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GLOBAL LIGHTLIKE MANIFOLDS AND HARMONICITY

Dedicated to Prof. Layos Tamassy on his 80th birthday

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Abstract

A new class of semi-Riemannian and lightlike manifolds (including globally null) is constructed by using a hypersurface of an orientable Riemannian manifold, endowed with the second fundamental form instead of a metric induced from the ambient space. We show the existence (or non-existence) of harmonic tensor fields and harmonic maps and extend to the semi-Riemannian and lightlike case a result of Chen-Nagano [4]. Then we deal with general lightlike submanifolds immersed in a semi-Riemannian manifold and propose a definition of minimal lightlike submanifolds, which generalize the one given in [7] in the Minkowski space $\mathbf{R}_1^4$. Several examples are given throughout.

0. Introduction

Since the middle of the twentieth century Riemannian geometry has created a substantial influence on several main areas of mathematical sciences. For example, see Berger's recent survey book [2], with voluminous bibliography. Primarily, semi-Riemannian (in particular global Lorentzian) geometry [10] has its roots in global Riemannian geometry, with many similarities. On the other hand, the situation is quite different for lightlike (null) manifolds, as one fails to use, in the usual way, the theory of non-degenerate geometry.

To deal with this anomaly, lightlike manifolds have been studied by several ways corresponding to their use in a given problem. In 1996, Duggal-Bejancu [7] published a book on general theory of lightlike submanifolds of semi-Riemannian manifolds and their applications to general relativity. They introduced a non-degenerate screen distribution to construct a lightlike transversal vector bundle which is non-intersecting to its lightlike tangent bundle and developed local geometry of lightlike curves, hypersurfaces and submanifolds. Having a different approach than presented in [7], this paper has two objectives.

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The first objective is to produce new examples of certain types of lightlike manifolds. For this reason, we start with a hypersurface H of an orientable Riemannian manifold $(\tilde{M}, \tilde{g})$ endowed with the second fundamental form B instead of the metric g^H induced from the ambient space and we deal with a semi-Riemannian or a lightlike (H, B). In particular, H is a globally null manifold which admits a global null vector field and a complete Riemannian hypersurface (Definition 1). We also construct a hypersurface L of a proper semi-Riemannian (H, B) on which B is lightlike. Then, we study harmonic properties of some geometric objects on semi-Riemannian and lightlike manifolds. We recall the concept of harmonic tensor field [4] and extend it in two different ways. We show that if the null distribution $\mathrm{Rad}(TH)$ of the lightlike manifold (H, B) is Killing, then B is harmonic w.r.t. the Riemannian metric g^H over $\mathrm{Rad}(TH)$. We also prove a characterization result (Theorem 2.1) between harmonic tensors and harmonic maps for a semi-Riemannian manifold (H, B). For the second objective, we study a general lightlike submanifold M^m immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \bar{g})$. We first prove (Theorem 3.1) that if $\phi : (M_1, g_1) \to (M_2, g_2)$ is an immersion between semi-Riemannian manifolds and if $\phi^* g_2$ is a semi-Riemannian (resp. lightlike) metric on M_1, then ϕ is harmonic iff $\phi^* g_2$ is a harmonic tensor w.r.t. g_1 and $\mathrm{trace}_{g_1} h = 0$, where h denotes the second fundamental form of the immersed semi-Riemannian submanifold $(M_1, \phi^* g_2)$ (resp. lightlike submanifold) in (M_2, g_2). We give an example of a globally null hypersurface, which is non-compact and another example of a 3-dimensional compact lightlike submanifold in an 8-dimensional semi-Euclidean space $(\mathbf{R}^8_4, \langle\ \rangle)$. Because of the difficulty coming from the degenerate metric, a definition of minimality was given in [7] only for a hypersurface of a Minkowski space $\mathbf{R}^4_1$. Here we introduce the general notion of *minimal lightlike submanifolds* immersed in an arbitrary semi-Riemannian manifold (Definition 2) and study their existence or non-existence.

1. A class of lightlike manifolds

Let (M, s) be a real paracompact smooth manifold endowed with a symmetric $(0,2)$-tensor field s which has a constant index on M. For any $x \in M$, let $\mathrm{Rad}\, T_xM = \{u \in T_xM / s(u, v) = 0, \forall v \in T_xM\}$ denote the radical subspace of T_xM. Then M is called *a lightlike manifold* [7] with *a lightlike metric* s if the mapping $x \in M \to \mathrm{Rad}(T_xM)$ that assigns to each $x \in M$ the radical subspace $\mathrm{Rad}(T_xM)$ of T_xM, defines a non-zero smooth distribution $\mathrm{Rad}(TM)$. If the distribution $\mathrm{Rad}(TM)$ is zero, then (M, s) is called *semi-Riemannian* [10] and is *proper* provided $\pm s$ is not Riemannian.

Definition 1 [5]. *A lightlike manifold is called globally null if it admits a globally null vector field and a complete Riemannian hypersurface.*

Let $(\tilde{M}^{m+1}, \tilde{g})$, with $m \geq 2$, be an orientable smooth Riemannian manifold with an orientable hypersurface H^m. From the orientation, there exists a unique globally defined unit normal vector field, say $\mathbf{n} \in \Gamma(T\tilde{M})$. Let $\tilde{\nabla}$ and ∇ be the Levi-Civita connections on $\tilde{M}$ and H respectively. Then, the Gauss-Weingarten formulas are

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y)\mathbf{n},$$
$$\tilde{\nabla}_X \mathbf{n} = -A_{\mathbf{n}} X,$$

for any tangent vectors X and Y of H. Here $B(\bullet, \bullet)\mathbf{n}$ is *the second fundamental form tensor* and B is *the second fundamental form*, related with *the shape operator* $A_{\mathbf{n}}$ by

$$B(X, Y) = \tilde{g}(A_{\mathbf{n}} X, Y), \quad \forall X, Y \in \Gamma(TH). \tag{1.1}$$

The eigenvalues of $A_{\mathbf{n}}$ in a point $p \in H$ are called *the principal curvatures of H in p.*

Now, we consider H endowed not with the Riemannian structure inherited from $(\tilde{M}, \tilde{g})$, but with the symmetric $(0,2)$-tensor field B which is its second fundamental form. Therefore, (H, B) is one of the following:

(1) proper semi-Riemannian;
(2) lightlike;
(3) Riemannian (for instance, H umbilical in $(\tilde{M}, \tilde{g})$);
(4) None of the above three (for example, the saddle $z = x^2 - y^2 \in \mathbf{R}^3$).

Next we deal with the first two cases, which are useful for our purpose.

Case (1). The manifold (H, B) is semi-Riemannian iff it has nowhere zero principal curvatures and the same number of negative ones (by taking into account their multiplicity) in each point of H.

Example 1. Let S^3 be endowed with the standard inner product induced from $\mathbf{R}^4$. For any $\theta \in \left(0, \frac{\pi}{2}\right)$, identify the torus $S^1 \times S^1$ with the tori

$$T_\theta = \{(\cos\theta \cos u, \cos\theta \sin u, \sin\theta \cos v, \sin\theta \sin v) \in S^3 / u, v \in [0, 2\pi]\},$$

which are Clifford tori, in particular the one corresponding to $\theta = \frac{\pi}{4}$. The second fundamental form of any such torus is proper semi-Riemannian on T_θ.

This example can be generalized as follows:

Example 2. If H is a minimal hypersurface of $(\tilde{M}, \tilde{g})$, then its second fundamental form is either proper semi-Riemannian or lightlike on (possibly some open subsets of) H.

Case (2). If the manifold (H, B) is lightlike, then we have $\mathrm{Rad}(TH) = \ker A_{\mathbf{n}}$. Moreover, H is globally null iff H has a global zero principal curvature in each point and there exists a complete hypersurface of H on which all principal curvatures are positive.

Remark 1. If instead of being positive we let the principal curvatures to have only the same sign, then in the negative case by changing the orientation of H, we choose the inward unit normal vector field $-\mathbf{n}$ instead of the outward one $\mathbf{n}$ and then the above statement remains valid since B changed to $-B$, as in the next:

Example 3. The hypercylinder

$$\mathscr{C} = \{(x_1, \ldots, x_{m+1}) \in \mathbf{R}^{m+1} / x_1^2 + \cdots + x_m^2 = 1\}$$

in $\mathbf{R}^{m+1}$, endowed with its second fundamental form derived from the inward unit normal vector field $\mathbf{n} = -(x_1\partial_1, + \cdots + x_m\partial_m)$, is globally null and $\mathrm{Rad}(T\mathscr{C}) = \mathrm{span}\{\partial_{m+1}\}$.

Proposition 1.1. *Let (H, B) be a proper semi-Riemannian hypersurface of $(\tilde{M}, \tilde{g})$ and L be a hypersurface of (H, B) on which B is lightlike.*

(i) *Then $\varrho_x = \{\xi \in T_x L / A_{\mathbf{n}}\xi \in T_x^{\perp} L\} \neq \{0\}$, $\forall x \in L$, where $T^{\perp}L$ denotes the orthogonal of TL in TH with respect to $\tilde{g}$.*

(ii) *The map $x \to \varrho_x$ defines a 1-dimensional distribution on L.*

(iii) *Let Σ be a complementary distribution of ϱ in TL, that is, $TL = \Sigma \oplus \varrho$. Then, there exists a unique vector bundle $\mathrm{tr}(TL)$ of rank 1 over L such that to every non-zero (local) null section $\xi \in \varrho$, there is a unique null section $N \in \mathrm{tr}(TL)$ for which $\tilde{g}(A_{\mathbf{n}}N, \xi) = 1$ and $A_{\mathbf{n}}N$ is orthogonal to N and Σ, with respect to $\tilde{g}$.*

Proof. From the definition of a lightlike manifold and (1.1) we have that $\varrho = \mathrm{Rad}(TL)$ which yields (i). Then (ii) holds since the manifold (H, B) is semi-Riemannian and L is a hypersurface of it, which is lightlike with respect to B. The statement (iii) follows by using (1.1) and proceeding exactly as presented in [7, Theorem 1.1, page 79], which complete the proof.

Example 4. Let $H = \{x \in \mathbf{R}^{m+1} / x_1^2 + \cdots + x_m^2 - x_{m+1}^2 = 1\}$ be a hyper quadric in the Euclidean space $\mathbf{R}^{m+1}$ and denote by B its second fundamental form derived from the unit normal vector field

$$\mathbf{n} = -(x_1\partial_1 + \cdots + x_m\partial_m - x_{m+1}\partial_{m+1})/\sigma, \quad \sigma = \sqrt{1 + 2x_{m+1}^2}.$$

It turns out that (H, B) is Lorentzian, since the principal curvatures of H are $1/\sigma$ (with the multiplicity $m-1$) and $-(1/\sigma^3)$. The timelike 1-dimensional distribution on H is spanned by the vector field

$$x_{m+1}(x_1\partial_1 + \cdots + x_m\partial_m) + (1 + x_{m+1}^2)\partial_{m+1}.$$

For any $c \in \mathbf{R}$, we restrict h to

$$L_c = \{x \in H / x_{m+1} - x_m = c(\gamma + 1)\}, \quad \gamma = \sqrt{x_1^2 + \cdots + x_{m-1}^2},$$

to obtain a family of lightlike manifolds which are globally null. For instance, when $m = 3$, then $\mathrm{Rad}(TL_c)$, Σ and $\mathrm{tr}(TL_c)$ are spanned respectively by $x_2\partial_1 - x_1\partial_2$, $\gamma W - V$ *and* $\gamma W + V$, where $V = x_1x_3\partial_1 + x_2x_3\partial_2 - \gamma\partial_3$ and $W = x_1x_4\partial_1 + x_2x_4\partial_2 + x_3x_4\partial_3 + (\sqrt{1 + x_2^2})\partial_4$.

Remark 2. Proposition 1.1 can be generalized for a lightlike submanifold of a semi-Riemannian manifold (H, B). We deal with this case in section 3.

2. Harmonic tensor fields

We recall that B. Y. Chen and T. Nagano introduced in [4, page 297] the concept of (relatively) harmonic tensor field. If M is a manifold endowed with both a Riemannian tensor field σ and a symmetric $(0, 2)$-tensor field s and ∇^σ denotes the Levi-Civita connection of σ, then s is called *a harmonic tensor w.r.t.* σ if for any $X \in \Gamma(TM)$ it satisfies:

$$\mathrm{trace}_\sigma(\nabla^\sigma_X s) = 2(\mathrm{div}\, s)(X), \tag{2.1}$$

where the divergence is defined by $(\mathrm{div}\, s)(X) = \mathrm{trace}_\sigma(\nabla^\sigma_\bullet s)(\bullet, X)$. More general, if D is a distribution on M, we say that *s is harmonic w.r.t. σ over D* if (2.1) is satisfied for any $X \in \Gamma(D)$. Another way to generalize the above concept is to take σ to be semi-Riemannian, instead of Riemannian, as we use later on.

Under the previous notations, H is a hypersurface embedded in $(\tilde{M}, \tilde{g})$. In this section we assume that the manifold H is endowed with both the Riemannian metric g^H induced from $(\tilde{M}, \tilde{g})$ and its second fundamental form B and let ∇ denote the Levi-Civita connection of g^H.

Proposition 2.1. *Let the manifold (H, B) be lightlike. Then the distribution* $\mathrm{Rad}(TH)$ *is Killing (i.e. each vector field in* $\mathrm{Rad}(TH)$ *is Killing w.r.t. B) iff we have*:

$$(\nabla_X B)(Y, Z) = (\nabla_Y B)(X, Z) + (\nabla_Z B)(X, Y), \tag{2.2}$$
$$\forall X \in \Gamma(\mathrm{Rad}(TH)),\ Y, Z \in \Gamma(TH).$$

Moreover, in that case B is harmonic w.r.t. g^H over $\mathrm{Rad}(TH)$.

Proof. We have:

$$\begin{aligned}(\mathscr{L}_X B)(Y, Z) &= XB(Y, Z) - B(\nabla_X Y, Z) + B(\nabla_Y X, Z)\\ &\quad - B(Y, \nabla_X Z) + B(Y, \nabla_Z X)\\ &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) - (\nabla_Z B)(X, Y),\end{aligned}$$

since $B(X, U) = 0$, $\forall X \in \Gamma(\mathrm{Rad}(TH))$, $U \in \Gamma(TH)$. Thus, (2.2) is satisfied iff any $X \in \Gamma(\mathrm{Rad}\ TH)$ is Killing. The last statement follows from (2.1) and the above equivalence, which complete the proof.

Remark 3. In 1959, Reinhart [12] introduced a class of Riemannian foliated manifolds with bundle-like metrics which are now called as Reinhart manifolds. A lightlike manifold is a Reinhart lightlike manifold if and only if its radical distribution is Killing [7, page 49].

Example 5. The hypercylinder in Example 1.3 is a Reinhart lightlike manifold, since its tangent space is spanned by ∂_{m+1} and all vector fields tangent to the unit sphere S^{m-1}, which shows that the radical distribution is Killing.

Example 6. If (H, B) is lightlike and the second fundamental form B is parallel (i.e. $\nabla B = 0$), then from Proposition 2.1 it follows that (H, B) is a Reinhart lightlike manifold.

The concept of harmonic maps constitutes a very useful tool for both Global Analysis and Differential Geometry (see the harmonic maps and harmonic morphisms bibliographies [3] and [9], respectively). Among them only few ones deal with semi-Riemannian case and even less with the lightlike case (i.e. Duggal [6], Pambira [11]). We provide some work in Section 3. A map $\phi : (M_1, g_1) \to (M_2, g_2)$ between semi-Riemannian manifolds is *harmonic* (resp. *totally geodesic*) if its tension field $\tau(\phi) = \operatorname{div} d\phi = \operatorname{trace}_{g_1} \nabla\, d\phi$ (resp. $\nabla\, d\phi$) is identically zero. If (x^i) are local coordinates on M_1, then

$$\tau(\phi) = g_1^{ij}\left[\nabla^{M_2}_{\partial/\partial x^i}\, d\phi\left(\frac{\partial}{\partial x^j}\right) - d\phi\left(\nabla^{M_1}_{\partial/\partial x^i}\frac{\partial}{\partial x^j}\right)\right],$$

where ∇^{M_1} and ∇^{M_2} denote the Levi-Civita connections of g_1 and g_2 respectively. *The stress-energy tensor field of* ϕ is given by $S(\phi) = e(\phi) g_1 - \phi^* g_2$, where $e(\phi) : M_1 \to \mathbf{R}_+$ is *the energy density of* ϕ, defined by [1]

$$e(\phi) = \frac{1}{2}\operatorname{trace}_{g_1} \phi^* g_2.$$

A harmonic map is called minimal if it is an isometric immersion. For instance, in the context of this paper, the inclusion map $i : (H, g^H) \to (\tilde{M}, \tilde{g})$ is minimal (resp. totally geodesic) iff $\operatorname{trace}_{g^H} B = 0$ (resp. $B = 0$).

Proposition 2.2. *If the manifold* (H^m, B) *is semi-Riemannian, then the inclusion map* $I : (H, B) \to (\tilde{M}, \tilde{g})$ *is never harmonic.*

Proof. Let $\tilde{\nabla}$, ∇ and D denote the Levi-Civita connections of $\tilde{g}$, g^H and B, respectively. If we suppose that I is harmonic, then for any local coordinates (x^i) on H, we have

$$B^{ij}\left(\tilde{\nabla}_{\partial/\partial x^i}\frac{\partial}{\partial x^j} - D_{\partial/\partial x^i}\frac{\partial}{\partial x^j}\right) = 0,$$

which is equivalent to

$$B^{ij}\left(\tilde{\nabla}_{\partial/\partial x^i}\frac{\partial}{\partial x^j}-\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^j}\right)=B^{ij}\left(\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^j}-D_{\partial/\partial x^i}\frac{\partial}{\partial x^j}\right)=0.$$

Thus, $B^{ij}B_{ij}=m=0$, which is a contradiction so the proof is complete.

Remark 4. More general, to study the harmonicity of an immersion $\phi:(M_1,g_1)\to(M_2,g_2)$ between semi-Riemannian manifolds in terms of the second fundamental form of the immersed submanifold (M_1,ϕ^*g_2) in (M_2,g_2), one should take into account that ϕ^*g_2 can be degenerate on M_1. We deal with this general case in Section 3.

Theorem 2.1. *If the manifold* (H,B) *is semi-Riemannian, then the following assertions are equivalent*:

(1) *the identity map* $1_H:(H,B)\to(H,g^H)$ *is harmonic*;
(2) g^H *is harmonic w.r.t.* B;
(3) $\mathrm{trace}_B(\mathscr{L}_ZB)=\mathrm{trace}_B(\mathscr{L}_{A_\mathbf{n}Z}g^H)$, $\forall Z\in\Gamma(TH)$;
(4) *the stress-energy tensor* $S(1_H)$ *is divergence free*.

Proof. The equivalence (1) ⇔ (2) holds for the metric B, which is semi-Riemannian, in a similar way as in the Riemannian context [4, page 296], where the identity map is harmonic iff the metric on the target is a harmonic tensor w.r.t. the metric on the domain. The same for the equivalence (1) ⇔ (4), which holds as in the Riemannian case [1, Proposition 3.4.7], where a diffeomorphism between Riemannian manifolds is harmonic iff it is divergence free. To prove the equivalence (1) ⇔ (3) we use the definition of the Levi-Civita connections ∇ and D of g^H and B respectively, which yield:

$$\begin{aligned}2B(\nabla_XY-D_XY,Z)&=2[g^H(\nabla_XY,A_\mathbf{n}Z)-B(D_XY,Z)]\\&=(\mathscr{L}_ZB)(X,Y)-(\mathscr{L}_{A_\mathbf{n}Z}g^H)(X,Y),\quad\forall X,Y,Z\in\Gamma(TH).\end{aligned}$$

To above formula we apply the trace operator w.r.t. B and since B is semi-Riemannian, we use that 1_H is harmonic iff $B(\tau(1_H),Z)=0$, $\forall Z\in\Gamma(TH)$.

Corollary 2.1. *If the manifold* (H^m,B) *is semi-Riemannian, then any two of the following assertions imply the third*:

(i) *the identity map* $1_H:(H,B)\to(H,g^H)$ *is harmonic*;
(ii) *the energy density* $e(1_H)$ *is constant*;
(iii) $\mathrm{div}\,g^H=0$, *where* div *denotes the divergence operator w.r.t.* B.

Proof. From Theorem 2.1 it follows that 1_H is harmonic iff the stress-energy tensor field $S(1_H)=e(1_H)B-g^H$ is divergence free (on the domain manifold (H,B)). If D denotes the Levi-Civita connection of B, then:

$$\begin{aligned}[\operatorname{div} S(1_H)](X) &= \operatorname{trace}_B(D_\bullet S)(\bullet, X) \\ &= \operatorname{trace}_B(\bullet e(1_H))B(\bullet, X) - \operatorname{trace}_B(D_\bullet g^H)(\bullet, X) \\ &= \sum_{i=1}^{m} \varepsilon_i(u_i e(1_H))B(u_i, X) - (\operatorname{div} g^H)(X), \quad \forall X \in \Gamma(TH),\end{aligned}$$

where $\{u_i\}$ is an orthonormal frame on (H^m, B) for $1 \le i \le m$ and $\varepsilon_i = B(u_i, u_i) = \pm 1$. Replacing X consecutively by each u_i we complete the proof.

Remark 5. Under the condition that (H, B) is semi-Riemannian, Proposition 2.2 and Corollary 2.1 hold if g^H and B are interchanged.

3. Minimal lightlike submanifolds

A submanifold M^m immersed in a semi-Riemannian manifold $(\overline{M}^{m+n}, \bar{g})$ is called *a lightlike submanifold* if it is a lightlike manifold w.r.t. the metric g induced from $\bar{g}$ and the radical distribution $\operatorname{Rad}(TM)$ is of rank r, where $1 \le r \le m$. We note that $\operatorname{Rad}(TM) = TM \cap TM^\perp$, where

$$TM^\perp = \bigcup_{x \in M} \{u \in T_xM / \bar{g}(u, v) = 0, \forall v \in T_xM\}.$$

By following Duggal-Bejancu [7], let $S(TM)$ be *a screen distribution* which is a semi-Riemannian complementary distribution of $\operatorname{Rad}(TM)$ in TM, i.e.

$$TM = S(TM) \perp \operatorname{Rad}(TM)$$

and let $[S(TM)]^\perp$ be its complementary orthogonal vector bundle in $T\overline{M}|_M$. We consider *a screen transversal vector bundle* $s(TM^\perp)$, which is a semi-Riemannian complementary vector bundle of $\operatorname{Rad}(TM)$ in $TM^\perp$, i.e.

$$TM^\perp = \operatorname{Rad}(TM) \perp s(TM^\perp).$$

Since for any local basis $\{\xi_i\}$ of $\operatorname{Rad}(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal complement of $s(TM^\perp)$ in $[S(TM)]^\perp$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$, it follows that there exists a *lightlike transversal vector bundle* $\operatorname{ltr}(TM)$ locally spanned by $\{N_i\}$ [7, Theorem 1.3, page 144]. Then, the following decomposition holds:

$$T\overline{M}|_M = S(TM) \perp [\operatorname{Rad}(TM) \oplus \operatorname{ltr}(TM)] \perp s(TM^\perp) \tag{3.1}$$

and the above direct sum $\operatorname{Rad}(TM) \oplus \operatorname{ltr}(TM)$ is semi-Riemannian. If the transversal vector bundle is denoted by

$$\operatorname{tr}(TM) = \operatorname{ltr}(TM) \perp s(TM^\perp), \tag{3.2}$$

then we have

$$T\overline{M}|_M = TM \oplus \operatorname{tr}(TM).$$

The following four cases occur:

Case 1: r-lightlike submanifold. $r < \min\{m, n\}$;

Case 2: Co-isotropic submanifold. $r = n < m$;

Case 3: Isotropic submanifold. $r = m < n$;

Case 4: Totally lightlike submanifold. $r = m = n$.

Under the above notations for $(M, g, S(TM), s(TM^{\perp}))$, the Levi-Civita connection $\overline{\nabla}$ of $(\overline{M}, \bar{g})$ satisfies the Gauss-Weingarten type formulas:

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + h(X, Y) \\ \overline{\nabla}_X V &= -A_V X + \nabla^t_X V, \quad \forall X, Y \in \Gamma(TM),\ V \in \Gamma(\mathrm{tr}(TM)), \end{aligned} \tag{3.3}$$

where $\{\nabla_X Y, A_V X\}$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$, respectively. We note that the induced linear connection ∇ is torsion free and the transversal ∇^t is a linear connection. The second fundamental form h is a symmetric $\mathscr{F}(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(\mathrm{tr}(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$. From (3.2) we use the following decomposition:

$$\begin{aligned} h(X, Y) &= h^{\ell}(X, Y) + h^s(X, Y), \\ \nabla^t_X V &= D^{\ell}_X V + D^s_X V, \quad \forall X, Y \in \Gamma(TM),\ V \in \Gamma(\mathrm{tr}(TM)), \end{aligned} \tag{3.4}$$

where $\{h^{\ell}(X, Y), D^{\ell}_X V\}$ and $\{h^s(X, Y), D^s_X V\}$ belong to $\Gamma(\mathrm{ltr}(TM))$ and $\Gamma(s(TM^{\perp}))$ respectively.

THEOREM 3.1. *Let $\phi : (M_1, g_1) \to (M_2, g_2)$ be an immersion between semi-Riemannian manifolds. If $\phi^* g_2$ is a semi-Riemannian* (*resp. lightlike*) *metric on M_1, then ϕ is harmonic iff $\phi^* g_2$ is a harmonic tensor w.r.t. g_1 and* $\mathrm{trace}_{g_1} h = 0$, *where h denotes the second fundamental form of the immersed semi-Riemannian submanifold $(M_1, \phi^* g_2)$* (*resp. lightlike submanifold $(M_1, \phi^* g_2, S(TM_1), s(TM_1^{\perp}))$*) *in (M_2, g_2).*

Proof. Let ∇^{M_1} and ∇^{M_2} denote the Levi-Civita connections of g_1 and g_2, respectively. Let ∇ denote the Levi-Civita (resp. linear) connection of $\phi^* g_2$ according as the manifold $(M_1, \phi^* g_2)$ is semi-Riemannian (resp. lightlike). For any local coordinates (x^i) on M_1, we have:

$$\begin{aligned} \tau(\phi) &= g_1^{ij}\left[\nabla^{M_2}_{\partial/\partial x^i}\, d\phi\left(\frac{\partial}{\partial x^j}\right) - d\phi\left(\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^j}\right) + d\phi\left(\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^j} - \nabla^{M_1}_{\partial/\partial x^i}\frac{\partial}{\partial x^j}\right)\right] \\ &= g_1^{ij}\left[d\phi\left(\nabla_{\partial/\partial x^i}\frac{\partial}{\partial x^j} - \nabla^{M_1}_{\partial/\partial x^i}\frac{\partial}{\partial x^j}\right) + h\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right]. \end{aligned}$$

This gives a decomposition of $\tau(\phi) \in \phi^{-1}(TM_2)$, by taking into account first the identification $\phi^{-1}(TM_2) = TM_2|_{M_1}$ and then the splitting of $TM_2|_{M_1}$ into the orthogonal $TM_1 \perp TM_1^{\perp}$ or direct sum $TM_1 \oplus \mathrm{tr}(TM_1)$, according as $(M_1, \phi^* g_2)$ is semi-Riemannian or lightlike, which complete the proof.

Remark 6. (i) If $(M_1, \phi^* g_2, S(TM_1), s(TM_1^\perp))$ is lightlike, then the above theorem is independent on the choice of the screen distribution $S(TM_1)$, but it depends on the choice of the transversal bundle $\mathrm{tr}(TM_1)$; (ii) In the Riemannian case, the above theorem is proved in [4].

PROPOSITION 3.1. *Let* $(M, g, S(TM), s(TM^\perp))$ *be a lightlike submanifold of* $(\overline{M}, \bar{g})$. *Then*:
(i) $h^\ell = 0$ *on* $\mathrm{Rad}(TM)$;
(ii) $h^s = 0$ (*in Cases* 2, 4) *and* $h^s = 0$ *on* $\mathrm{Rad}(TM)$ *iff* $\mathscr{L}_W \bar{g} = 0$ *on* $\mathrm{Rad}(TM)$, $\forall W \in \Gamma(s(TM^\perp))$ (*in Cases* 1, 3).

Proof. From the definition of $\overline{\nabla}$ we have:

$$\begin{aligned}(3.5)\qquad \bar{g}(\overline{\nabla}_{\xi'}\xi'', K) = \xi'\bar{g}(\xi'', K) + \xi''\bar{g}(\xi', K) - K\bar{g}(\xi', \xi'') + \bar{g}([\xi', \xi''], K)\\ + \bar{g}([K, \xi'], \xi'') - \bar{g}([\xi'', K], \xi'),\\ \forall \xi', \xi'' \in \Gamma(\mathrm{Rad}(TM)),\ K \in \Gamma(T\overline{M}|_M).\end{aligned}$$

(i) Suppose h^ℓ is not identically zero on $\mathrm{Rad}(TM)$ and let $\xi^{(1)}, \xi^{(2)} \in \Gamma(\mathrm{Rad}(TM))$ such that $h^\ell(\xi^{(1)}, \xi^{(2)}) \neq 0$. As the direct sum $\mathrm{Rad}(TM) \oplus \mathrm{ltr}(TM)$ is semi-Riemannian and $h^\ell(\xi^{(1)}, \xi^{(2)})$ is a non-zero section of the lightlike vector bundle $\mathrm{ltr}(TM)$, there exists $\xi \in \Gamma(\mathrm{Rad}(TM))$ such that $\bar{g}(h^\ell(\xi^{(1)}, \xi^{(2)}), \xi) = 1$. If in (3.5) we substitute $K = \xi$, $\xi' = \xi^{(1)}$, $\xi'' = \xi^{(2)}$, then from (3.3) and (3.4) we obtain:

$$\bar{g}(h^\ell(\xi^{(1)}, \xi^{(2)}), \xi) = \bar{g}(\overline{\nabla}_{\xi^{(1)}}\xi^{(2)}, \xi) = 0,$$

which is a contradiction that yields the statement. (ii) From (3.5) we obtain:

$$\bar{g}(\overline{\nabla}_{\xi'}\xi'', W) = -(\mathscr{L}_W \bar{g})(\xi', \xi''), \quad \forall \xi', \xi'' \in \Gamma(\mathrm{Rad}(TM)),\ W \in \Gamma(s(TM^\perp)).$$

Using (3.3), (3.4) and $s(TM^\perp)$ semi-Riemannian, we complete the proof.

Example 7. Let $(\mathbf{R}_1^4, \langle\ \rangle)$ be the Minkowski space with signature $(+,+,+,-)$ w.r.t. the canonical basis $(\partial_1, \ldots, \partial_4)$. Then the manifold $(M, \langle\ \rangle_{|_M}, S(TM))$ is a lightlike hypersurface, given by an open subset of the lightlike cone

$$M = \{t(\cos u \cos v, \cos u \sin v, \sin u, 1) \in \mathbf{R}_1^4 / t > 0, u \in (o, \pi/2), v \in [0, 2\pi]\},$$

where $S(TM) = \mathrm{span}\{e_1 = -\sin u \cos v\partial_1 - \sin u \sin v\partial_2 + \cos u\partial_3,$

$$e_2 = -\sin v\partial_1 + \cos v\partial_2\}.$$

We note that e_1 and e_2 are orthonormal,

$$\mathrm{Rad}(TM) = \mathrm{span}\{\xi = \cos u \cos v\partial_1 + \cos u \sin v\partial_2 + \sin u\partial_3 + \partial_4\},$$

$$\mathrm{ltr}(TM) = \mathrm{span}\left\{N = \frac{1}{2}(\cos u \cos v\partial_1 + \cos u \sin v\partial_2 + \sin u\partial_3 - \partial_4)\right\},$$

and $\langle \xi, N \rangle = 1$. We have $h(e_1, e_1) = -\left(\frac{1}{t \cos u}\right)N$, $h(e_2, e_2) = -\left(\frac{1}{t \cos u}\right)N$ and $h(e_1, e_2) = 0$. It turns out that the open subset of the lightlike cone $(M, \langle\ \rangle_{|_M})$ is globally null, since ξ is globally defined and $S(TM)$ is a spacelike integrable distribution.

Different from above non-compact hypersurface, in the next example we construct a lightlike submanifold of codimension > 1, which is compact.

Example 8. Let $(\mathbf{R}^8_4, \langle\ \rangle)$ be the semi-Euclidean space with the signature $(-,-,+,+,-,-,+,+)$ w.r.t. the canonical basis $\{\partial_1, \ldots, \partial_8\}$. Then $(M, \langle\ \rangle_{|_M}, S(TM), s(TM^{\perp}))$ is a compact lightlike submanifold, given by

$$\begin{aligned} M = T^2 \times S^1 = \{&(\cos u \cos v, \cos u \sin v, \sin u \cos w, \sin u \sin w, \\ &\sin u \cos v, \sin u \sin v, \cos u \cos w, \cos u \sin w)/u, v, w \in [0, 2\pi]\}, \end{aligned}$$

where
$$\begin{aligned} S(TM) = \mathrm{span}\{&e_1 = -\cos u \sin v\partial_1 + \cos u \cos v\partial_2 \\ &- \sin u \sin v\partial_5 + \sin u \cos v\partial_6, \\ &e_2 = -\sin u \sin w\partial_3 + \sin u \cos w\partial_4 - \cos u \sin w\partial_7 + \cos u \cos w\partial_8\}. \end{aligned}$$

Here e_1 is timelike, e_2 is spacelike and

$$\begin{aligned} \mathrm{Rad}(TM) &= \mathrm{span}\{\xi = -\sin u \cos v\partial_1 - \sin u \sin v\partial_2 + \cos u \cos w\partial_3 \\ &\qquad + \cos u \sin w\partial_4 + \cos u \cos v\partial_5 + \cos u \sin v\partial_6 \\ &\qquad - \sin u \cos w\partial_7 - \sin u \sin w\partial_8\}, \\ \mathrm{ltr}(TM) &= \mathrm{span}\{N = \cos u \cos v\partial_1 + \cos u \sin v\partial_2 + \cos u \cos w\partial_3 \\ &\qquad + \cos u \sin w\partial_4 + \sin u \cos v\partial_5 + \sin u \sin v\partial_6 \\ &\qquad - \sin u \cos w\partial_7 - \sin u \sin w\partial_8\}, \\ s(TM^{\perp}) &= [\mathrm{span}\{e_1, e_2, e_3, e_4\}]^{\perp}, \end{aligned}$$

where

$$\begin{aligned} e_3 &= \frac{1}{2}[(\cos u + \sin u)(\cos v\partial_1 + \sin v\partial_2) + (\sin u - \cos u)(\cos v\partial_5 + \sin v\partial_6)], \\ e_4 &= \frac{1}{2}[(\cos u - \sin u)(\cos v\partial_1 + \sin v\partial_2) + 2\cos u(\cos w\partial_3 + \sin w\partial_4) \\ &\qquad + (\cos u + \sin u)(\cos v\partial_5 + \sin v\partial_6) - 2\sin u(\cos w\partial_7 + \sin w\partial_8)] \end{aligned}$$

are timelike and spacelike, respectively and e_1, e_2, e_3, e_4 are mutually orthogonal. We have

$$\begin{aligned} h(e_1, e_1) &= (-\cos u + \sin u)(\cos v\partial_1 + \sin v\partial_2) - \cos u(\cos w\partial_3 + \sin w\partial_4) \\ &\quad - (\sin u + \cos u)(\cos v\partial_5 + \sin v\partial_6) \\ &\quad + \sin u(\cos w\partial_7 + \sin w\partial_8) \in s(TM^\perp), \\ h(e_2, e_2) &= -(\sin u \cos w\partial_3 + \sin u \sin w\partial_4 \\ &\qquad + \cos u \cos w\partial_7 + \cos u \sin w\partial_8) \in s(TM^\perp), \quad h(e_1, e_2) = 0. \end{aligned}$$

From [7, page 166], a lightlike submanifold M of $\overline{M}$ is called *totally geodesic* if any geodesic of M w.r.t. an induced linear connection ∇ is a geodesic of $\overline{M}$ w.r.t. the Levi-Civita connection $\overline{\nabla}$. For example, any lightlike curve of a semi-Riemannian manifold and any lightlike hyperplane of a semi-Euclidean space are totally geodesic lightlike submanifolds.

Remark 7. A more general notion, precisely the one of minimal lightlike submanifold M of a semi-Riemannian manifold $\overline{M}$ was not introduced yet, as far as we know. In the semi-Riemannian context, a minimal isometric immersion is a particular harmonic map. In [11], a harmonic map ϕ between lightlike manifolds is defined with the assumption that ϕ is radical preserving (i.e. ϕ maps the radical of the domain into the radical of the target). This does not apply here to define minimality, since an isometric immersion from M to $\overline{M}$ is not radical preserving. In [6], harmonic maps from a semi-Riemannian manifold into a lightlike manifold are defined only when the target is a Riemannian hypersurface of a globally null manifold. This also does not apply here to define minimality, since our domain M is lightlike. In [7, page 131], a minimal lightlike submanifold is defined only in the particular case when M is a hypersurface of the Minkowski space $\overline{M} = \mathbf{R}^4_1$. We introduce here the notion of minimal lightlike submanifolds in a general context.

From now on we work in Case 1 or 2 so that $S(TM)$ is non-zero. In view of Proposition 3.1, we introduce the following:

Definition 2. *We say that a lightlike submanifold* $(M, g, S(TM), s(TM^\perp))$ *isometrically immersed in a semi-Riemannian manifold* $(\overline{M}, \bar{g})$ *is minimal if*:
(i) $h^s = 0$ *on* $\mathrm{Rad}(TM)$ *and*
(ii) $\mathrm{trace}\, h = 0$, *where trace is written w.r.t.* g *restricted to* $S(TM)$.

We note that in Case 2, the condition (i) is trivial. From Proposition 3.1, this definition is independent of $S(TM)$ and $s(TM^\perp)$, but it depends on the choice of the transversal bundle $\mathrm{tr}(TM)$.

As in the semi-Riemannian case [1, page 435], any lightlike totally geodesic submanifold is minimal. Next, we construct a proper lightlike minimal submanifold which is not totally geodesic.

Example 9. Let $(\mathbf{R}^4_1, \langle\ \rangle)$ be the Minkowski space with the signature $(+, +, +, -)$ w.r.t. the canonical basis $(\partial_1, \ldots, \partial_4)$ and let

$$S_1^3 = \{p \in \mathbf{R}_1^4 / \langle p, p \rangle = 1\}$$

be the 3-dimensional unit pseudosphere of index 1, which is a Lorentzian hypersurface of $(\mathbf{R}_1^4, \langle\ \rangle)$. We denote by $(\overline{M} = S_1^3 \times \mathbf{R}_1^2, g)$ the semi-Riemannian cross product, where $\mathbf{R}_1^2$ is semi-Euclidean space with the signature $(+,-)$ w.r.t. the canonical basis $\{\partial_5, \partial_6\}$ and g is the inner product of $\mathbf{R}_2^6 = \mathbf{R}_1^4 \times \mathbf{R}_1^2$ restricted to $\overline{M}$. Then the submanifold $(M, g_{|_M}, S(TM), s(TM^\perp))$ is a minimal lightlike submanifold of $\overline{M}$ given by

$$M = S^1 \times \mathscr{H} \times \mathbf{R} = \{(p, t, t) \in S_1^3 \times \mathbf{R}_1^2 / t \in \mathbf{R},$$
$$p = \frac{\sqrt{2}}{2}(\cos\theta, \sin\theta, \cosh\varphi, \sinh\varphi) \in S_1^3, \theta \in [0, 2\pi], \varphi \in \mathbf{R}\},$$

where $\mathscr{H}$ is the hyperbola and

$$S(TM) = \operatorname{span}\{e_1 = -\sin\theta\partial_1 + \cos\theta\partial_2, e_2 = \sinh\varphi\partial_3 + \cosh\varphi\partial_4\}.$$

Here $\varepsilon_1 = g(e_1, e_1) = 1$, $\varepsilon_2 = g(e_2, e_2) = -1$ and

$$\operatorname{Rad}(TM) = \operatorname{span}\{\xi = \partial_5 + \partial_6\},$$
$$\operatorname{ltr}(TM) = \operatorname{span}\left\{N = \frac{1}{2}(\partial_5 - \partial_6)\right\},$$
$$s(TM^\perp) = [\operatorname{span}\{e_1, e_2, \partial_5, \partial_6\}]^\perp$$
$$= \operatorname{span}\left\{W = \frac{\sqrt{2}}{2}(\cos\theta\partial_1 + \sin\theta\partial_2 - \cosh\varphi\partial_3 - \sinh\varphi\partial_4)\right\},$$

where $\{e_1, e_2, \partial_5, \partial_6, W\}$ is an orthonormal basis of $\overline{M}$. Let $\bar{p} = \frac{\sqrt{2}}{2}(\cos\theta\partial_1 + \sin\theta\partial_2 + \cosh\varphi\partial_3 + \sinh\varphi\partial_4)$ be the position vector of an arbitrary point p of S_1^3, which is normal to S_1^3 in $\mathbf{R}_1^4$. Since the canonical Levi-Civita connection ∇^c of $\mathbf{R}_1^4$ satisfies $\nabla^c_{e_1} e_1 = -\frac{1}{2}(W + \bar{p})$ and $\nabla^c_{e_2} e_2 = \frac{1}{2}(-W + \bar{p})$, it follows that $h(e_1, e_1) = -\frac{1}{2}W$, $h(e_1, e_2) = 0$, $h(e_2, e_2) = -\frac{1}{2}W$, from which

$$\operatorname{trace}_{g|_{S(TM)}} h = \varepsilon_1 h(e_1, e_1) + \varepsilon_2 h(e_2, e_2) = h(e_1, e_1) - h(e_2, e_2) = 0.$$

We also have $h(\xi, \xi) = 0$ and, therefore, M is a minimal lightlike submanifold of $\overline{M}$, which is not totally geodesic.

Note that the Examples 7 and 8 are not minimal submanifolds. The classical notion of minimality is connected to the geometric interpretation of being an extremal of the volume functional [8, page 391]. Here we relate the classical minimality (in semi-Riemannian case) with the minimality introduced in the lightlike case by Definition 2, as follows:

Theorem 3.2. *Let $(M, g, S(TM), s(TM^\perp))$ be a lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \bar{g})$, with $S(TM)$ integrable. If its leaves are minimal (semi-Riemannian) submanifolds of $(\overline{M}, \bar{g})$ and $h^s = 0$ on* $\operatorname{Rad}(TM)$, *then*

M is a lightlike minimal submanifold of $\overline{M}$. Conversely, if M is a lightlike minimal submanifold of $\overline{M}$, then $\mathrm{Rad}(TM)$ *contains the mean curvature vector field of any leaf of $S(TM)$.*

Proof. Let $i : \Sigma \to \overline{M}$ denote the inclusion map of any leaf Σ of $S(TM)$. The tension field $\tau(i)$ of Σ can be decomposed from (3.1) into:

$$\tau(i) = \tau^\star(i) + \mathrm{trace}_{g_{|S(TM)}}\, h,$$

where $\tau^\star(i) \in \mathrm{Rad}(TM)$ and h is defined by (3.3). Since Σ is minimal in $\overline{M}$ iff the map i is harmonic, which means $\tau(i) = 0$, the statement follows from Definition 2.

We observe that Example 9 satisfies Theorem 3.2 with respect to the leaves $S^1 \times \mathscr{H}$ of $S(TM)$. Let $\Lambda = \{x \in \mathbf{R}_q^{n+1} / \langle x, x \rangle = 0\}$ be the lightlike cone in the semi-Euclidean space $(\mathbf{R}_q^{n+1}, \langle\ \rangle)$. Related to the non-existence result of compact minimal spacelike submanifolds isometrically immersed in semi-Euclidean spaces, we have the following:

Proposition 3.2. *There are no lightlike minimal isometric immersions $\phi : (M, \langle\ \rangle_{|M}, S(TM), s(TM^\perp)) \to (\mathbf{R}_q^{n+1}, \langle\ \rangle)$ with $\phi(M) \subset \Lambda$.*

Proof. Suppose there exists such a map ϕ. Then the function given by $p \in M \to \frac{1}{2}\langle \phi(p), \phi(p) \rangle \in \mathbf{R}$ is identically zero and hence,

$$0 = \frac{1}{2} X \langle \phi(p), \phi(p) \rangle = \langle X, \phi(p) \rangle \quad \text{and}$$

$$0 = \frac{1}{2} X(X \langle \phi(p), \phi(p) \rangle) = \langle \nabla^c_X X, \phi(p) \rangle + \langle X, X \rangle, \quad \forall p \in M,\ X \in \Gamma(TM),$$

where ∇^c is the canonical Levi-Civita connection of $(\mathbf{R}_q^{n+1}, \langle\ \rangle)$. If we replace X consecutively by e_a, where $\{e_a\}$ is the orthonormal basis of $S(TM)$, then from the minimality condition we have:

$$0 = \langle \mathrm{trace}_{\langle\ \rangle_{|S(TM)}}\, h, \phi(p) \rangle = -\left(\sum_a \varepsilon_a \langle e_a, e_a \rangle \right) < 0,$$

where h is given by (3.3) and $\varepsilon_a = \langle e_a, e_a \rangle$, $\forall a$. This contradiction completes the proof.

In support of Proposition 3.2, Example 7 of the lightlike submanifold M, which is an open subset of the lightlike cone Λ of $(\mathbf{R}_1^4, \langle\ \rangle)$ is never minimal.

Since Λ is a proper totally umbilical lightlike submanifold of $(\mathbf{R}_1^4, \langle\ \rangle)$, the Proposition 3.2 can be generalized, by using the definition of totally umbilical lightlike submanifolds [7, page 106], as follows:

Theorem 3.3. *There are no lightlike minimal submanifolds contained in a proper totally umbilical lightlike submanifold of a semi-Riemannian manifold.*

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