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INTEGRAL AND BP COHOMOLOGIES OF EXTRASPECIAL p-GROUPS FOR ODD PRIMES

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Abstract

For each odd prime p, we see $BP^{odd}(Bp_+^{1+4}) = 0$ where p_+^{1+4} is the extraspecial p-group of order p^5 and of exponent p.

1. Introduction

Let G be a compact group and BG its classifying space. All known examples of $BP^*(BG)$ are generated by even dimensional elements. Hence it is conjectured that $BP^{odd}(BG) = 0$. In this paper we give new examples of $BP^{odd}(BG) = 0$.

Throughout this paper, let p be an odd prime number. Let p_+^{1+2n} be the extraspecial p-group of order p^{1+2n} and exponent p. (For $p = 2$, the group 2_+^{1+2n} is the n-th central product of the dihedral group D_8 of order 8.) It is known that the Morava K-theory $K(k)^{odd}(BG) = 0$ for $G = p_+^{1+2}$, D_8 in [T-Y2] and for $G = 2_+^{1+4}$ in [S-Y]. By a theorem in [R-W-Y], we know $BP^{odd}(BG) = 0$ for these cases.

For $m \geq 1$ or $m = \infty$, let us write the central product by

$$G_m^n = \mathbf{Z}/p^m \times_{\mathbf{Z}/p} p_+^{1+2n}, \quad G_\infty^n = S^1 \times_{\mathbf{Z}/p} p_+^{1+2n}$$

so that $G_1^n = p_+^{1+2n}$.

Theorem 1.1. *The homology $H^*(BG_\infty^2; \mathbf{Z})$ has no higher p-torsion, i.e., all elements are just p-torsion or torsion free.*

Theorem 1.2. *For $m \geq 2$ or $m = \infty$, $K(k)^{odd}(BG_m^2) = 0$ for all k, and hence $BP^{odd}(BG_m^2) = 0$. For $m = 1$, we have $BP^{odd}(BG_1^2) = 0$.*

In §2, we recall the Hochschild-Serre spectral sequence converging to $H^*(BG_\infty^n; \mathbf{Z}/p)$, which was studied in [T-Y3]. In §3, we study the similar

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type spectral sequence but converging the integral cohomology $H^*(BG^n_\infty)$. D. Green also studied this spectral sequence [G]. Transferred elements are studied in §4. The exponent of $H^*(BG^n_m)$ is also studied in this section. For $m \geq 2$, $K(k)^{odd}(BG^2_m) = 0$ and $BP^{odd}(BG^2_m) = 0$ are proved in §5 and §6 respectively. Here we show $K(k)^*(BG^2_\infty) \cong K(k)^* \otimes H(H^*(BG^2_\infty; \mathbf{Z}/p); Q_k)$. The fact $BP^{odd}(BG^2_1) = 0$ is showed in §7. Here we use facts that $K(1)^{odd}(BG^2_1) = 0$ and that the Euler number of $K(1)^*(BG^2_1)$ is known, e.g., by Brunetti [B1]. In the last section, we study the relation $BP^*(BG^2_m)$ and the Chow ring $CH^*(BG^2_m)$.

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2. The central product of p^{1+2n}_+ and S^1

Hereafter we assume that p is an odd prime. The extraspecial p-group $G = p^{1+2n}_+$ is the group such that its exponent is p, its center is $C \cong \mathbf{Z}/p$ and there is the extension

(2.1) $$0 \to C \xrightarrow{i} G \xrightarrow{\pi} V \to 0$$

with $V = \bigoplus^{2n} \mathbf{Z}/p$. Throughout this section, we assume $G = p^{1+2n}_+$.

We can take generators $a_1, \dots, a_{2n}$, $c \in G$ such that $\pi(a_1), \dots, \pi(a_{2n})$ (resp. c) make a base of V (resp. C) such that

(2.2) $$[a_{2i-1}, a_{2i}] = c \quad \text{and} \quad [a_{2i-1}, a_j] = 1 \quad \text{if} \quad j \neq 2i.$$

Take the cohomologies

$$H^*(BC; \mathbf{Z}/p) \cong \mathbf{Z}/p[u] \otimes \Lambda(z), \quad \beta z = u,$$

$$H^*(BV; \mathbf{Z}/p) \cong \mathbf{Z}/p[y_1, \dots, y_{2n}] \otimes \Lambda(x_1, \dots, x_{2n}) = S_{2n} \otimes \Lambda_{2n} \quad \beta x_i = y_i,$$

identifying the dual of a_i (resp. c) with x_i (resp. z). Then from (2.2) the central extension (2.1) is expressed by

$$f = \sum_{i=1}^{n} x_{2i-1} x_{2i} \in H^2(BV; \mathbf{Z}/p).$$

Hence $\pi^* f = 0$ in $H^2(BG; \mathbf{Z}/p)$. Consider the spectral sequence

$$E_2^{*,*} = H^*(BV; H^*(BC; \mathbf{Z}/p)) \Rightarrow H^*(G; \mathbf{Z}/p).$$

Then the first nonzero differential is $d_2(z) = f$ since $\pi^*(f) = 0$. The next differential is

$$d_3(u) = \beta f = z(1) \quad \text{with} \quad z(1) = \sum y_{2i-1} x_{2i} - y_{2i} x_{2i-1}.$$

However this spectral sequence is quite difficult to compute and we consider more easy case.

Let $C_m = Z/p^m$ and $C_\infty = S^1$. Let us define the central product $G_m = G \times_C C_m$ so that its center is isomorphic to C_m.

Hereafter we always *assume* $p > n$ and let $\tilde{G} = G_\infty^n$.

We consider the spectral sequence

$$\tilde{E}_2^{*,*} = H^*(BV; H^*(BS^1; \mathbf{Z}/p)) = S_{2n} \otimes \Lambda_{2n} \otimes \mathbf{Z}/p[u] \Rightarrow H^*(B\tilde{G}; \mathbf{Z}/p).$$

Here $H^*(BS^1) \cong \mathbf{Z}[u]$. This spectral sequence $\tilde{E}_r^{*,*}$ is computed in [T-Y3] when $r < 2p(p-1)$ for general n and all r for $n = 2$. We recall some necessary facts and explain briefly how to compute this spectral sequence.

Given a graded $\mathbf{Z}/p$-algebra A and $z \in A^{odd}$, we define the homology $H(A, z)$ with the differential $d(a) = za$ since $z^2 = 0$. The first nonzero differential in $\tilde{E}_r^{*,*}$ is $d_3(u) = \beta f = z(1)$ from the naturality for $G \subset \tilde{G}$. Hence we want to compute $H(S_{2n} \otimes \Lambda_{2n}, z(1))$. For this, we use the following lemma taken from [T-Y3].

Lemma 2.1. *Let $y, z \in A$, and $|z| = odd$, $|y| = even$. Let us consider the $\mathbf{Z}/p$-algebra $A \otimes \Lambda(x)$ for $|x| = |z| - |y|$. Then we have an additive isomorphism*

$$H(A \otimes \Lambda(x), yx + z) \cong (H(A, z)/y)\{x\} \oplus \mathrm{Ker}(y \mid H(A, z))$$

where $\mathrm{Ker}(y \mid H(A, z))$ *is the $\mathbf{Z}/p$-submodule of $H(A, z)$ generated by the elements annihilated by the y-multiplication.*

From this lemma, we have $H(S_{2n} \otimes \Lambda_1, y_2 x_1) \cong S_{2n}/(y_2)\{x_1\}$. By induction we get

$$E_4^{*,2} \cong H(S_{2n} \otimes \Lambda_{2n}, z(1)) \cong Z/p\{x_1 \cdots x_{2n}\} = Z/p\{f^n\} \quad \text{since } n < p.$$

Since $\mathrm{Ker}(z) \cong \mathrm{Im}(z) \oplus H(A, z)$ for $z \in A^{odd}$, it is immediate that

Lemma 2.2. *There is an isomorphism $(A/z)/H(A, z) \cong \mathrm{Im}(z) \subset A$. In particular, if A is w-torsion free for $w \in A^{even}$, then so is $(A/z)/H(A, z)$.*

Apply this lemma with $A = S_{2n} \otimes \Lambda_{2n}$, $z = z(1)$, $w = y_1$. Since y_1 is injective on A, so is on $A/(z + H(A, z))$. Since f^n is y_i-torsion, there is no nonzero differential $d_r : \mathbf{Z}/p\{f^n u^s\} \to A/z$ for $r < 2p - 1$.

Next nonzero differential is the Kudo's transgression

$$d_{2p-1}(z(1) \otimes u^{p-1}) = \beta P^1 \beta f = w(1) \quad \text{with } w(1) = \sum y_{2i-1}^p y_{2i} - y_{2i}^p y_{2i-1}.$$

By the above lemma with $w = w(1)$, we know $\mathrm{Ker}(d_{2p-1} \mid \mathrm{Im}(z(1))) = 0$. Moreover we need

Lemma 2.3. $d_{2p-1}(f^n \otimes u^{p-1}) = nz(2) f^{n-1}$ *where* $z(2) = P^1 z(1) = \sum y_{2i-1}^p x_{2i} - y_{2i}^p x_{2i-1}$.

Proof. Since $\tilde{E}_r^{*,odd} = 0$, the Bockstein operation maps from $\tilde{E}_r^{*,even}$ to $\tilde{E}_r^{*+1,even}$. The element $\beta(f^n u^{p-1}) = n\beta(f) f^{n-1} u^{p-1}$ goes to $nw(1) f^{n-1}$ by

d_{2p-1}. Since $\beta(z(2)) = w(1)$, we know that $d_{2p-1}(f^n u^{p-1}) = nz(2)f^{n-1} + a$ with $a \in \mathrm{Ker}(\beta)$. Since $x_i f^n = 0$ and $x_i z(2) f^{n-1} = 0$ in $S_{2n} \otimes \Lambda_{2n}/(z(1))$, we know also $x_i a = 0$ and hence $\beta(x_i a) = y_i a = 0$ but $\mathrm{Ker}(y_i) = Z/p\{f^n\}$. This means $a = 0$. □

Therefore we have the theorem

THEOREM 2.4 ((3.7) in [T-Y3]). *There is an isomorphism* $u^p : \tilde{E}_{2p}^{*,*} \to \tilde{E}_{2p}^{*,*+2p}$ *and*

$$\tilde{E}_{2p}^{*,2j} \cong \begin{cases} S_{2n} \otimes \Lambda_{2n}/(z(1), w(1), z(2)f^{n-1}) & \text{if } j = 0 \bmod(p) \\ \mathbf{Z}/p\{f^n \otimes u^j\} & 1 \le j < p-1 \\ 0 & j = p-1. \end{cases}$$

By the transgression theorem, the next differential is $d_{2p+1}(u^p) = z(2)$. Let $E = S_{2n} \otimes \Lambda_{2n}/(z(1), w(1))$. We want to know $H(E/(z(2)f^{n-1}), z(2))$. First we note the additive isomorphism

$$H(E/(z(2)f^{n-1}), z(2)) \cong H(E, z(2)) \oplus \mathbf{Z}/p\{f^{n-1}\}.$$

By the similar but after some computations, we get

THEOREM 2.5 (Corollary 5.19 in [T-Y3]). *The term* $\tilde{E}_{2p+2}^{*,2p} \cong H(E/(z(2)f^{n-1}), z(2))$ *is generated by* $f^{n-1} \otimes u^p$ *as an* $S_{2n} \otimes \Lambda_{2n}$*-module and*

$$\beta : H(E, z(2))^{odd} \cong H(E, z(2))^{even}/(\mathbf{Z}/p\{f^n\})$$
$$H(E/(z(2)f^{n-1}), z(2))^{even} \cong S_{2n}/(y_i^p y_j - y_j^p y_i \mid i \ne j)\{f^{n-1}\} \oplus \mathbf{Z}/p\{f^n\}.$$

Let $w(2) = P^p w(1) = \sum y_{2i-1}^{p^2} y_{2i} - y_{2i-1} y_{2i}^{p^2}$. It is known that $(w(1), w(2))$ is a regular sequence in S_{2n} [T-Y1]. By using this fact and Lemma 2.2, we see

THEOREM 2.6. (1) *The multiplying by* $w(2)$ *is injective on*

$$E/(z(2)) + \mathbf{Z}/p\{f^{n-1}\} + H(E, z(2)) \cong E/(z(2) + S_{2n} \otimes \Lambda_{2n}\{f^{n-1}\})$$

(2) *The multiplying by* $w(2)$ *is zero on* $H(E/z(2)f^{n-1}, z(2))$.

By using the facts that $S_{2n}/(w(1))$ is $w(2)$-free but $H(E/(z(2)f^{n-1}), z(2))$ is not $w(2)$-free, we can prove (Section 6 in [T-Y3])

LEMMA 2.7. $\tilde{E}_{2p+2}^{*,*} \cong \tilde{E}_{2p(p-1)+1}^{*,*}$.

By the Kudo's transgression, $d_{2p(p-1)+1}(z(2)u^{p(p-1)}) = w(2)$. However for general n, it is unknown yet $d_{2p(p-1)+1}(f^{n-1}u^{p(p-1)})$.

Let us use the notation such that

$$a \doteq b \quad \text{means} \quad a = \lambda b \quad \text{for } 0 \ne \lambda \in \mathbf{Z}/p.$$

For $n=2$, we know

$$d_{2p(p-1)+1}(fu^{p(p-1)}) \doteq w_{12}(2)'\beta(x_1x_2) + w_{34}(2)'\beta(x_3x_4)$$

where $w_{ij}(2)' = (y_i^{p^2}y_j - y_j^{p^2}y_i)/(y_i^py_j - y_j^py_i)$. Hereafter let us write by $w(2)'$ the element $w_{12}(2)' - w_{34}(2)'$. When $n=2$, there is the another differential

$$d_{2p^3}(f^2u^{p^2-2}) \doteq z(3) = P^pz(2).$$

Thus we can compute $\tilde{E}_\infty^{*,*}$ for $n=2$.

Theorem 2.8 ([T-Y3]). *For the spectral sequence converging to $H^*(G_\infty^2;\mathbf{Z}/p)$, we have the isomorphisms*

$$\tilde{E}_\infty^{*,2pj} \cong \begin{cases} S_4 \otimes \Lambda_4/(z(1),z(2),z(3),w(1),w(2),w(2)'\beta(x_1x_2)), & j=0 \mod(p) \\ H(E/(z(2)f),z(2)) \quad 0<j<p-1 \mod(p) \\ \mathbf{Z}/p\{f^2\} \quad j=p-1 \mod(p), \end{cases}$$

$$\tilde{E}_\infty^{*,2j} \cong \begin{cases} \mathbf{Z}/p\{f^2\} \quad 0<j<p-1 \mod(p) \text{ and } j \neq p^j-2 \\ 0 \quad j=p-1 \mod(p) \text{ or } j=p^2-2. \end{cases}$$

Given $H^*(B\tilde{G};\mathbf{Z}/p)$ (or $H(B\tilde{G})$), to compute $H^*(BG_m^n;\mathbf{Z}/p)$ (or $H^*(BG_m^n)$) we use the following fibration induced from (2.1)

$$S^1 = \tilde{G}/G_m^n \to BG_m^n \to B\tilde{G}.$$

The induced spectral sequence is

$$E_2^{*,*} = H^*(B\tilde{G};H^*(S^1;\mathbf{Z}/p)) \cong H^*(B\tilde{G};\mathbf{Z}/p)\otimes\Lambda(z) \Rightarrow H^*(BG_m^n;\mathbf{Z}/p).$$

Let us write $d_2z=f'$. When $m=1$ this $f'=f$ but when $n>1$, $f'=0$ (see Proposition 3.17 in [Y2]).

Lemma 2.9. *As S_{2n}-modules, $H^*(BG_m^n;\mathbf{Z}/p)$ is isomorphic to*

$$\begin{cases} (\mathrm{Ker}(f)\,|\,H^*(B\tilde{G};\mathbf{Z}/p)\{z\} \oplus H^*(B\tilde{G};\mathbf{Z}/p)/(f) & \text{if } m=1 \\ H^*(B\tilde{G};\mathbf{Z}/p)\otimes\Lambda(z) & \text{if } m\geq 2. \end{cases}$$

3. Integral cohomology

We consider the integral coefficient spectral sequence

$$IE_2^{*,*} = H^*(BV;H^*(BS^1)) \Rightarrow H^*(B\tilde{G}).$$

This spectral sequence is also studied in [G] by Green. First we note that $H^*(BV) \cong \mathrm{Im}(\beta) \subset H^*(BV;\mathbf{Z}/p)$ since the cohomology $H(H^*(BV;\mathbf{Z}/p),\beta) \cong \mathbf{Z}/p\{1\}$. The cohomology $H(H^*(BV);z(1))$ is given by D. Green.

Lemma 3.1 ([G]). $H(H^*(BV),z(1)) \cong \mathbf{Z}\{p\} \oplus \mathbf{Z}/p\{z(1)f,\ldots,z(1)f^{n-1}\}$.

Proof. Let $V' \oplus (\mathbf{Z}/p)^2 \cong V$. By induction we assume that

$$H^+(H^*(BV'), z(1)) \cong \mathbf{Z}/p\{z(1), z(1)f, \dots, z(1)f^{n-2}\}.$$

Considering the spectral sequence

$$E_2^{*,*} = H(H^*(B(\mathbf{Z}/p)^2); H^*(BV')) \Rightarrow H^*(BV)$$

we can write $\mathrm{gr}\, H^*(BV) \cong A \oplus B \oplus \mathbf{Z}\{1\}$ where $A = E_2^{*,+} \cong H^*(BV')^+ \otimes \mathbf{Z}/p[y_1, y_2] \otimes \Lambda(x_1, x_2)$ and $B = E_2^{+,0} \cong (\mathbf{Z}/p[y_1, y_2] \otimes \Lambda(\beta))^+$ with $\beta = \beta(x_1x_2)$.

From Lemma 2.1, we have

$$H(A, z(1)) \cong H(H^*(BV')^+, z(1))\{x_1x_2\}$$

and $H(B, z(1)) \cong \mathbf{Z}/p\{\beta\}$ since $1 \notin B$ and $z(1) \mid B = \beta$. Thus we get

$$H(\mathrm{gr}\, H^*(BV)^+, z(1)) \cong \mathbf{Z}/p\{\beta, z(1)x_1x_2, \dots, z(1)f^{n-2}x_1x_2\}.$$

Since $z(1)f^i$ is really cycle for the differential $z(1)$, and we have the lemma from

$$H(H^*(BV)^+, z(1)) \cong H(H^*(BV), z(1)) \oplus \mathbf{Z}/p\{z(1)\}. \qquad \square$$

Corollary 3.2. *The term $IE_4^{*,2i}$ is isomorphic to*

$$\begin{cases} \mathbf{Z}\{1\} \oplus \beta H^*(BV; \mathbf{Z}/p)/(z(1)\beta H^*(BV; \mathbf{Z}/p)) & 2i = 0 \\ \mathbf{Z}\{p\} \oplus \mathbf{Z}/p\{z(1)f, \dots, z(1)f^{n-1}\} & 0 < 2i < 2(p-1) \\ \mathbf{Z}\{p\} \oplus z(1)\beta H^*(BV; \mathbf{Z}/p) \oplus \mathbf{Z}/p\{z(1)f, \dots, z(1)f^{n-1}\} & 2i = 2p-2. \end{cases}$$

We use the following notations. For an element $a \in E_\infty^{*,*}$ converging to $H^*(X)$ (or $H^*(X; Z/p)$), let us write by $\{a\}$ one of the correspondences elements in $H^*(X)$ (or $H^*(X; \mathbf{Z}/p)$). For an element $x \in H^*(X)$, let $[x] \in E_\infty^{*,*}$ be the corresponding nonzero element in the spectral sequence. Therefore $[\{a\}] = a$ for $a \neq 0$ but $x \equiv \{[x]\}$ modulo $\{E^{*+1,*}\}$.

Let $r : H^*(X) \to H^*(X; \mathbf{Z}/p)$ be the reduction map.

Lemma 3.3. *Let $1 \le s \le n$. Then $d_{2i+1}(p^{i-1}u^s) \doteq z(1)f^{i-1}u^{s-i}$ for all $i \le s$, and p^su^s generates $IE_{2s+2}^{0,2s} \cong IE_\infty^{0,2s}$. Moreover $r(\{p^su^s\}) = f^s$.*

Proof. By the naturality for the reduction map r, $d_3(u) = z(1)$ also in $IE_3^{*,*}$. Hence $pu \in E_4^{0,2}$ generates $E_\infty^{0,2}$ and $r(\{pu\}) \neq 0$. But it is easily seen that $\mathrm{Ker}(\beta)/\mathrm{Im}(\beta) \cap E_\infty^{2,0} \cong \mathbf{Z}/p\{f\}$. Thus we can take $r(\{pu\}) \doteq f$. For $s \le n$, we have

$$r(\{p^su^s\}) = r(\{pu\})^s \doteq \{f\}^s = f^s.$$

This means p^su^s generates $E_\infty^{0,2s}$, and by dimensional reason, we have $d_{2i+1}(p^{i-1}u^s) \doteq z(1)f^{i-1}u^{s-i}$ for all $i < s$. $\square$

Similarly, we have

Lemma 3.4. *Let* $1 \le i \le n$ *and* $n \le s \le p-1$. *Then* $d_{2i+1}(p^{i-1}u^s) \doteq z(1)f^{i-1}u^{s-i}$.

For the proof of this lemma, we prepare the following lemma.

Lemma 3.5. *Let* A *be a graded algebra acting the Bockstein* β *with* $H(A,\beta) = 0$. *Let* $z \in A^{odd}$ *with* $\beta z = 0$ *and write* $H(A,z) = H$ *and* $H(\beta A, z) = IH$. *Then*

$$H(A/(z+H),\beta) \subset z^{-1}IH, \quad \mathrm{Im}(\beta)(A/(z+H)) \cong \beta A/(z\beta A + IH)$$

identifying $z^{-1}IH$ *as the submodule of* $A/(z+H) \cong A/\mathrm{Ker}(z)$.

Proof. We note that

$$\mathrm{Ker}(\beta)\,|\,(A/(z+H)) \cong \mathrm{Ker}(\beta)\,|\,(A/\mathrm{Ker}(z)) \overset{\times z}{\cong} \mathrm{Im}(z) \cap \mathrm{Ker}(\beta) \subset A.$$

On the other hand,

$$\mathrm{Im}(\beta)(A/(z+H)) \overset{\times z}{\cong} \mathrm{Im}(\beta)(\mathrm{Im}(z)) \cong \mathrm{Im}(z)(\mathrm{Im}(\beta)) \cong \mathrm{Im}(z)(\mathrm{Ker}(\beta))$$

since $\beta(za) = z\beta(a)$ and $H(A,\beta) = 0$. Thus we get

$$\begin{aligned} H(A/(z+H),\beta) &\overset{\times z}{\cong} (\mathrm{Im}\, z \cap \mathrm{Ker}(\beta))/\mathrm{Im}(z)(\mathrm{Ker}(\beta)) \\ &\subset (\mathrm{Ker}(z) \cap \mathrm{Ker}(\beta))/\mathrm{Im}(z)(\mathrm{Ker}(\beta)) = H(\mathrm{Ker}(\beta), z). \end{aligned}$$

Moreover we have

$$\beta A/(z\beta A + IH) \cong \beta A/\mathrm{ker}(z) \overset{\times z}{\cong} \mathrm{Im}(z)(\mathrm{Im}\,\beta). \qquad \square$$

Let us write $A = E_2^{*,0} \cong S_{2n} \otimes \Lambda_{2n}$, $B = E_4^{*,0} \cong A/z(1)$, and $IA = IE_2^{+,0} \cong \beta A$, $IB = IE_4^{+,0} \cong IA/(z(1)IA)$. From the above lemma. We have

Corollary 3.6. $H(B^+,\beta) \cong Z/p\{f,\dots,f^n\}$ *and* $IB/IH \cong \beta B$ *where* $IH \cong \mathbf{Z}/p\{z(1)f,\dots,z(1)f^{n-1}\}$.

Proof. Here $H(A^+,\beta) = H = 0$ and hence

$$H(B^+,\beta) = H(A^+/z(1),\beta) = H(A^+/(z(1)+H),\beta) \subset z(1)^{-1}IH,$$

where $IH \cong \mathbf{Z}/p\{z(1)f,\dots,z(1)f^{n-1}\}$ is still given in Lemma 3.1. Since $\beta f = z(1) = 0$ in B, f^i are in $\mathrm{Ker}(\beta)$. $\square$

Let us write $\Delta = H(B^+,\beta) \cong Z/p\{f,\dots,f^n\}$.

Proof of Lemma 3.4. From Theorem 2.4, we know for $* < 2(p-1)$,

$$\mathrm{Ker}\,\beta(H^*(B\tilde{G};\mathbf{Z}/p)^+) \cong \beta B \oplus \Delta \oplus \mathbf{Z}/p\{f^n u,\dots,f^n u^{p-2}\}.$$

This module is also isomorphic to $H^*(BG)/p$. For each $i \le p-1$, $p^s u^i$ are in

$\tilde{E}_{\infty}^{0,*}$ for sufficient large s. Hence there is s' such that $r\{p^{s'}u^i\} \doteq \{f^n u^{i-n}\}$, when $n \le i \le p-1$. Moreover each element of form $z(1)f^k u^i$ must be killed in the spectral sequence $IE_{\infty}^{*,*}$. By dimensional reason, we have the lemma. q.e.d.

Next consider differentials for elements in $IE_4^{*,2(p-1)}$. The fact that $IE_4^{*,2(p-1)} \cong \mathrm{Ker}(z(1)) \cap IA$ is y_i-torsion free implies that there does not exist differential such that $d_r(x) \ne 0 \in E_r^{*,2(p-1)}$ for $4 \le r \le 2p-1$ since $z(1)f^i u^s$ is y_j-torsion. Similarly since $A/(z+H) \cong B/H$ is y_j-torsion free, and so is $IB/IH \cong \beta B/H$. Hence each element $z(1)f^i u^s$ does not go by differential into a nonzero element in IB/IH.

For the element $w(1) \in IE_{\infty}^{*,0}$, since $r(w(1)) = 0 \in \tilde{E}_{\infty}^{*,0}$, we have $w(1) = \lambda p\{p^{s'}u^{p+1}\}$ in $H^*(B\tilde{G})$ where note $|z(1)f^i u^s| = odd$. But $w(1)$ is p-torsion also in $H^*(B\tilde{G})$ and $\{p^{s'}u^{p+1}\}$ is torsion free and $\lambda = 0$. Therefore there is an element z with $d_r(z) = w(1)$ in $IE_r^{*,*}$. By dimensional reason or by naturality, we have

$$d_{2p-1}(z(1)u^{p-1}) = w(1).$$

Similarly we get

$$d_{2p-1}(z(1)f^{i-1}u^{p-1}) = \beta(z(2)f^{i-1}) = w(1)f^{i-1} - (i-1)z(2)z(1)f^{i-2}.$$

Recall that $E = S_{2n} \otimes \Lambda_{2n}/(z(1), w(1))$. Let us write $IE = IE_{2p+1}^{+,0} = IB/(w(1)IB, \Gamma)$ where

$$\Gamma = \mathbf{Z}/p\{\beta(z(2)f), \dots, \beta(z(2)f^{n-1})\}.$$

Lemma 3.7. $IE_{2p+1}^{0,+} \cong IE \subset E/(z(2)f^{n-1}) \cong \tilde{E}_{2p+1}^{0,*}$.

Proof. Let $x \in IB$ and $x = 0 \in E$. Then $x = \beta(x') = w(1)a$ in B. Hence $w(1)\beta a = 0$. Here $\mathrm{Ker}(w(1)) \cong \mathrm{Im}\, z(1) \oplus \mathbf{Z}/p\{f^n\}$. So $\beta a = \lambda f^n$ but it does not hold $\lambda \ne 0$ in B, indeed, $f^n \notin \beta B$. Thus $\beta a = 0$. This means $a = \beta a' + \sum \lambda_i f^i$. Therefore

$$\begin{aligned} w(1)a &= w(1)\beta a' + \sum w(1)\lambda_i f^i \\ &= w(1)\beta a' + \sum \lambda_i (w(1)f^i - iz(2)z(1)f^{i-1}) \quad \text{in } B \end{aligned}$$

since $z(1) = 0 \in B$. Thus we see that $x \in (w(1)IB, \Gamma)$ and $x = 0$ in IE. □

Lemma 3.8. $H(E^+/z(2), \beta) \cong \Delta$.

Proof. Let $x \in \mathrm{Ker}(\beta \,|\, E/z(2))$. Since $E/(z(2)) = B/(w(1), z(2))$, this means

$$\beta x = z(2)a + w(1)b \quad \text{in } B.$$

Take more β, and we get

$$0 = \beta^2 x = w(1)a - z(2)\beta a + w(1)\beta b.$$

Multiply by $z(2)$, we have $z(2)w(1)(a + \beta b) = 0$. Here we note that $\mathrm{Ker}(w(1))$ in B is isomorphic to $\mathrm{Ker}(z(1)) \cong \mathrm{Im}\, z(1) + Z/p\{f^n\}$ in A. This fact is shown from that the Kudo's transgression $d_{2p-1} : \mathrm{Im}\, z(1) \to B$ via. $z(1) \mapsto w(1)$ is injective. Hence $w(1)x = 0$ in B means that $z(1)x = 0$ in A. By dimensional reason $|z(2)| > |f^n|$, we have $z(2)(a + \beta b) = 0$. Thus

$$\beta x = z(2)(-\beta b) + w(1)b = \beta(z(2)b).$$

Hence $\beta(x - z(2)b) = 0$ in B. Since $H(B, \beta) = \Delta$, we have

$$x - z(2)b \in \mathrm{Im}\, \beta + \Delta \quad \text{in } B.$$

Thus $x \in \mathrm{Im}\, \beta + \Delta$ in $E/z(2)$. □

From Theorem 2.5, Lemma 2.7 and the above lemma, we see;

Corollary 3.9. *When $* < 2p^2 - 2p$, each element of $H^*(B\tilde{G})$ is torsion free or just p-torsion.*

From the above corollary, the map $r : IE_\infty^{m,0} \to \tilde{E}_\infty^{m,0}$ is injective for $0 < m < 2p^2 - 2p$. In particular elements $\beta(z(2)x) \in \tilde{E}_{2p-1}^{*,0}$, $* \leq 2(p+n) < 4p - 1$ must be target $d_r(z)$ for some $z \in \tilde{E}_r^{*,*}$ by arguments before Lemma 3.7. By the naturality, we see

$$d_{2p+1}(\beta(x)u^p) = \beta(xz(2)) \quad \text{in } IE_{2p+1}^{*,*}$$

from the fact $d_{2p+1}(xu^p) = xz(2)$ in $\tilde{E}_r^{*,*}$. For the cases $|\beta(x)| \geq 4p - 1$, we can write $x = \sum x'\beta(x'')$ with $|x'| < 2p$ and we also have $d_{2p+1}(u^p\beta(x)) = z(2)\beta(x)$.

Let $F = \tilde{E}_{2p+2}^{*,0} \cong E/(z(2))$ and $IF = IE/(\beta(z(2)E))$.

Lemma 3.10. *$IF \subset F$ and $\beta H(E, z(2)) \cong IH(IE, z(2))$ where $IH(IE, z(2)) = \{\beta(x) \in IE \mid \beta(z(2)x) = 0\}/(\beta(z(2)E))$.*

Proof. Let $\beta(x) \in IE$ and $\beta(x) = 0 \in F$. From the proof of Lemma 3.7, we can see that $\beta(x) = z(2)\beta b - w(1)b = \beta(z(2)b) \in E$. So $x = 0 \in IF$. □

From the above corollary and lemma, we show that all nonzero elements in $IE_r^{*,+}$ $* \neq 0 \bmod(p)$ must be killed.

Lemma 3.11. *When $s \leq n$, we get*

$$\begin{cases} d_{2i+1}(p^{i-1}u^{s+p}) \doteq z(1)f^{i-1}u^{s+p-i} & \text{if } s \geq i \\ d_{2s+1}(p^s u^{s+p}) = 0 \\ d_{2i+3}(p^{i-1}u^{s+p}) \doteq z(1)f^i u^{s+p-i-1} & n \geq i > s+1. \end{cases}$$

Corollary 3.12.

$$IE^{*,2j}_{2p(p-1)+1} \cong \begin{cases} IF \cong \mathbf{Z}\{1\} \oplus \beta E/(\beta(z(2)E)) & j=0 \\ \mathbf{Z}\{p^j u^j\} & 0<2j<2n \\ \mathbf{Z}\{p^n u^j\} & 2n \le 2j \le 2p-2 \end{cases}$$

$$IE^{*,2p+2j}_{2p(p-1)+1} \cong \begin{cases} \beta H(E,z(2)) \oplus \mathbf{Z}\{p^{n-1}u^p\} & j=0 \\ \mathbf{Z}\{p^{n-1}u^{p+j}\} & 0<2j<2n \\ \mathbf{Z}\{p^n u^{p+j}\} & 2n \le 2j \le 2p-2. \end{cases}$$

Now we consider the case $n=2$. Recall $w(2)/(y_1^p y_2 - y_1 y_2^p) = w_{12}(2)' - w_{34}(2)'$ and write it by $w(2)'$ so that $d_{2p(p-1)}(fu^{p(p-1)}) \doteq w(2)'\beta(x_1x_2) \in \tilde{E}_r^{*,0}$. Hence $w(2)'\beta(x_1x_2) = pa$ in $H^*(\tilde{G};\mathbf{Z})$. But nonzero elements in $IE^{*,s}_{2p(p-1)+1}$ for $0<s<2p^2$ are even dimensional from Cor. 3.12 and Theorem 2.5. Hence $w(2)'\beta(x_1x_2)=0$ also in $H^*(\tilde{G};\mathbf{Z})$. By dimensional reason we have

$$d_{2(p-1)p+3}(pu^{p(p-1)+1}) \doteq w(2)'\beta(x_1x_2) \quad \text{in } IE_r^{*,0}.$$

Define

$$G = E_\infty^{*,0} \cong F/(w(2), w(2)'\beta(x_1x_2), z(3))$$
$$\text{and} \quad IG = IF/(w(2)\{1, IF\}, w(2)'\beta(x_1x_2)IF).$$

Lemma 3.13. *When $n=2$, $IG=\beta G$ and $IG \cong IE_\infty^{+,0}$. Moreover $H(G,\beta) \cong \Delta \oplus \mathbf{Z}/p\{w(2)'x_1x_2\}$.*

Proof. Let $x=0 \in G$ and $x=\beta x' \in F$. Then in F,

$$\beta(x') = w(2)a + w(2)'\beta(x_1x_2)c + z(3)d \quad \text{for } c \in H(E;Z(2)),\ d \in \mathbf{Z}/p.$$

By dimensional reason, we see $d=0$. First consider the case $|x| = even$. Applying β, we see

$$w(2)\beta a + w(2)'\beta(x_1x_2)\beta(c) = 0.$$

Here $|c| = odd$ and $c=0$ otherwise $w(2)'\beta(x_1x_2)\beta(c) \neq 0 \bmod (w(2))$ from Theorem 2.5 and Theorem 2.6 (2). Thus $w(2)\beta(a)=0$. Hence for this case, we can prove the lemma by the arguments similar to those of the proof of Lemma 3.9.

Let $|x| = odd$. Then $|c| = even$ and also from Theorem 2.5, $\beta c = 0$ and $\beta(x_1x_2)c = \beta(x_1x_2c)$. Therefore we can prove the lemma similarly to the case $|x| = even$. □

Remark. The fact $H(G,\beta) \cong \Delta \oplus \mathbf{Z}/p\{w(2)'x_1x_2\}$ is also proved in Section 4 below.

Thus we get the results for the case $n=2$.

Theorem 3.14. *When $n=2$*

$$IE_\infty^{*,2pj} \cong \begin{cases} \mathbf{Z}\{1\} \oplus IG & \text{if } j = 0 \mod(p) \\ \mathbf{Z}\{p\} \oplus IH(E, z(2)) & 0 < j < p-1 \mod(p) \\ \mathbf{Z}\{p\} & j = p-1 \mod(p), \end{cases}$$

For $j \neq 0 \mod(p)$,

$$IE_\infty^{*,2j} \cong \begin{cases} \mathbf{Z}\{p\} & j = 1 \mod(p),\ j \neq p(p-1)+1 \mod(p^2) \\ \mathbf{Z}\{p^2\} & 2 \le j \le p-1 \mod(p) \text{ or } j = p(p-1)+1 \mod(p^2). \end{cases}$$

Corollary 3.15. *All elements in $H^*(BG_\infty^2)$ are just p-torsion or torsion free.*

Corollary 3.16. *The reduced map $r : H^*(BG_\infty^2) \to H^*(BG_\infty^2; \mathbf{Z}/p)$ is given by*

$$\begin{cases} r\{pu^{sp}\} \doteq \{f^2 u^{sp-2}\} & 1 \le s \le p-1 \\ r\{pu^{sp+1}\} \doteq \{fu^{sp}\} & 0 \le s \le p-2 \\ r\{p^2 u^{p(p-1)+1}\} \doteq \{w(2)' x_1 x_2\} \\ r\{p^2 u^{sp+j}\} \doteq \{f^2 u^{sp+j-2}\} & 2 \le j \le p-1. \end{cases}$$

Now we study the integral cohomology of the finite groups G_m^n. The integral version of the spectral sequence is

$$IE_2^{*,*} = H^*(B\tilde{G}) \otimes \Lambda(z) \Rightarrow H^*(BG_m^n). \tag{3.2}$$

Here the differential is $d_2(z) = f' \doteq \{p^m u\} \in H^*(B\tilde{G})$. This fact is proved by the naturality to the restriction maps

$$S^1 \to B\mathbf{Z}/p^m \to BS^1$$

and by the isomorphism $H^*(B\mathbf{Z}/p^m) \cong \mathbf{Z}[u]/(p^m u)$. Similarly to the mod p case, we have the isomorphism

$$H^*(BG_m^n) \cong (\mathrm{Ker}\, f' \,|\, H^*(B\tilde{G})) \oplus H^*(B\tilde{G})/(f').$$

For the integral case, $d_2(z) \neq 0$ even if $m \ge 2$. Let $p^{m(i)} u^i$ generate $IE_\infty^{0,*}$. Since

$$d_2\{p^{m(i-1)} u^{i-1} z\} \doteq \{p^{m(i-1)} u^{i-1}\}\{p^m u\} = \{p^{m(i-1)+m} u^i\},$$

we have

$$p^{m(i-1)+m-m(i)} \,|\, \exp(H^*(BG_m^n))$$

where $\exp(H^*(BG_m^n))$ is the exponent of $H^*(BG_m^n)$.

Since each element of $H^*(BG_\infty^2)$ is just p-torsion or torsion free, and $m(p^2) = 0$ and $m(p^2-1) = 2$, we easily see that

Corollary 3.17. $\exp(H^*(BG_m^2)) = p^{m+2}$.

This fact is extended for all $n < p$ in Corollary 4.7 bellow.

4. Transfers

In this section, we study about generators $p^{m(i)}u^i \in IE_\infty^{0,2i}$. We can take $\{p^s u^s\}$ as a Chern class $c_s(\xi)$ where ξ is a one dimensional representation with $\xi(x) = e^{2\pi xi}$ for $x \in \mathbf{R}/\mathbf{Z} \cong S^1$ and $\xi(a_j) = 1$. Moreover $\{p^n u^n\}$ is represented by transfer.

Let A^{odd} be the maximal abelian subgroup of $\tilde{G}$ generated by

$$A^{odd} = \langle a_1, a_3, \dots, a_{2n-1} \rangle \times S^1$$

so that

$$H^*(BA^{odd}; \mathbf{Z}/p) \cong \mathbf{Z}/p[y_1, y_3, \dots, y_{2n-1}] \otimes \Lambda(x_1, x_3, \dots, x_{2n-1}) \otimes \mathbf{Z}/p[u].$$

Consider the transfer

$$\mathrm{tr}(i) = \mathrm{Cor}_{A^{odd}}^{\tilde{G}}(u^i) \in H^*(B\tilde{G}) = H^*(BG_\infty^n).$$

Since $[\tilde{G}; A^{odd}] = p^n$, we have $\mathrm{tr}(i) \,|\, S^1 = p^n u^i$. Moreover $r(\mathrm{tr}(i))$ is x_{even}-torsion and y_{even}-torsion because by the Frobenius formula

$$y_{even}\,\mathrm{tr}(i) = y_{even}\,\mathrm{Cor}(u^i) = \mathrm{Cor}(i_{odd}^*(y_{even})u^i) = 0$$

where $i_{odd} : A^{odd} \to \tilde{G}$ is the inclusion and $i_{odd}^*(y_{even}) = 0$.

Lemma 4.1. *If $i \le n(p-1)$, then $r(\mathrm{tr}(i)) \doteq \{f^n u^{i-n}\}$ or 0.*

Proof. The transfer $r(\mathrm{tr}(i))$ is y_{even}-torsion, and x_{even}-torsion in $H^*(B\tilde{G}; \mathbf{Z}/p)$, and also in $E_{2p^2+1}^{*,*}$ for $i < 2p^2$. Hence $r(\mathrm{tr}(i))$ is $w(2) = \sum y_{2i-1}^p y_{2i} - y_{2i}^p y_{2i-1}$-torsion in $E_{2p^2+1}^{*,*}$. From Theorem 2.6, there is no nonzero such torsion element in

$$E_{2p^2+1}^{*',*}/(H(E, z(2)) + Z/p\{f^{n-1}\}) \quad \text{for } *' < 2p(p-1)+1.$$

Also from Theorem 2.5 and Lemma 4.3 (2) below, the nonzero y_{even}-torsion elements of degree less than $2n(p-1)+1$ is only the f^n in $H(E, z(2)) + \mathbf{Z}/p\{f^{n-1}\}$. □

From Corollary 3.12, we have (see also Lemma 4.6 bellow)

Corollary 4.2. *If $n \le i \le (p-1)$ $\mathrm{mod}(p)$, then $r(\mathrm{tr}(i)) \doteq \{f^n u^{i-n}\}$.*

Proof. Recall $\mathrm{tr}(i) \,|\, S^1 = p^n u^i$. From Corollary 3.12, we know $\mathrm{tr}(i) \ne 0$ $\mathrm{mod}(p)$ in $H^*(B\tilde{G})$. Hence $r(\mathrm{tr}(i)) \ne 0$. Thus we have the corollary from the above lemma. □

Lemma 4.3. *Given $k \ge 1$, let $a \in S = S_l/(y_i^{p^k} y_j - y_i y_j^{p^k} \,|\, 1 \le i < j \le l)$. Then we have*

(1) *if $y_i a = 0$ for all i, then $a = 0$.*

(2) *if* $a \in \mathrm{Ideal}(y_1 \cdots y_s)$ *and* $y_i a = 0$ *for all* $1 \le i \le s$, *then* $|a| \ge s(p^k - 1) + 1$.

Proof. Replacing $y_i y_j^{p^k}$ by $y_i^{p^k} y_j$ for $i < j$, we can uniquely write an element $a \in S$ as

$$(a) \qquad a = \sum \lambda_I y_I = \sum \lambda y_1^{i_1} \cdots y_l^{i_l}$$

where $I = (i_1, \dots, i_l) = (0, \dots, 0, i_{m(I)}, i_{m(I)+1}, \dots, i_l)$ with $i_{m(I)} \ne 0$ and $0 \le i_s \le p^k - 1$ for all $m(I) + 1 \le s \le l$.

For the proof of (1), let $\tilde{I}$ be the smallest I for $\lambda_I \ne 0$ by the lexicographic order (i.e., $I > J$ if there is s such that $i_k = j_k$ for all $k < s$ and $i_s > j_s$). Then $y_{m(\tilde{I})} a \ne 0$ because $y_{m(\tilde{I})} y_I > y_{m(\tilde{I})} y_{\tilde{I}}$ for $I > \tilde{I}$. This shows (1).

Suppose that $y_i a = 0$ for $l - s \le i \le l$. Then $\tilde{i}_l \ge p^k - 1$, otherwise $y_l y_{\tilde{I}}$ becomes the smallest in $y_l y_I$, and hence $y_l a \ne 0$. Since $a \in \mathrm{Ideal}(y_{l-s} \cdots y_l)$, we know $\tilde{i}_l = p^k - 1$ if $s \ge 1$. Next applying y_{l-1} on a implies $\tilde{i}_{l-1} = p^k - 1$ if $s \ge 2$. Continue this arguments, we know $\tilde{i}_t = p^k - 1$ for $l - s \le t \le l$. This shows (2). □

For a finite group G, an element $x \in H^*(BG; \mathbf{Z}/p)$ is said to be essential if it restricts trivially to all proper subgroups of G. We consider essential elements for $G = G_1^n = p_+^{1+2n}$. Similar arguments are also done by Minh ([Mi]).

Proposition 4.4. *If* $n < i < (p-1)$, *then* $\mathrm{tr}(i) \in H^*(BG_1^n; \mathbf{Z}/p)$ *is essential.*

Proof. Any maximal subgroup M of G_1^n is isomorphic to $G_1^{n-1} \times \mathbf{Z}/p$. Let $\langle M, g \rangle = G_1^n$. Suppose that $A^{odd} = A \subset M$. Then by the double coset formula,

$$\mathrm{tr}(i) \mid M = \sum_{k=0}^{p-1} \mathrm{Cor}^M_{g^k A g^{-k} \cap M} (g^{k*} u^i) = \mathrm{Cor}^M_A \left(\sum_k g^{k*} u^i \right).$$

Let $H^*(M; \mathbf{Z}/p) \cong H^*(BG_1^{n-1}; \mathbf{Z}/p) \otimes \mathbf{Z}/p[y] \otimes \Lambda(x)$ so that $g^*(u) = u + y$. Then

$$\sum_k g^{k*} u^i = \sum_k (y + ky)^i = \sum_{j=0}^{i} \binom{i}{j} \left(\sum k^j \right) u^i y^{i-j} = 0 \mod(p)$$

since $\sum_{k=0}^{p-1} k^j = 0 \mod(p)$ for $j < p - 1$.

Next suppose that $\langle A, M \rangle = G$. Let $\tilde{A} = A \cap M$. Then $\tilde{A} \cong (Z/p)^s$ for $s \le n$. Since all maximal elementary abelian p-subgroup of G_1^n have the rank $= n + 1$, there is a subgroup $A \cap M \subset B \subset M$ with $B \cong (Z/p)^{n+1}$. By the double coset formula, we also have

$$\mathrm{tr}(i) \mid M = \mathrm{Cor}^M_{\tilde{A}}(u^i) = \mathrm{Cor}^M_B \, \mathrm{Cor}^B_{\tilde{A}}(u^i).$$

Since $B \cong \tilde{A} \times (\mathbf{Z}/p)^{n+1-s}$, we see $\mathrm{Cor}^B_{\tilde{A}}(-) = 0$. □

Let $A' = \langle A, G_1^1 \rangle \subset G_1^n$. Then from Corollary 3.12 and Theorem 2.4, we see

$$\{f^n u^{p-n}\} \doteq \mathrm{Cor}_{A'}^{G_1^n}(u_p) \quad \text{where } u_p = \{u^p\} \in H^*(BG_1^1; \mathbf{Z}/p) \subset H^*(BA'; \mathbf{Z}/p).$$

Proposition 4.5. *For $n \geq 2$, the element $\mathrm{Cor}_{A'}^{G_1^n}(u_p) \in H^*(BG_1^n; \mathbf{Z}/p)$ is essential.*

Proof. Suppose that $A' \subset M$. Then by the double coset formula

$$\mathrm{Cor}_{A'}^{G_1^n}(u_p) \mid M = \sum_{k=0}^{p-1} \mathrm{Cor}_{g^k A' g^{-k} \cap M}^{M}(g^{k*} u_p) = \mathrm{Cor}_{A'}^{M}\left(\sum_k g^{k*} u_p\right).$$

It is known that $u_p \mid \langle a_1, c \rangle = u^p - y_1^{p-1} u$ [L]. Hence

$$g^* u_p \mid A = (u+y)^p - y_1^{p-1}(u+y) = (u^p + y_1^{p-1} u) + (y^p - y_1^{p-1} y).$$

From this equation we can prove (for details, see [L])

$$g^* u_p = u_p + y^p - \chi y \quad \text{where } \chi = \mathrm{Cor}_{\langle a_1, c \rangle}^{G_1^1}(u^{p-1}) + y_2^{p-1}.$$

Here we identify $\mathrm{Cor}_{\langle a_1, c \rangle}^{G_1^1}(-) = \mathrm{Cor}_A^{A'}(-)$ since $\langle a_1, c \rangle \times (\mathbf{Z}/p)^{n-1} \cong A$ and $G_1^1 \times (\mathbf{Z}/p)^{n-1} \cong A'$. Thus we get $\sum_k g^{k*} u_p = 0$ since $g^* \chi = \chi$.

Next suppose that $\langle A', M \rangle = G$. Let us write $\tilde{A} = A' \cap M$. If $\mathrm{rank}_p(\tilde{A}) \leq n$, then we can take B as the proof of Proposition 4.4. Similarly we get $\mathrm{Cor}_{\tilde{A}}^{B}(-) = 0$ for the above case. Hence let $\tilde{A} \cong (\mathbf{Z}/p)^{n+1}$ and this implies $\tilde{A} = A$. Also by the double coset formula

$$\mathrm{Cor}_{A'}^{G_1^n}(u_p) \mid M = \mathrm{Cor}_A^M(u^p - y_1^{p-1} u) = \mathrm{Cor}_A^M(u^p) - y_1^{p-1} \mathrm{Cor}_A^M(u).$$

But the above formula is zero by the following reason. We take $\tilde{A} = A \subset B \subset M$ such that $B \cong G_1^1 \times (\mathbf{Z}/p)^{n-1}$. Here let us reorder i of a_i so that $B \supset G_1^1 = \langle c, a_3, a_4 \rangle$. The restrictions

$$\mathrm{Cor}_{\langle a_3, c \rangle}^{G_1^1}(u^p) \mid \langle a_3^\lambda a_4, c \rangle = \mathrm{Cor}_{\langle c \rangle}^{\langle a_3^\lambda a_4, c \rangle}(u^p) = 0 \quad \text{for } 0 \leq \lambda \leq p-1,$$

$$\mathrm{Cor}_{\langle a_3, c \rangle}^{G_1^1}(u^p) \mid \langle a_3, c \rangle = \sum_{k=0}^{p-1} (a_4)^{*k}(u^p) = \sum (u + k y_3)^p = 0$$

implies $\mathrm{Cor}_{\langle a_3, c \rangle}^{G_1^1}(u^p) = 0$ (in fact, there is no essential element of degree $2p$ in $H^*(BG_1^1; \mathbf{Z}/p)$). Moreover $\mathrm{Cor}_{\langle a_3, c \rangle}^{G_1^1}(u) = f = 0$. Hence we know

$$\mathrm{Cor}_{\tilde{A}}^{B}(u^p) = 0 \quad \text{and} \quad \mathrm{Cor}_{\tilde{A}}^{B}(u) = 0. \qquad \square$$

Remark 4.1. For the group G_1^2, we note that

$$\{w(2)' x_1 x_2\} \doteq (\{p u^{p(p-1)}\}\{pu\}) \doteq r(\{p u^{p(p-1)}\}) f \doteq \{f^2 u^{p(p-1)-2}\} f.$$

There contains errors in Theorem 8.18 in [T-Y3]. The elements $z_{p(p-1)-1} z$ and $\{w(2)' x_1 x_1\}$ in $H^*(BG_1; \mathbf{Z}/p)$ should be deleted. Ignoring the assumption $|b| \neq$

$2p(p-1)+2$ in Lemma 8.18 occurred the errors. Hence $\eta = 0$ in Prop. 6 in [Mi], while the main theorem in [Mi] is of course correct.

Remark 4.2. Considering the restriction to G_∞^1 and using the arguments in Lemma 7.3 below, we can prove

$$\{fu^p\} \,|\, (G_1^1 \times \mathbf{Z}/p) \doteq \mathrm{tr}(2)\, y_3^{p-1}.$$

Now we study $m(i)$ for $n \le i \le p-1 \mod(p)$.

Lemma 4.6. *Let $n \le i \le p-1 \mod(p)$. Then for the group G_∞^n, the number $m(i) = n$, that is, $p^n u^i$ generates $IE_\infty^{0,*}$.*

Proof. By induction, we assume the above fact for n. Consider the map of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_\infty^n & \longrightarrow & G_\infty^{n+1} & \longrightarrow & \mathbf{Z}/p \oplus \mathbf{Z}/p & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle j_1} & & \uparrow{\scriptstyle j_2} & & \uparrow & & \\ 0 & \longrightarrow & S^1 \oplus (\mathbf{Z}/p)^n & \longrightarrow & G_\infty^1 \oplus (\mathbf{Z}/p)^n & \longrightarrow & \mathbf{Z}/p \oplus \mathbf{Z}/p & \longrightarrow & 0 \end{array}$$

and the induced spectral sequences

$$E(n+1)_2^{*,*} = H^*(B(\mathbf{Z}/p \oplus \mathbf{Z}/p); H^*(BG_\infty^n)) \Rightarrow H^*(BG_\infty^{n+1})$$
$$E(1')_2^{*,*} = H^*(\mathbf{Z}/p \oplus \mathbf{Z}/p; H^*(B(S^1 \oplus (\mathbf{Z}/p)^n)) \Rightarrow H^*(B(G_\infty^1 \oplus (\mathbf{Z}/p)^n)).$$

The differential of the transferred element is

$$\begin{aligned} d_2(\mathrm{tr}(i)) &= d_2(j_{1!}(u^i)) = j_{2!} d_2(u^i) \\ &= j_{2!}(iu^{i-1} \otimes z_{12}(1)) = ij_{1!}(u^{i-1}) \otimes z_{12}(1) = i\,\mathrm{tr}(i-1) \otimes z_{12}(1) \end{aligned}$$

where $j_!$ is the transfer map induced from an injection j.

We assume that $n+1 \le i \le p-1 \mod(p)$, and so $n \le i-1 \mod(p)$. This means that $\mathrm{tr}(i-1)$ generates $IE_\infty^{0,2(i-1)}$ and $\mathrm{tr}(i-1) \ne 0$ by inductive assumption. Thus $\mathrm{tr}(i)$ is not a permanent cycle in $E(n+1)_r^{*,*}$. Hence $p^{n+1}u^i$ generates $IE_\infty^{0,2i}$ for $H^*(BG_\infty^{n+1})$. □

Corollary 4.7 ([T-Y2] Theorem 5.2). $p^{m+n} \,|\, \exp(H^*(BG_m^n))$.

Proof. Note that $IE_\infty^{0,*}$ is generated by $p^n u^{p^n-1}$ (resp. u^{p^n}) when $* = 2(p^n-1)$ (resp. $* = 2p^n$). Therefore the differential in (3.2) is

$$d_2((p^n u^{p^n-1}) \otimes z) = p^n u^{p^n-1} p^m u = p^{m+n} u^{p^n}.$$ □

5. Morava K-theory

In this section, we compute the Morava K-theory of the group $\tilde{G}$. Let us write the infinitive term $E_\infty^{*,0}$ by A, i.e.,

$$A = S_4 \otimes \Lambda_4/(w(1), w(2), z(1), z(2), z(3), w(2)'\beta(x_1x_2)).$$

Write by A_i, $0 \le i \le 4$, the S_4-submodule of A generated by i-th product $x_{j_1}\cdots x_{j_i}$ of odd degree generators. In particular, $A_0 = S_4/(w(1), w(2))$.

We consider the additive decomposition

$$A_0 = B_0 \oplus C_0 \quad \text{with } B_0 = A_0/(w_{12}(1)),\ C_0 = A_0\langle w_{12}(1)\rangle$$

where $A_0\langle w\rangle$ means the A_0-*submodule* of A generated by w (while $A_0\{w\}$ means the *free* A_0-module). Here we have

$$B_0 \cong S_{12}/(w_{12}(1)) \otimes S_{34}/(w_{34}(1)), \quad C_0 \cong S_4/(w(1), w(2)')\{w_{12}(1)\}$$

where $S_{ij} = \mathbf{Z}/p[y_i, y_j]$, $w_{ij}(1) = y_i^p y_j - y_i y_j^p$, so that $w(1) = w_{12}(1) + w_{34}(1)$ and $w(2) = w(2)'w_{12}(1)$.

We also consider the decomposition of A_+ such that

$$A_+ = B_+ \oplus C_+ \quad \text{with } B_+ = A_+/(z_{12}(1), z_{12}(2), x_1x_2, x_3x_4),$$
$$C_+ = A\langle z_{12}(1), z_{12}(2), x_1x_2, x_3x_4\rangle$$

Let $B = B_0 \oplus B_+$ and $C = C_0 \oplus C_+$ so that $A = B \oplus C$. Let us write

$$B_{ij} = B_{ij,0} \oplus B_{ij,+} \quad \text{where } B_{ij,0} = S_{ij}/(w_{ij}(1)),$$

$$\begin{aligned} B_{ij,+} &= S_{ij}\{x_i, x_j\}/(z_{ij}(1), z_{ij}(2), x_ix_j) \\ &\cong S_{ij}\{x_i, x_j\}/(y_ix_j - y_jx_i, y_i^p x_j - y_j x_i^p, x_ix_j) \cong S_{ij}/(y_{ji})\{x_i\} \oplus \mathbf{Z}/p[y_j]\{x_j\} \end{aligned}$$

so that $B \cong B_{12} \otimes B_{34}$. Here

$$y_{ji} = w_{ji}(1)/y_i = y_j^p - y_i^{p-1}y_j.$$

The Q_k-action is given by $Q_kx_i = y_i^{p^k}$. Hence $Q_k : B_{ij,+} \to B_{ij,0} = S_{ij}/(w_{ij}(1))$ is injective since $w_{ij}(1) = y_iy_{ji}$. Then we can easily compute the Q_k homology

$$H(B_{ij}, Q_k) \cong S_{ij}/(y_i^{p^k}, y_j^{p^k}, w_{ij}(1)).$$

By Kunneth formula, we have;

Lemma 5.1.

$$H(B; Q_k) \cong S_4/(y_1^{p^k}, \dots, y_4^{p^k}, w_{12}(1), w_{34}(1)).$$

Next we will study $H(C; Q_k)$. Recall

$$C_0 = S_4/(w(1), w(2)')\{w_{12}(1)\} \quad C_+ = (S_4 \otimes \Lambda_4)\langle z_{12}(1), z_{12}(2), x_1x_2, x_3x_4\rangle.$$

For ease of notation, let us write $D = S_4/(w(1), w(2)')$. We already know $z_{12}(1)$ generates the D-module in C_+ since $\beta(x_1x_2) = z_{12}(1)$.

Lemma 5.2. $w(2)'z_{12}(2) = 0$.

Proof. In S_{12}, we have $P^1w_{12}(1) = 0$ and

$$P^1 w_{12}(2) = P^1(y_1^{p^2} y_2 - y_1 y_2^{p^2}) = y_1^{p^2} y_2^p - y_1^p y_2^{p^2} = w_{12}(1)^p.$$

Since $w_{12}(2) = w_{12}(2)' w_{12}(1)$, we get $P^1 w_{12}(2)' = w_{12}(1)^{p-1}$. Hence in C, we get

$$0 = P^1(w(2)' z_{12}(1)) = (w_{12}(1)^{p-1} - w_{34}(1)^{p-1}) z_{12}(1) + w(2)' P^1 z_{12}(1)$$

The first term of the righthand side of the above equation is zero since $w_{12}(1) = -w_{34}(1)$ in A. The fact $P^1 z_{12}(1) = z_{12}(2)$ implies the result $w(2)' z_{12}(2) = 0$. □

Thus we get the map $Q_1 : D\langle z_{12}(1)\rangle \to D\{w_{12}(1)\}$. Here $Q_1(z_{12}(1)) = w_{12}(1)$ and its image is a D-free module. Therefore this map is an isomorphism, i.e., $z_{12}(1)$ generates a free D-module. Since $Q_0(z_{12}(2)) = w_{12}(1)$, $z_{12}(2)$ also generates a free D-module. Moreover $Q_0(z_{12}(1)) = 0$ and $Q_1(z_{12}(2)) = 0$. This means that $D\langle z_{12}(1)\rangle$ and $D\langle z_{12}(2)\rangle$ have no intersection except for zero. Thus we have

Lemma 5.3. $C_1 \cong D\{z_{12}(1)\} \oplus D\{z_{12}(2)\}$.

Next consider the module C_2, Note that $x_1 z_{12}(1) = -y_2 x_1 x_2$ and $x_3 z_{12}(1) = -x_3 z_{34}(1) = y_4 x_3 x_4$. Similar fact holds for $z_{12}(2)$. Thus we get

$$C_2 = S_4\langle x_1x_2, x_3x_4\rangle \cong S_4\langle x_1x_2, f\rangle.$$

We have the map $Q_0 : D\langle x_1x_2\rangle \to D\{z_{12}(1)\}$ with $Q_0(x_1x_2) = z_{12}(1)$. While $w(2)' x_1 x_2 \neq 0$, but the fact $y_1 x_1 x_2 = x_1 z_{12}(1)$ $(y_3 x_1 x_2 = -y_3 x_3 x_4 = -x_3 z_{34}(1))$ implies that $S_4^+\langle x_1x_2\rangle$ is a D-module. Hence we have the isomorphism

Lemma 5.4. *There is an additive isomorphism*

$$C_2 \cong D\{x_1x_2\} \oplus \mathbf{Z}/p\{w(2)' x_1 x_2\} \oplus S_4/(w_{ij}(1) \mid i < j)\{f\}.$$

Proof. We already know the module $S_4\langle f\rangle$ from Theorem 2.5. The kernel of the map $Q_0 : C_2 \to D\{z_{12}(1)\}$ is direct sum of $\mathbf{Z}/p\{w(2)' x_1 x_2\}$ and the S_4-module generated by f. □

The generators $x_{i_1} x_{i_2} x_{i_3} \in C_3$ are represented as $x_i f$, e.g., $x_1 x_2 x_3 = f x_3$. The S_4-submodule generated by $x_i f$, $1 \le i \le 4$ is still given in Theorem 2.5

$$C_3 \cong H(E, z(2))^{odd} \cong S_4\{x_i f \mid 1 \le i \le 4\}/(y_{ji} x_i f \mid i \neq j).$$

We also note that $Q_0 : C_3 \to S_4^+/(w_{ij}(1) \mid i < j)\{f\}$ is an isomorphism. The fact

$$C_4 \cong \mathbf{Z}/p\{x_1x_2x_3x_4 = f^2\}$$

is also given in Theorem 2.5.

First note that $Q_k(f^2) = 0$, since this element is represented as the transfer.

Lemma 5.5. $H(S_4/(w_{ij}(1))\{f\} \oplus C_3; Q_k) \cong S_4/(w_{ij}(1), y_i^{p^k})\{f\}$.

Proof. Exchanging $y_iy_j^p$ by $y_i^py_j$ if $i > j$, each element $a \in S_4/(w_{ij}(1) = y_i^py_j - y_iy_j^p)$ is uniquely represented as

$$a = \sum a_I y_I \quad \text{with } a_I \in \mathbf{Z}/p,\ y_I = y_m^{i_m} \cdots y_4^{i_4},\ m < \cdots < 4$$

such that $i_m \neq 0$ and $0 \leq i_j < p$ for all $m < j$.

Similarly using the relation $0 = y_{ji}x_i = (y_j^p - y_i^{p-1}y_j)x_i$, each element $b \in C_3$ is uniquely written as

$$b = \sum b_I z_I f \quad \text{with } b_I \in \mathbf{Z}/p,\ z_I = x_m y_m^{i_m} \cdots y_4^{i_4},\ m < \cdots < 4$$

such that $0 \leq i_j < p$ for all $m < j$. The Q_k action is given by

$$Q_k(b) = \sum b_I y_m^{p^k + i_m} y_{m+1}^{i_{m+1}} \cdots y_4^{i_4} f.$$

Hence if $b \neq 0$ in C_3, then $Q_k(b) \neq 0$ also in $S_4/(w_{ij}(1))\{f\}$. This proves the lemma. □

Lemma 5.6. *Let k be an algebraic closed field of* $\mathrm{ch}(k) = p$. *For each $\lambda \in k$, the sequence $(w(1), w(2)', y_3 - \lambda y_4, y_4)$ is regular in $S_4 \otimes k$.*

Proof. The sequence is regular if and only if the dimension of the variety

$$\dim_k \mathrm{Var}(w(1), w(2)', y_3 - \lambda y_4, y_4) = 4 - 4 = 0.$$

Letting $y_3 = y_4 = 0$, we only need to show $\dim_k \mathrm{Var}(w_{12}(1), w_{12}(2)') = 0$ where $w_{12}(2)' = (y_1^{p^2}y_2 - y_1y_2^{p^2})/(y_1^py_2 - y_1y_2^p)$. The regularity of $(w_{12}(1), w_{12}(2)')$ in S_2 is well known, in fact, these elements are Dickson invariants

$$\mathbf{Z}/p[y_1, y_2]^{SL_2(Z/p)} = \mathbf{Z}/p[w_{12}(1), w_{12}(2)'].$$ □

Let us write $w_{12}(k) = (y_1^{p^k}y_2 - y_1y_2^{p^k}) = Q_kz_{12}(1) = Q_kQ_0(x_1x_2)$.

Lemma 5.7. *Suppose that $aw_{12}(k) + bw_{12}(k-1)^p = 0$ in $S_4/(w(1), w(2))$. Then*

$$a = (w_{12}(k-1)^p/w_{12}(1))c, \quad b = (w_{12}(k)/w_{12}(1))c \quad \textit{for } c \in S_4/(w(1), w(2)').$$

Proof. When $k \leq 2$, the theorem is almost immediate. We assume $k \geq 3$. Suppose that $aw_{12}(k)/w_{12}(1) + bw_{12}(k-1)^p/(w_{12}(1)) = 0$ in $D = S_4/(w(1), w(2)')$. We have the decomposition

$$w_{12}(k)/w_{12}(1) = \prod_{\lambda \in F_{p^k} - F_p} (y_2 - \lambda y_1).$$

Let $y_2 - \lambda y_1 = 0$ for $\lambda \in F_{p^k} - F_p$. Then by the supposition we get

$$0 = bw_{12}(k-1)^p/w_{12}(1) = b\lambda' y_1^{p^k - p}$$

in $S_4 \otimes \bar{F}_p/(w(1), w(2)', y_2 - \lambda y_1)$ and $\lambda' \neq 0 \in \bar{F}_p$ because $F_{p^k} - F_p$ and $F_{p^{k-1}} - F_p$ have no intersection in $\bar{F}_p$. Since $(w(1), w(2)', y_2 - \lambda y_1, y_1)$ is regular, we have $b = 0$ in $S_4 \otimes \bar{F}_p/(w(1), w(2)', y_2 - \lambda y_1)$ and we can take $b = (y_2 - \lambda y_1)c' \in S_4 \otimes \bar{F}_p/(w(1), w(2)')$. Continuing this argument for all other $\lambda \in F_{p^k} - F_p$ and we get $b = w_{12}(k)/w_{12}(1)c$.

Apply the similar arguments for $y_2 - \mu y_1$, $\mu \in F_{p^{k-1}}$, we get the lemma. □

Lemma 5.8. *The homology $H(C_0 \oplus C_1 \oplus D\{x_1x_2\}; Q_k)$ is isomorphic to*

$$D/(w_{12}(k)/w_{12}(1), w_{12}(k-1)^p/w_{12}(1))\{w_{12}(1)\}.$$

Proof. We will show that the following sequence is exact

$$0 \to D\{x_1x_2\} \overset{Q_k}{\to} D\{z_{12}(1), z_{12}(2)\} \overset{Q_k}{\to} D\{w_{12}(1)\}.$$

The Q_k-operations are given

$$Q_k(z_{12}(1)) = Q_k(y_1x_2 - y_2x_1) = y_1y_2^{p^k} - y_2y_1^{p^k} = -w_{12}(k)$$
$$Q_k(z_{12}(2)) = Q_k(y_1^px_2 - y_2^px_1) = (y_1^py_2^{p^k} - y_1^{p^k}y_2^p) = -w_{12}(k-1)^p.$$

Hence if $c_1 = az_{12}(1) + bz_{12}(2) \in C_1$ is in the kernel $\mathrm{Ker}(Q_k)$, then from Lemma 5.7, we have

$$c_1 = c(w_{12}(k-1)^p/(w_{12}(1))z_{12}(1) + w_{12}(k)/(w_{12}(1))z_{12}(2)) \quad \text{with } c \in D$$

which is just $cQ_k(x_1x_2)$, indeed,

$$Q_0Q_k(x_1x_2) = w_{12}(k) \quad \text{and} \quad Q_1Q_k(x_1x_2) = w_{12}(k-1)^p$$

imply that

$$Q_k(x_1x_2) = w_{12}(k-1)^p/(w_{12}(1))z_{12}(1) + w_{12}(k)/(w_{12}(1))z_{12}(2)$$

since $Q_1z_{12}(1) = w_{12}(1)$, $Q_0z_{12}(2) = w_{12}(1)$, and $Q_0z_{12}(1) = 0$, $Q_1z_{12}(2) = 0$. □

Since $w_{12}(k)$ and $w_{12}(k-1)^p$ are in $\mathrm{Ideal}(y_i^{p^k})$ in A_0 we have

Corollary 5.9. $H(B \oplus C_0 \oplus C_1 \oplus D\{x_1x_2\}, Q_k) \cong S_4/(w(1), w(2), y_i^{p^k})$.

Corollary 5.10. *$H(E_\infty^{*,0}, Q_k)$ is generated as an S_4-module by* 1, $w(2)'x_1x_2$, f *and* f^2.

Recall the isomorphism $E_\infty^{*,ps} \cong C_3 \oplus S_4/(w_{ij}(1)\{f\} \oplus \mathbf{Z}/p\{f^2\}$. Hence its cohomology is still given in Lemma 5.5. As for elements $\{f^2u^s\}$, we may assume that its Q_k-action is trivial because $H(E_\infty^{*,0}; Q_k)$ is generated by even dimensional elements. Thus we get

Theorem 5.11. *There is an isomorphism*

$$H(H^*(BG_\infty^2;\mathbf{Z}/p);Q_k) \cong \mathbf{Z}/p[u^{p^2}] \otimes (S_4/(w(1),w(2),y_i^{p^k}) \oplus \mathbf{Z}/p\{w(2)'x_1x_2\} \oplus \bigoplus_s \mathbf{Z}/p\{f^2u^s\} \oplus \bigoplus_t S_4/(w_{ij}(1),y_i^{p^k})\{fu^{pt}\})$$

where $0 \le s \ne (p-1) \bmod p$ *and* $s \ne p^2-2$ *and* $0 \le t \le p-2$. *Thus this homology is generated by even dimensional elements. Hence we have*

$$K(k)^*(BG_\infty^2) \cong K(k)^* \otimes H(H^*(BG_\infty^2;\mathbf{Z}/p),Q_k).$$

Next consider the cases of finite groups G_m^2, $m \ge 2$. By arguments after (3.2), we see

$$H(H^*(BG_m^2;\mathbf{Z}/p),Q_k) \cong H(H^*(BG_\infty^2;\mathbf{Z}/p),Q_k) \otimes \Lambda(z).$$

We consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(BG_m^2;K(k)^*) \Rightarrow K(k)^*(BG_m^2).$$

Recall $K(k)^* \cong \mathbf{Z}/p[v_k,v_k^{-1}]$. Since the first nonzero differential is the form $d_{2p^k-1}(x) = v_k \otimes Q_k(x)$, we still have the $E_{2p^k}^{*,*}$-term. Since all elements in $H(H^*(BG_\infty^2;\mathbf{Z}/p),Q_k)$ are permanent cycles in the above spectral sequence, we only need to study d_rz.

Consider the injection $\mathbf{Z}/p^m \subset G_m^2$. The Morava K-theory is

$$K(k)^*(B\mathbf{Z}/p^m) \cong K(k)^*[u]/([p^m](u)).$$

Here $[p](u) = v_ku^{p^k}$ implies $[p^m](u) = v_k^{1+p^k+\cdots+p^{(m-1)k}}u^{p^{mk}}$. Thus in the Atiyah-Hirzebruch spectral sequence converging $K(k)^*(B\mathbf{Z}/p)$, the differential

$$d_{2p^{mk}-1}(z) = v_k^{1+p^k+\cdots+p^{(m-1)k}}u^{p^{mk}}.$$

Thus we get

Theorem 5.12. *Let* $m \ge 2$. *Then*

$$K(k)^*(BG_m^2) \cong K(k)^*(BG_\infty^2)/(u^{p^{mk}}).$$

6. *BP*-theory

Let $BP^*(-)$ be the Brown-Peterson cohomology theory with the coefficient ring $BP^* = \mathbf{Z}/p[v_1,\dots]$, $|v_i| = -2(p^n-1)$. Since $K(k)^{odd}(BG_m^2) = 0$ for $m \ge 2$, we also have $BP^{odd}(BG_m^2) = 0$ from the theorem by Ravenel-Wilson-Yagita [R-W-Y]. In this section we will study $BP^*(BG_m^2)$ more explicitly.

Recall that $\tilde{E}_r^{*,*}$ (resp. $IE_r^{*,*}$) is the Hochschild-Serre spectral sequence converging to $H^*(BG_\infty^2;\mathbf{Z}/p)$ (resp. $H^*(BG_\infty^2)$). From Lemma 3.13, we already know that $IE_\infty^{+,0} \cong \beta E_\infty^{+,0} \oplus \mathbf{Z}/p\{f,f^2,w(2)'x_1x_2\}$. The decomposition $\tilde{E}_\infty^{+,0} \cong B^+ \oplus C$ is given in §5 with

$$H(B^+,\beta) \cong 0 \quad \text{and} \quad H(C;\beta) \cong \mathbf{Z}/p\{w(2)'x_1x_2, f, f^2\}.$$

Note that B and C are closed under the Bockstein operation. The Bockstein images of C is

$$\beta C \cong D\{w_{12}(1), z_{12}(1)\} \oplus (S_4^+/(w_{ij}(1)\{f\}).$$

Here $Q_1z_{12}(1) = w_{12}(1)$. The Bockstein of B is

$$S_4/(w_{12}(1), w_{34}(1)) \oplus S_4\langle \beta\{x_sx_t \,|\, 1 \leq s \leq 2, 3 \leq t \leq 4\}\rangle.$$

Lemma 6.1. *If* $0 \neq x \in B_2$ *in the notation in* §5, *then* $Q_1Q_0x \neq 0$ *in* B_0.

Proof. Each element x in B_2 is expressed as (recall the arguments before Lemma 5.1)

$$x = a_{13}x_1x_3 + a_{14}x_1x_4 + a_{23}x_2x_3 + a_{24}x_2x_4$$

where $a_{13} \in S_4/(y_{21}, y_{43})$, $a_{23} \in S_{34}/(y_{43}) \otimes \mathbf{Z}/p[y_2]$, $a_{14} \in S_{12}/(y_{12}) \otimes \mathbf{Z}/p[y_4]$, $a_{24} \in \mathbf{Z}/p[y_2, y_4]$.

Suppose that $Q_1Q_0x = 0$ in $B_0 = S_4/(w_{12}(1), w_{34}(1))$. First let $y_1 = y_3 = 0$. Then $Q_1Q_0x = Q_1Q_0a_{24}x_2x_4 = a_{24}w_{24}(1)$. But $w_{24}(1) = y_2^py_4 - y_2y_4^p$ is a nonzero divisor in $\mathbf{Z}/p[y_2, y_4]$. Hence $a_{24} = 0$.

Next let $y_1 = 0$. Then $Q_1Q_0x = a_{23}w_{23}(1)$. But $y_2 - \lambda y_3$ is a nonzero divisor in $S_{34}/(y_{34}) \otimes \mathbf{Z}/p[y_2]$ because the dimension of the variety

$$\mathrm{Var}(y_{43}, y_2 - \lambda y_3) = \bigcup_{\mu}(y_4 - \mu y_3, y_2 - \lambda y_3)$$

is just one. Hence $a_{23} = 0$. Similarly letting $y_3 = 0$, we have $a_{14} = 0$.

Lastly, consider $Q_1Q_0x_1x_3$. The dimension of the variety is

$$\mathrm{Var}(y_{21}, y_{34}, y_1 - \lambda y_3) = \bigcup_{\mu,\mu'} \mathrm{Var}(y_2 - \mu y_1, y_4 - \mu' y_3, y_1 - \lambda y_3)$$

is also just one. Hence $w_{13}(1) = y_1^py_3 - y_1y_3^p$ is also nonzero divisor in $S_4/(y_{21}, y_{43})$. So $a_{13} = 0$. □

Since $Q_1Q_0(x_sx_t) = w_{st}(1)$ and $Q_1(z_{12}(1)) = w_{12}(1)$, we have;

Corollary 6.2. $Q_1(IE_\infty^{+,0}) \cong S_4\langle w_{ij}(1) \,|\, i < j\rangle$.

We also known $IE_\infty^{+,2p} \cong S_4/(w_{ij}(1))\{f\}$. Considering the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(BG_\infty^2; BP^*) \Rightarrow BP^*(BG_\infty^2).$$

The first nonzero differential is $d_{2p-1}(x) = v_1 \otimes Q_1(x)$. The term $E_{2p}^{*,*}$ is generated by even dimensional elements. Hence we have;

Theorem 6.3. *The graded ring* gr $BP^*(BG_\infty^2)$ *is isomorphic to*

$$(BP^* \otimes S_4/(w(1), w(2), v_1 w_{ij}(1)) \oplus BP^* \otimes (F \oplus U)) \otimes \mathbf{Z}[\{u^{p^2}\}]$$

where $F = S_4^+/(w_{ij}(1))\{fu^{sp} \mid 1 \le s \le p-2\}$, $U = \mathbf{Z}\{u_t \mid 0 \le t \le p^2-1\}$,

$$u_t = \begin{cases} \{1\} & t = 0 \\ \{pu^t\} & t = 0, 1 \mod(p) \text{ and } 0 < t \ne p(p-1)+1 \\ \{p^2 u^t\} & 2 \le t \le p-1 \mod(p) \text{ or } t = p(p-1)+1. \end{cases}$$

Corollary 6.4. *All BP^*-linear relations in $BP^*(BG_\infty^2)$ are deduced from the relations in $BP^*(BV)$.*

Proof. Since $[p](y_i) = py_i + v_1 y_i^p + \cdots = 0$ in $BP^*(BV)$, we have the relation in $BP^*(BV)$,

$$y_j[p](y_i) - y_i[p](y_j) = v_1(y_i^p y_j - y_i y_j^p) + \cdots = v_1 w_{ij}(1) + \cdots = 0. \qquad \square$$

We consider the cases of finite groups G_m^2, $m \ge 2$. Recall that

$$H^*(BG_m^2) \cong IE_\infty^{+,0} \otimes \Lambda(z) \oplus IE_\infty^{0,*}/(\{p^m u\}).$$

We easily see that $IE_\infty^{0,*}/(\{p^m u\})$ is generated by u_t, $0 \le t \le p^2-1$ and $\{u^{p^2}\}$ with

$$\exp(u_t) = \begin{cases} p^{m-1} & (t = 2 \mod(p) \text{ but } t \ne p(p-1)+2) \\ & \text{or } (t = 1) \text{ or } (t = p(p-1)+1) \\ p^m & (3 \le t \le p-1 \mod(p)) \text{ or } (1 \mod(p) \text{ but} \\ & t \ne 1, \ne p(p-1)+1) \text{ or } (t = p(p-1)+2) \\ p^{m+1} & (t = ps, 0 < s < p) \\ p^{m+2} & (t = p^2). \end{cases}$$

Theorem 6.5. *For $m \ge 2$, we have the isomorphism*

$$\mathrm{gr}\, BP^*(BG_m^2) \cong \mathrm{gr}\, BP^*(BG_\infty^2)/(v_1^{s_1} y_i u^{p^m}, v_2^{s_2} w_{ij}(1) u^{p^{2m}}, \exp(u_t) u_t)$$

where $s_k = 1 + p^k + \cdots + p^{(m-1)k}$.

Proof. We consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(BG_m^2; BP^*) \Rightarrow BP^*(BG_m^2).$$

The first nonzero differential is $d_{2p-1}(x) = v_1 \otimes Q_1(x)$. The $2p$-term is

$$E_{2p}^{*,*} \cong (BP^* \otimes S_4/(w(1), w(2), v_1 w_{ij}(1)) \oplus BP^* \otimes (F)) \otimes \Lambda(z)$$

$$\oplus BP^* \otimes U/(\exp(u_t) u_t) \otimes \tilde{\mathbf{Z}}/(p^{m+2})[u^{p^2}]$$

where $\tilde{\mathbf{Z}}/(p^{m+2})[u^{p^2}]$ means $\mathbf{Z}[u^{p^2}]/(p^{m+2} u^{p^2})$. By $K(1)^*(-)$ theory, the next nonzero differential is $d_{2p^m-1}(y_i z) = v_1^{s_1} y_i u^{p^m}$. The last nonzero differential is $d_{2p^{2m}-1}(w_{ij}(1) z) = v_2^{s_2} w_{ij}(1) u^{p^{2m}}$ from $K(2)^*(-)$ theory. Thus we get the theorem. $\square$

7. $BP^*(Bp_+^{1+4})$

In this section, we will study the BP-theory of the case $m = 1$, i.e., $G_1^2 = p_+^{1+4}$. The integral cohomology is (the integral version of Lemma 2.9)

$$\text{(7.1)} \qquad \text{gr}\, H^*(BG_1^2) \cong ((\text{Ker}(f) \mid H^*(BG_\infty^2))\{z\} \oplus H^*(BG_\infty^2)/(f).$$

Recall (see §6 also)

$$\begin{cases} IE_\infty^{+,0}/(f) \cong \beta B \oplus D\{w_{12}(1), z_{12}(1)\} \\ \text{Ker}(f) \mid IE_\infty^{+,0} \cong S_4\langle w_{st}(1), \beta(x_s x_t) \mid 1 \le s \le 2 < t \le 4\rangle \oplus \beta C \\ IE_\infty^{+,2ps}/(f) \cong \text{Ker}(f) \mid IE_\infty^{+,2ps} \cong S_4^+/(w_{ij}(1)\{fu^{ps}\}) \quad 1 \le s \le p-2. \end{cases}$$

Hence from Lemma 6.1 and the arguments before the lemma, we have

$$\begin{cases} H(IE_\infty^{+,0}/(f), Q_1) \cong S_4^+/(w_{ij}(1)), \\ H(\text{Ker}(f) \mid IE_\infty^{+,0}, Q_1) \cong S_4^+/(w_{ij}(1))\{fz\}, \\ H(IE_\infty^{+,2ps} \otimes \Lambda(z), Q_1) \cong S_4^+/(w_{ij}(1)\{fu^{ps}\}\Lambda(z). \end{cases}$$

Thus we can prove that

Lemma 7.1. *The homology $H(\text{gr}\, H^*(BG_1^2), Q_1)$ is isomorphic to*

$$((S_4^+/(w_{ij}(1)) \otimes (\Lambda(fz) \oplus \mathbf{Z}/p\{u^p f, \dots, u^{p(p-2)} f\} \otimes \Lambda(z)) \oplus U) \otimes \tilde{\mathbf{Z}}/p^3[u^{p^2}].$$

We will study the Atiyah-Hirzebruch spectral sequence

$$\text{(7.2)} \qquad E_2^{*,*} = H^*(\text{gr}\, H^*(BG_1^2); \tilde{K}(1)^*) \Rightarrow \tilde{K}(1)^*(BG_1^2)$$

where $\tilde{K}(1)^*(-)$ is the integral K-theory with the coefficient ring $\tilde{K}(1)^* = \mathbf{Z}_{(p)}[v_1, v_1^{-1}]$. The first nonzero differential is also

$$d_{2p-1}(x) = v_1 \otimes Q_1(x)$$

but $Q_1(x)$ is considered as an element in $\text{gr}\, H^*(BG_1^2)$. We want to prove the following lemma;

Lemma 7.2. *$d_{2p-1}(y_i fzu^{p(s-1)}) \doteq v_1 y_i fu^{ps}$ for $1 \le s \le p-2$ and hence $Q_1(y_i fzu^{p(s-1)}) \doteq y_i fu^{ps}$ in $H^*(BG_1^2; \mathbf{Z}/p)$.*

To prove this lemma, we prepare some lemmas. For a compact group G, it is known that $\tilde{K}(1)^{odd}(BG) = 0$ and $\tilde{K}(1)^*(BG)$ is torsion free by the Atiyah theorem. Hence $K(1)^{odd}(BG) = 0$. Moreover it is given

$$\dim_{K(1)^*} K(1)^*(BG_1^2) = p^4 + p - 1$$

by Brunetti [B1]. In [B2], he also showed that the Euler characteristic for $K(n)^*$-theory has the property $\chi_{n,p}(G_2^2) = p^n \chi_{n,p}(G_1^2)$. Indeed, from Theorem 5.11 and Theorem 5.12, we know $\dim_{K(1)^*} K(1)^*(BG_2^2) = p^5 + p^2 - p$.

Given $\lambda_i \in F_p^{\times}$, $1 \le i \le 4$ with $\lambda_1\lambda_2 = \lambda_3\lambda_4$, let $g = g(\lambda_1, \dots, \lambda_4)$ be the automorphism of G_1^2 defined by

$$a_i \mapsto a_i^{\lambda_i}, \quad c \mapsto c^{\lambda_1\lambda_2}.$$

Then the induced map g^* defines the automorphism of $H^*(BG_1^2)$, and moreover the automorphism of the Hochschild-Serre spectral sequence converging to $H^*(BG_1^2)$ so that

$$y_i \mapsto \lambda_i y_i \quad u \mapsto \lambda_1\lambda_2 u.$$

Indeed this gives the (weight) decomposition of the spectral sequence.

For a sequence $I = (i_1, \dots, i_4)$, let $y^I = y_1^{i_1} \cdots y_4^{i_4}$. Suppose that in $\tilde{K}(1)^*(BG_1^2)$, there is a relation

$$(*) \qquad py^I\{u^{pt}\} = v_1^s \sum_K a_K y^K \mod(p^2, v_1^{s+1}),$$

where $y^K \neq 0 \in S_4/(y_i^p)$, $0 \neq a_K \in \mathbf{Z}/p$. Let $J = K - I$. Applying g^* on the above equation, we have

$$(\lambda_1\lambda_2)^t = \lambda_1^{j_1}\lambda_2^{j_2}\lambda_3^{j_3}(\lambda_1\lambda_2/\lambda_3)^{j_4}.$$

Hence we get

$$j_1 = j_2, \quad j_3 = j_4, \quad t = j_1 + j_3 \mod(p-1).$$

On the other hand, by dimensional reason,

$$2t = |u^{pt}| = |v_1^s| + |y^J| = 4i_1 + 4(t - i_1) \mod(2(p-1)).$$

This means $t = 0 \mod(p-1)$. Similar facts hold for the differentials since the action g^* is compatible with the differentials of the spectral sequence. Thus we get

Lemma 7.3. *If* $(*)$ *holds or* $d_r(y^I zfu^{(t-2)p}) =$ *righthandside of* $(*)$, *then* $t = 0 \mod(p-1)$ *and letting* $J = K - I$,

$$j_1 = j_2 = p - 1 - j_3 = p - 1 - j_4 \mod(p-1).$$

Lemma 7.4. *In* $(*)$, *letting* $t = 0$, *we have* $s \ge 2$.

Proof. If $s = 1$, then by [Y1], there is an element $x \in H^*(BG_1^2; \mathbf{Z}/p)$ such that $Q_0(x) = y^I$ and $Q_1 x = y^K$. But $Q_1 x_i = y_i^p \in S_4 \otimes \Lambda_4$ so this contradicts to $y^K \neq 0$ in $S_4/(y_i^p)$. □

Let us write by IV the vector space in $S_4/(y_i^p)$

$$IV = \{y \in S_4/(y_i^p) \mid \deg(y) > 4(p-1)\} \oplus \mathbf{Z}/p\{(y_1y_2)^j(y_3y_4)^{p-1-j} \mid 0 \le j \le p-1\}.$$

Lemma 7.5. $\dim_{\mathbf{Z}/p}(S_4/(y_i^p, IV)) > (p^4 + p - 1)/2$.

Proof. First note that $\dim_{\mathbf{Z}/p}(S_4/(y_i^p)) = p^4$. Since the largest degree of $S_4/(y_i^p)$ is $8(p-1)$, by the duality the t-dimensional homogeneous parts are

$$\dim_{\mathbf{Z}/p}(S_4/(y_i^p))^t = \dim_{\mathbf{Z}/p}(S_4/(y_i^p))^{8(p-1)-t}.$$

The degree of $(y_1y_2)^j(y_3y_4)^{p-1-j}$ is of course $4(p-1)$ and it generates a p-dimensional $\mathbf{Z}/p$-vector space. Note $\deg(IV) \geq 4(p-1)$. The $4(p-1)$-homogeneous parts of $S_4/(y_i^p)$ is quite large, e.g., $\dim_{\mathbf{Z}/p}(S_4/(y_i^p))^{4(p-1)} > p^2$. Since

$$\begin{aligned}\dim_{\mathbf{Z}/p}&\{y \mid \deg(y) \leq 4(p-1)\}\\ &= 1/2\dim_{\mathbf{Z}/p}\{S_4/(y_i^p)\} + 1/2\dim_{\mathbf{Z}/p}\{y \mid \deg(y) = 4(p-1)\},\end{aligned}$$

we know

$$\dim_{\mathbf{Z}/p}(S_4/(y_i^p, IV)) > p^4/2 + p^2/2 - p > (p^4 + p - 1)/2. \qquad \square$$

Lemma 7.6. *As $K(1)^*$-modules, we have the injection*

$$K(1)^* \otimes S_4/(y_i^p, IV) \subset K(1)^*(BG_1^2).$$

Proof. First we note that additively $\tilde{K}(1)^* \otimes S_4/(y^p, IV) \subset \tilde{K}(1)^*(BG_1^2)$, because all targets of differentials are in IV by dimensional reasons and Lemma 7.3. If $0 \neq y \in S_4/(y_i^p, IV)$ is zero in $K(1)^*(BG_1^2)$, then there is $y' \in \tilde{K}(1)^*(BG_1^2)$ such that

$$py' = v_1^s y \quad \text{for } s \leq 2.$$

But this does not happen from Lemma 7.3 and the definition of IV. $\square$

Lemma 7.7. *If $d_{2p-1}(y_izf) = 0$, then $d_{4p-3}(y_izf) = 0$ in the spectral sequence* (7.2).

Proof. From Lemma 7.1, we can write

$$d_{4p-3}(y_1fz) = v_1^2 \sum b_J y^J y_1 u^p f \mod(v_1^3).$$

If $|J| \geq 0$ and if there is $j_i \neq 0 \mod(p-1)$, then from Lemma 7.3, we see $|y^J| \geq 4(p-1)$, and this contradicts to the dimensional reason. Hence all $j_i = 0 \mod(p-1)$ if $j_1 \geq 0$. If $j_1 = -1$, there is the case $y^J y_1 = y_2^{p-2} y_3 y_4$ by the similar arguments. Let us write

$$(**) \qquad d_{4p-3}(y_1fz) = v_1^2\left(\left(\sum_i b_i y_i^{p-1}\right) y_1 + b' y_2^{p-2} y_3 y_4\right) f u^p \mod(v_1^3).$$

We consider the (twisted) automorphism tw defined by

$$tw : a_1 \leftrightarrow a_3, \quad a_2 \leftrightarrow a_4, \quad c \mapsto c,$$

which induces

$$tw^* : y_1 \leftrightarrow y_3, \quad y_2 \leftrightarrow y_4, \quad u \mapsto u$$

on the spectral sequence. Applying tw^* on $(**)$, we get

$$d_{4p-3}(y_3 fz)$$
$$= v_1^2((b_1 y_3^{p-1} + b_2 y_4^{p-1} + b_3 y_1^{p-1} + b_4 y_2^{p-1})y_3 + b' y_4^{p-2} y_1 y_2) fu^p \mod(v_1^3).$$

Since $y_3 d_{4p-3}(y_1 fz) = y_1 d_{4p-3}(y_3 fz)$, we know $b_4 = b_2$ and $b' = 0$. We also have the other twisted map, e.g., $tw' : a_1 \leftrightarrow a_4$. Similarly, we get $b_1 = b_2 = b_3 = b_4$.

We consider the other automorphism f_λ of G_1^2 defined by

$$f_\lambda : a_3 \mapsto a_3 a_4^\lambda, \quad f_\lambda : a \mapsto a \quad \text{for } a = a_i,\ i \neq 3 \text{ or } c$$

which induces

$$f_\lambda^* : y_4 \mapsto y_4 + \lambda y_3, \quad f_\lambda^* : y \mapsto y \quad \text{for } y = y_i,\ i \neq 4 \text{ or } u.$$

Apply f_λ^* on $(**)$ with $b_i = b$. Then the left hand side of $(f_\lambda^* - \mathrm{id.})(**)$ is zero, but the righthand side is

$$v_1^2 b((y_4 + y_3)^{p-1} - y_4^{p-1}) y_1 fu^p \neq 0, \quad \text{if } b \neq 0.$$

Hence b must be zero. □

Proof of Lemma 7.2. If $d_{2p-1}(y_i zf) \neq 0$, then it is $\lambda v_1 y_i fu^p$ for $\lambda \neq 0 \in \mathbf{Z}/p$ by the dimensional reason. Suppose $d_{2p-1}(y_i zf) = 0$. Then from above lemma, $d_{4p-3}(y_i fz) = 0$. This means that all nonzero element in $\tilde{K}(1)^* \otimes S_4/(y_i^p, IV)$ are not targets of differentials. By arguments similar to the proof of Lemma 7.6 and Lemma 7.4, we can show

$$K(1)^* \otimes S_4/(y_i^p, IV)\{1, fu^p\} \subset K(1)^*(BG_1^2).$$

The dimension of the left hand side $K(1)^*$-vector space is larger than $p^4 + p - 1$ by Lemma 7.5. This contradicts to the result of $\dim_{K(1)^*} K(1)^*(BG_1^2)$ by Brunetti. Thus we get $d_{2p-1}(y_i zf) \doteq \{y_i u^p f\}$. By the induction on s we get the lemma. q.e.d.

Therefore we get

$$E_{2p}^{*,*} \cong \tilde{K}(1)^* \otimes (S_4^+/(w_{ij}(1)\{1, u^{p(p-2)} zf\} \oplus U) \otimes \mathbf{Z}/p^3[u^{p^2}].$$

From Theorem 5.12, we know $u^{p^2} = 0 \in K(1)^*(BG_2^2)$. Hence so in $K(1)^*(BG_1^2)$. However from Lemma 7.3, there is no $y' \in \tilde{K}(1)^*(BG_1^2)$ such that $py' = v_1^s y_i u^{p^2}$ since $y' \in S_4^+/(w_{ij}(1))$ or $y' \in U$. (Note that there is such $y' \in U$ for $v_1^s u^{p^2}$.) Hence for some s, the element $v_1^s y_i u^{p^2}$ is a target of differential in the spectral sequence. By dimensional reason we have

$$d_{4p-3}(y_i u^{p(p-2)} fz) \doteq v_1^2 y_i u^{p^2}.$$

Thus we get;

LEMMA 7.8.

$$\mathrm{gr}\ \tilde{K}(1)^*(BG_1^2) \cong \tilde{K}(1)^* \otimes (S_4^+/(w_{ij}(1)) \oplus U \otimes \tilde{\mathbf{Z}}/p^3[u^{p^2}]),$$
$$\mathrm{gr}\ K(1)^*(BG_1^2) \cong K(1)^* \otimes (S_4/(y_i^p) \oplus \mathbf{Z}/p\{u_3, \dots, u_{p+1}\}).$$

Proof. We study elements in U. In $H^*(BG_\infty^2)$, we have

$$u_1 = \{pu\} = f, \quad u_2 = \{p^2u^2\} = f^2,$$

which are zero in $H^*(BG_1^2)$ from (7.1). Relations $p^3u^i = v_1p^2u^{p+i-1}$ in $\tilde{K}(1)^*(B\langle c\rangle)$ give that for U in $\tilde{K}(1)^*(BG_1^2)$, e.g., $pu_3 = v_1u_{p+2}$, $pu_4 = v_1u_{p+3}, \dots$ □

Note that $\dim_{K(1)^*} K(1)^*(BG_1^2)$ is in fact $p^4 + p - 1$.

THEOREM 7.9. *The BP^*-algebra* gr $BP^*(BG_1^2)$ *is isomorphic to the quotient of the free BP^*-algebra*

$$BP^* \otimes (S_4^+/(w(1), w(2)) \oplus U \oplus S_4^+/(w_{ij}(1))\{fu^p, \dots, fu^{p(p-2)}\}) \otimes \tilde{\mathbf{Z}}/p^3[u_{p^2}]$$

by the following relations

$$(v_1w_{ij}(1), v_1y_ifu^{sp}, v_1^2y_iu_{p^2}, v_2w_{ij}(1)u_{p^2})$$

Proof. We consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{*,*} = H^*(BG_1^2; BP^*) \Rightarrow BP^*(BG_1^2).$$

The first nonzero differential is $d_{2p-1}(x) = v_1 \otimes Q_1(x)$, which was still given in the arguments for $\tilde{K}(1)^*$-theory.

$$E_{2p}^{*,*} \cong BP^* \otimes (U \oplus S_4^+/(w(1), w(2), v_1w_{ij}(1)) \oplus S_4\langle w_{ij}(1)\rangle/(v_1)\{z\}$$
$$\oplus S_4^+/(w(1), v_1)\{fu^p, \dots, fu^{p(p-2)}\} \oplus S_4^+/(w_{ij}(1)\{fzu^{p(p-2)}\}) \otimes \tilde{\mathbf{Z}}/p^3[u^{p^2}].$$

Here note that $BP^*/(v_1) \otimes S_4\langle w_{ij}(1)\rangle\{z\}$ remains, while it disappears for $\tilde{K}(1)^*$-theory.

The next nonzero differential is $d_{4p-3}(y_iu^{p(p-2)}fz) \doteq v_1^2y_iu^{p^2}$ same as the $\tilde{K}(1)^*$-theory. The last nonzero differential is

$$d_{2p^2-1}w_{ij}(1)z \doteq v_2w_{ij}(1)u^{p^2}$$

which is given from $K(2)^*$-theory and (v_2-version of) Lemma 7.3. □

Proof of Remark 4.2. First we consider the element $\{fu^p\}$ in $H^*(BG_\infty^2; \mathbf{Z}/p)$. Recall that [L]

$$H^{even}(BG_\infty^1)/p \cong (S_2/(w_{12}(1)) \oplus \mathbf{Z}/p\{\mathrm{tr}(1), \dots, \mathrm{tr}(p-1)\}) \otimes \mathbf{Z}/p[u_p]$$

where $\mathrm{tr}(i) = \mathrm{Cor}_{\langle a_1, c\rangle}^{G_\infty^1}(u^i)$ and $u_p = \{u^p\}$. Since

$$\operatorname{Im}\rho(BP^*(BG^1_\infty \times B\mathbf{Z}/p)) \cong H^{even}(BG^1_\infty) \otimes \mathbf{Z}/p[y_3]$$

for the Thom map $\rho : BP \to H\mathbf{Z}_{(p)}$, we can write

$$(*) \qquad \{fu^p\} \mid G^1_\infty \times \mathbf{Z}/p = \sum a(i', i'', J)\ \mathrm{tr}(i')u_p^{i''} y^J.$$

By arguments similar to the proof of Lemma 7.3, we have

$$i' + i'' + j_1 = 2, \quad j_1 = j_2, \quad j_3 = 0 \mod(p-1).$$

Hence by the dimensional reason, we can write

$$(*) = \mathrm{tr}(1)u_p + \mathrm{tr}(2)ay_3^{p-1}.$$

here we use the fact $y_i\ \mathrm{tr}(1) = y_2\ \mathrm{tr}(i) = 0$ for $i < p-1$.

Now we consider the conjugation map a_4^* on $H^*(BG^2_\infty; \mathbf{Z}/p)$ (or $H^*(BG^1_\infty \times \mathbf{Z}/p; \mathbf{Z}/p)$, $H^*(B(\langle a_1, a_3, c\rangle); \mathbf{Z}/p)$) which induces

$$a_4^* : u \mapsto u + y_3, \quad y_i \mapsto y_i.$$

This action is invariant on the cohomology of G^2_∞, and so is on $(*)$

$$(a_4^* - 1)\ \mathrm{tr}(1)u_p = \mathrm{Cor}(u + y_3)a_4^* u_p - \mathrm{Cor}(u)u_p = \mathrm{tr}(1)(a_4^* - 1)u_p.$$

We already know $u_p \mid \langle a_1, c\rangle = u^p - y_1^{p-1}u$. By the same argument as the proof of Proposition 4.5, we have

$$(a_4^* - 1)u_p = y_3^p - \chi y_3 \quad \text{where } \chi = \mathrm{Cor}^{G_1^1}_{\langle a_1, c\rangle}(u^{p-1}) + y_2^{p-1}.$$

On the other hand

$$(a_4^* - 1)\ \mathrm{tr}(2) = \mathrm{Cor}((u + y_3)^2) - \mathrm{Cor}(u^2) = 2\ \mathrm{tr}(1)y_3.$$

Hence we have

$$(a_4^* - 1)(*) = \mathrm{tr}(1)(y_3^p - \chi y_3) + 2a\ \mathrm{tr}(1)y_3^p.$$

Here it is known that $\chi\ \mathrm{tr}(1) = 0$([L]). Thus $a = -1/2$ and we get $(*) = \mathrm{tr}(1)u_p - 1/2\ \mathrm{tr}(2)y_3^{p-1}$. Consider the restriction $(*) \mid G_1^1 \times \mathbf{Z}/p$, and we have the remark since $\mathrm{tr}(1) = 0$ in $H^*(BG_1^1; \mathbf{Z}/p)$. q.e.d.

8. Algebraic cobordism and Chow ring

Let X be a smooth algebraic variety over $\mathbf{C}$. Recently Levine-Morel [L-M1,2] defined an algebraic cobordism $\Omega^*(X)$ having following properties.

(1) There is the natural map $\rho : \Omega^*(X) \to MU^*(X)$ such that $\Omega^* = \Omega^*(pt) \cong MU^*(pt)$ where $MU^*(-)$ is the complex cobordism theory.

(2) $\Omega^*(X) \otimes_{\Omega^*} \mathbf{Z} \cong CH^{*/2}(X)$; the classical Chow ring.

(3) $\Omega^*(X) \otimes_{\Omega^*} \tilde{K}(1)^* \cong K_0(X) \otimes \tilde{K}(1)^*$; where $K_0(X)$ is the Grothendieck group of algebraic bundle over the variety X.

Let G be an algebraic group over $\mathbf{C}$, Totaro [To1,2] defines the Chow ring $CH^*(BG)$ of the classifying space as a limit of algebraic varieties. He conjectured that

$$CH^{*/2}(BG)_{(p)} \cong BP^*(BG) \otimes_{BP^*} \mathbf{Z}_{(p)}.$$

In particular he showed above conjecture for $* \leq 4$ ([To2] Corollary 3.5).

Recall that except for elements in $F = S_4^+/(w_{ij}(1))\{fu^{ps}\}$ in Theorem 6.3, all elements in $BP^*(BG_\infty^2)$ are represented by transferred Chern classes, and hence come from the algebraic cobordism where transfers and Chern classes exist. Hence we only need to see whether fu^{ps} are in the Chow ring or not.

Theorem 8.1. *$\{fu^p\} \in BP^*(BG_\infty^2)$ comes from the algebraic cobordism.*

Corollary 8.2. *When $p = 3$, the natural maps $\rho : \Omega^*(BG_m^2) \to BP^*(BG_m^2)$ are epic for all $m \geq 1$ or $m = \infty$.*

Proof of Theorem 8.1. By Totaro (Theorem 3.1 in [To2]), $K_0(BG) \otimes \tilde{K}(1)^* \cong \tilde{K}(1)^*(BG)$. From Theorem 6.3, fu^{ps} is nonzero in $\tilde{K}(1)^*(BG_\infty^2)$. Hence from (3) there is $f_s \in \Omega^*(BG_\infty^2)$ with $\rho(f_s) = v_1^t fu^{ps}$. Now consider the case $s = 1$. Note that $\Omega^*(X)$ is generated by positive degree elements as a Ω^*-module from (2). Hence $t = 0, 1$. If $t = 1$, then $|f_s| = 4$ and this contradicts to Totaro's conjecture for $* = 4$. Thus $t = 0$ and we have the theorem. q.e.d.

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