue $-\dfrac{d}{2\pi}i$ at $\{+\infty\}$ and $\dfrac{d}{2\pi}i$ at $\{+\infty\}$, we get a function

$$f_d(z) = \int_\gamma \Psi^d_{+\infty,-\infty} \circ \Pi.$$

This $f_d(z)$ is a periodic function which satisfies $f_d(z+c) = f_d(z) + d$ and the image domain is a periodic vertical slit domain with period $d$. By Theorem 4 in [5] each boundary component of $f_d(\hat{R})$ lies on a line segment parallel to the imaginary axis. It satisfies that

$$f_d(iy) = \frac{-id}{2\pi}\int_1^{e^{-2\pi y/c}} \frac{dw}{w} + O(1) = \frac{-id}{2\pi}[\log w]_1^{e^{-2\pi y/c}} + O(1) = \frac{d}{c}iy + O(1)$$

Particularly, $f_c(z)$ is a periodic function satisfying $f_c(z+c) = f_c(z) + c$ and behave as $z + c$ when imaginary part of $z$ is sufficiently large.

Similarly, by using meromorphic differential with $\Lambda_{hse}$-behavior, we can get a conformal mapping to a periodic horizontal slit domain.

## 4. Remarks to the charge simulation method for conformal mapping

Koebe's mapping theorem is as follows. Any multiply connected domain $G$ in the extended complex domain is mapped to a parallel slit domain. Let $z_0 \in G$ and consider the class $F(G, z_0)$ of univalent meromorphic functions on $G$ such that each function $g$ in $F(G, z_0)$ has the following Laurent development at $z_0$:

$$g(z) = \begin{cases} z + \sum_{n=1}^\infty a_n(g) z^{-n} & (z_0 = \infty) \\ \dfrac{1}{z - z_0} + \sum_{n=1}^\infty a_n(g)(z - z_0)^n & (z_0 \in C). \end{cases}$$

Then there exists uniquely $f_h \in F(G, z_0)$ ($f_v \in F(G, z_0)$) such that

$$\sup_{g \in F(G, z_0)} \Re\, a_1(g) = a_1(f_h), \quad \left( \inf_{g \in F(G, z_0)} \Re\, a_1(g) = a_1(f_v) \right),$$

and $f_h$ ($f_v$) gives a conformal mapping from $G$ onto a horizontal (vertical) slit domain and is called an extremal horizontal (vertical) slit mapping. It is difficult to know the extremal values $\Re\, a_1(f_h)$, $\Re\, a_1(f_v)$ for a given domain $G$ generally. We are concerned to know their approximated values by numerical method.

Here we introduce the charge simulation method for conformal mappings by K. Amano and show some examples. Let $M_\ell$ be a Jordan domain considered as a conductor, $M = \bigcup_{\ell=1}^{n} M_\ell$, and $G = \hat{\mathbf{C}} - M$ be an $n$-multiply connected domain. Roughly speaking, charge simulation method is approximation of the real part of conformal mapping from $G$ to a vertical slit domain by a finite sum of logarithmic function (green function) whose poles (charges) are set in the conductor $M$. We take the approximation function $F_h$ ($F_v$) of $f_h$ ($f_v$) as follows: let $\alpha$ denote $h$ or $v$ and

$$F_\alpha(z) = z + \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell} Q_{\ell,i}^{\alpha} \log(z - \zeta_{\ell,i}),$$

where $\zeta_{\ell,i}$ is a charge point in $M_\ell$ and $Q_{\ell,i}^{\alpha}$ is amount of charge at $\zeta_{\ell,i}$. By the condition that $F_\alpha(z)$ must be single value on $G$, it is needed that

$$\int_{\partial M_\ell} dF_\alpha = 0, \quad \sum_{i=1}^{N_\ell} Q_{\ell,i}^{\alpha} = 0.$$

Hence we can write the following form

$$\begin{aligned} F_\alpha(z) - z &= \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell - 1} Q_\ell^i(\alpha) \log \frac{z - \zeta_{\ell,i}}{z - \zeta_{\ell,i+1}} \\ &= \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell - 1} Q_\ell^i(\alpha) \left( \log \left| \frac{z - \zeta_{\ell,i}}{z - \zeta_{\ell,i+1}} \right| + i \arg \frac{z - \zeta_{\ell,i}}{z - \zeta_{\ell,i+1}} \right), \end{aligned}$$

where $Q_\ell^i(\alpha) = \sum_{k=1}^{i} Q_{\ell,k}^{\alpha}$. Since the boundary $\partial M_m$ is mapped on a slit, binding condition is required at points $\{z_{m,j} = x_{m,j} + i y_{m,j}\}_{j=1,2,\ldots N_m}$ on every $\partial M_m$

$$\operatorname{Im} F_h(z_{m,j}) = V_m, \quad \operatorname{Re} F_v(z_{m,j}) = U_m,$$

where $V_m$ ($U_m$) is the imaginary (real) part of the point on horizontal (vertical) slit on which $\partial M_m$ is mapped. We have the following simultaneous equations of dimension $\sum_{\ell=1}^{n} N_\ell$ for unknown number $\{Q_\ell^i(\alpha)\}_{\ell=1,2,\ldots,n,\ i=1,2,\ldots,N_\ell-1}$ and $\{V_m\}_{m=1,2,\ldots n}$ ($\{U_m\}_{m=1,2,\ldots n}$):

$$\sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell - 1} Q_\ell^i(h) \arg \frac{z_{m,j} - \zeta_{\ell,i}}{z_{m,j} - \zeta_{\ell,i+1}} - V_m = -y_{m,j} \quad (m = 1, 2, \ldots, n,\ j = 1, 2, \ldots, N_m),$$

$$\left(\sum_{\ell=1}^{n}\sum_{i=1}^{N_\ell-1} Q_\ell^i(v)\log\left|\frac{z_{m,j}-\zeta_{\ell,i}}{z_{m,j}-\zeta_{\ell,i+1}}\right| - U_m = -x_{m,j}\ (m=1,2,\dots n,\ j=1,2,\dots,N_m)\right).$$

The solution of simultaneous linear equations gives the required mapping.

*Example.*

We give the boundary of material as the following form:

$$\partial M_m = \{x+iy; x = r_m\cos(t)+x_m, y = r_m\sin(t)+y_m\},$$

where $x_m + iy_m$ is a center of $M_m$ and for an angle $t \in [0, 2\pi]$ the radius

$$r_m(t) = a_m + b_m\cos(t) + c_m\cos(2t) + d_m\cos(3t) + b'_m\sin(t) + c'_m\sin(2t) + d'_m\sin(3t).$$

The charge points are located at

$$\{\rho_m r_m\cos(t_j) + x_m + i(\rho_m r_m\sin(t_j) + y_m)\}_{j=0,1\dots,N_m-1},$$

and binding boundary points are located at

$$\{r_m\cos(t_j) + x_m + i(r_m\sin(t_j) + y_m)\}_{j=0,1\dots,N_m-1},$$

where $0 < \rho_m < 1$, $t_j = \dfrac{2\pi j}{N_m}$, $N_m$ is the number of charge points on $M_m$.

*Example* 1.

When $G = \{z; |z| > 2\} \cup \{\infty\}$, we have

$$f_h(z) = z + \frac{4}{z},\quad f_v(z) = z - \frac{4}{z}\quad \text{and}\quad a_1(f_h) = 4,\quad a_1(f_v) = -4.$$

For $\rho_1 = 0.6$, we have

| $N_1$ | $a_1(F_h)$ | $a_1(F_v)$ | $a_1(F_h) - a_1(F_v)$ |
|---|---|---|---|
| 3 | 4.0801702521770426 | −3.8709989168130141 | 7.9511691689900567 |
| 10 | 3.9999866712311313 | −4.0000005587595524 | 7.9999872299906833 |
| 20 | 3.9999999997985189 | −3.9999999999406501 | 7.9999999997391686 |
| 30 | 3.9999999999999977 | −3.9999999999999995 | 7.9999999999999973 |
| 31 | 3.9999999999999995 | −4.0000000000000035 | 8.0000000000000035 |
| 32 | 3.9999999999999995 | −3.9999999999999924 | 7.9999999999999920 |
| 33 | 3.9999999999999982 | −4.0000000000000035 | 8.0000000000000017 |
| 34 | 4.0000000000000017 | −4.0000000000000079 | 8.0000000000000106 |
| 35 | 4.0000000000000017 | −4.0000000000000017 | 8.0000000000000035 |
| 40 | 4.0000000000000115 | −4.0000000000000133 | 8.0000000000000248 |
| 50 | 3.9999999999999928 | −3.9999999999996660 | 7.9999999999996589 |
| 60 | 3.9999999999999809 | −4.0000000000024620 | 8.0000000000024424 |
| 80 | 3.9999999999343924 | −3.9999999998757065 | 7.9999999998100985 |
| 100 | 3.9999999862123849 | −3.9999999836610617 | 7.9999999698734463 |

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When $N_1$ is about 30, we got best approximation. It is judged visually and suggested from the data of $a_1(F_h)$ and $a_1(F_v)$.

*Example* 2.
Let $n = 7$, $N_m = 20$,

| $\backslash m$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $x_m$ | 0 | −7 | −10 | 2 | 10 | −9 | −2 |
| $y_m$ | 0 | −10 | 6 | −10 | −2 | −2 | 9 |
| $a_m$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $b_m$ | 0 | 0 | 0 | 0.1 | −0.1 | −0.1 | 0.1 |
| $c_m$ | 0 | 0.1 | 0 | −0.1 | 0.1 | 0.1 | −0.1 |
| $d_m$ | 0 | 0 | 0.1 | 0.1 | −0.1 | −0.1 | 0.1 |
| $b'_m$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $c'_m$ | 0 | −0.1 | 0 | −0.1 | 0.1 | −0.1 | −0.1 |
| $d'_m$ | 0 | 0 | −0.1 | 0.1 | −0.1 | −0.1 | 0.1 |
| $\rho_m$ | 0.6 | 0.4 | 0.5 | 0.5 | 0.4 | 0.4 | 0.5 |

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From these data we show the figure of $G$, horizontal slit domain $F_h(G)$, vertical slit domain $F_v(G)$ and $(F_h + F_v)(G)$. In figure $G$, white holes are conductors $\{M_m\}$, points in the holes are charge points and each shadowed annular domain is a boundary neighborhood. The slits of $F_h(G)$ are looked like straight but there are non straight slits in $F_v(G)$. It is known that the complement of $(f_h + f_v)(G)$ consists of convex sets. However the white holes in Figure $(F_h + F_v)(G)$ are not always convex. Visual estimation of approximation of mapping by such a theoretical fact is effective.

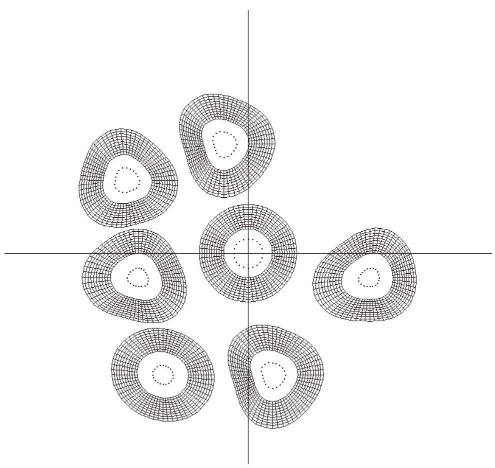

Figure of $G$

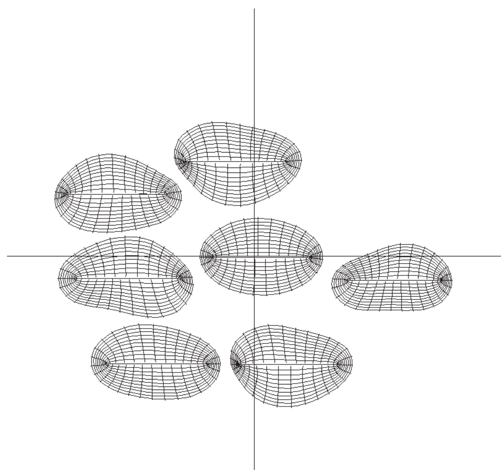

Figure of $F_h(G)$

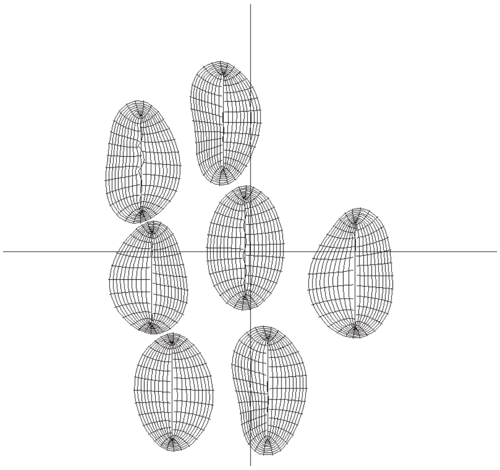

Figure of $F_v(G)$

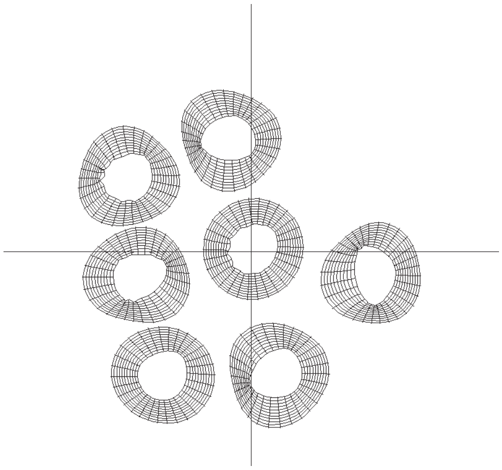

Figure of $(F_h + F_v)(G)$

We have the location of horizontal slits of $F_h(G)$ and the location of vertical slits of $F_v(G)$ as follows:

| | |
|---|---|
| $V_1 = -0.1271785729$ | $U_1 = -0.4395905987$ |
| $V_2 = -8.7250929885$ | $U_2 = -6.3244275738$ |
| $V_3 = 4.9936400236$ | $U_3 = -8.8404996986$ |
| $V_4 = -8.7776892726$ | $U_4 = 1.4094530922$ |
| $V_5 = -1.9696910082$ | $U_5 = 8.4893133217$ |
| $V_6 = -1.8096235983$ | $U_6 = -7.9712762687$ |
| $V_7 = 7.6064915766$ | $U_7 = -2.1892279424$ |

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We get

| $a_1(F_h)$ | $a_1(F_v)$ | $a_1(F_h) - a_1(F_v)$ |
|---|---|---|
| 28.452087696850522746672 | −28.436246840677430469668 | 56.888334537527953216340 |

,

where $a_1(F_h) - a_1(F_v)$ is called span of $G$. By these numerical experiments, it seems that the approximation $F_h$ is better than that of $F_v$ and that the extremal values $a_1(f_h)$ and $a_1(f_v)$ play a role of getting better approximated slit mappings. The location of charge points $\{\zeta_{\ell,i}\}$ and chosen boundary points $\{z_{m,j}\}$ is important factor in this charge simulation method. It should be that $\{z_{m,j}\}$ are located according to the geometrical form of the boundary $\partial G$. As for $\{\zeta_{\ell,i}\}$ we note the following. The value $\operatorname{Re} a_1(f_\alpha)$ giving as an extremal value is a remarkable quantity. The first coefficient $a_1(F_\alpha)$ of Laurent development of $F_\alpha$ is as follows:

$$a_1(F_\alpha) = \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell - 1} Q_\ell^i(\alpha)(\zeta_{\ell,i+1} - \zeta_{\ell,i}).$$

When a charge point $\zeta_{k,s} = \xi_{k,s} + i\eta_{k,s}$ is a little moved, the behavior of value $\mathrm{Re}\, a_1(F_\alpha)$ is given by

$$\frac{\partial}{\partial \xi_{k,s}} \Re\, a_1(F_\alpha) = Q_k^{s-1}(\alpha) - Q_k^s(\alpha) + \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell - 1} \frac{\partial Q_\ell^i(\alpha)}{\partial \xi_{k,s}} (\xi_{\ell,i+1} - \xi_{\ell,i})$$

$$\frac{\partial}{\partial \eta_{k,s}} \Re\, a_1(F_\alpha) = \sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell - 1} \frac{\partial Q_\ell^i(\alpha)}{\partial \eta_{k,s}} (\xi_{\ell,i+1} - \xi_{\ell,i}).$$

On the other hand, the following is satisfied for $m = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, N_m$,

$$\sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell - 1} \frac{\partial Q_\ell^i(h)}{\partial \xi_{k,s}} \arg \frac{z_{m,j} - \zeta_{\ell,i}}{z_{m,j} - \zeta_{\ell,i+1}} - \frac{\partial V_m}{\partial \xi_{k,s}} + \frac{Q_k^s(h)(y_{m,j} - \eta_{k,s}) - Q_k^{s-1}(h)(y_{m,j} - \eta_{k,s})}{(x_{m,j} - \xi_{k,s})^2 + (y_{m,j} - \eta_{k,s})^2} = 0,$$

$$\sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell - 1} \frac{\partial Q_\ell^i(h)}{\partial \eta_{k,s}} \arg \frac{z_{m,j} - \zeta_{\ell,i}}{z_{m,j} - \zeta_{\ell,i+1}} - \frac{\partial V_m}{\partial \eta_{k,s}} - \frac{Q_k^s(h)(x_{m,j} - \xi_{k,s} - Q_k^{s-1}(h)(x_{m,j} - \xi_{k,s})}{(x_{m,j} - \xi_{k,s})^2 + (y_{m,j} - \eta_{k,s})^2} = 0,$$

$$\sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell - 1} \frac{\partial Q_\ell^i(v)}{\partial \xi_{k,s}} \log\left|\frac{z_{m,j} - \zeta_{\ell,i}}{z_{m,j} - \zeta_{\ell,i+1}}\right| - \frac{\partial U_m}{\partial \xi_{k,s}} - \frac{Q_k^s(v)(x_{m,j} - \xi_{k,s}) - Q_k^{s-1}(v)(x_{m,j} - \xi_{k,s})}{(x_{m,j} - \xi_{k,s})^2 + (y_{m,j} - \eta_{k,s})^2} = 0,$$

$$\sum_{\ell=1}^{n} \sum_{i=1}^{N_\ell - 1} \frac{\partial Q_\ell^i(v)}{\partial \eta_{k,s}} \log\left|\frac{z_{m,j} - \zeta_{\ell,i}}{z_{m,j} - \zeta_{\ell,i+1}}\right| - \frac{\partial U_m}{\partial \eta_{k,s}} - \frac{Q_k^s(v)(y_{m,j} - \eta_{k,s}) - Q_k^{s-1}(v)(y_{m,j} - \eta_{k,s})}{(x_{m,j} - \xi_{k,s})^2 + (y_{m,j} - \eta_{k,s})^2} = 0.$$

Unknown numbers $\left\{\frac{\partial Q_\ell^i(h)}{\partial \xi_{k,s}}, \frac{\partial Q_\ell^i(v)}{\partial \xi_{k,s}}, \frac{\partial V_m}{\partial \xi_{k,s}}, \frac{\partial U_m}{\partial \xi_{k,s}}\right\}$ and $\left\{\frac{\partial Q_\ell^i(h)}{\partial \eta_{k,s}}, \frac{\partial Q_\ell^i(v)}{\partial \eta_{k,s}}, \frac{\partial V_m}{\partial \eta_{k,s}}, \frac{\partial U_m}{\partial \eta_{k,s}}\right\}$ are sought as the solutions of these simultaneous equations. These are substituted in the expressions of $\frac{\partial}{\partial \xi_{k,s}} \Re\, a_1(F_\alpha)$ and $\frac{\partial}{\partial \eta_{k,s}} \Re\, a_1(F_\alpha)$ and their sign of $\dfrac{\partial}{\partial \xi_{k,s}} \Re\, a_1(F_\alpha)$ and $\dfrac{\partial}{\partial \eta_{k,s}} \Re\, a_1(F_\alpha)$ are known. Thus we are able to control the value $\Re\, a_1(F_\alpha)$ by the location of charge points. For example, when $\dfrac{\partial}{\partial \xi_{k,s}} \Re\, a_1(F_\alpha) > 0$, $\Re\, a_1(F_\alpha)$ becomes larger as $\zeta_{k,s}$ is moved to real positive direction. Since $f_h$ gives the maximal value $\Re\, a_1(f_h)$ in its extremal problem, if $\zeta_{k,s}$ is moved as $\Re\, a_1(F_h)$ becomes larger, it is expected that the changed $F_h$ will give well approximated mapping of $f_h$. Although the mapping in the class $F(G, z_0)$ must be univalent, $F_h$ is not always univalent. This is a knotty point.

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