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ON QUADRATIC GENERATION OF IDEALS DEFINING PROJECTIVE TORIC VARIETIES

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Abstract

For any ample line bundle L on a projective toric variety of dimension n, it is known that the line bundle $L^{\otimes i}$ is normally generated if i is greater than or equal to $n-1$. We prove that $L^{\otimes i}$ is also normally presented if i is greater than or equal to $n-1$. Furthermore we show that $L^{\otimes i}$ is normally presented for $i \geq [n/2]+1$ if L is normally generated.

Introduction

Mumford showed in [M] that for ample invertible sheaf L generated by its global sections on a projective algebraic variety X, the k times twisted sheaf $L^{\otimes k}$ defines an embedding of X as an intersection of quadrics for sufficiently large k. In order to describe the precise statement we need recall the definition of normal generation and normal presentation following Mumford.

Definition. Let L be an ample invertible sheaf on a projective variety X. Then L is said to be *normally generated* if the map

$$H^0(X,L)^{\otimes k} \to H^0(X,L^{\otimes k})$$

is surjective for all $k \geq 1$.

A normally generated invertible sheaf L is said to be *normally presented* if the map

$$I_2(L) \otimes H^0(X,L^{\otimes(k-2)}) \to I_k(L)$$

is surjective for all $k \geq 2$, where $I_k(L)$ denotes the kernel of the multiplication map $\mathrm{Sym}^k H^0(X,L) \to H^0(X,L^{\otimes k})$. In other words, the defining ideal $I = \bigoplus_{k\geq 0} I_k(L)$ of the image of X mapped by $H^0(X,L)$ in $\boldsymbol{P}(H^0(X,L)^*)$ is generated by quadrics.

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By using vanishing of cohomology groups Mumford proved that for an ample invertible sheaf L generated by global sections $L^{\otimes k}$ is normally generated and presented for sufficiently large k. And he proved that for a nonsingular complete curve X of genus g, an invertible sheaf L with $\deg L \geq 2g+1$ is normally generated and that L with $\deg L \geq 3g+1$ is normally presented. He also proved that for an ample invertible sheaf L on an abelian variety X, the tensor power $L^{\otimes k}$ is normally generated and presented for $k \geq 4$. Fujita improved in [Fj] the case of curves so that L is normally presented if $\deg L \geq 2g+2$.

In this paper we consider only the case that X is a toric variety. When X is a toric variety of dimension two, Koelman proved in [K1], [K2], [K3] that any ample invertible sheaf L is normally generated and decided when L is normally presented. When X is toric and $\dim X = n \geq 3$, we proved in [NO] that $L^{\otimes i}$ is normally generated for $i \geq n-1$ and is normally presented for $i \geq n$. More precicely we proved that the multiplication map

$$H^0(X, L^{\otimes i}) \otimes H^0(X, L) \to H^0(X, L^{\otimes (i+1)})$$

is surjective for $i \geq n-1$. By employing an analogous argument of [M] we showed that $L^{\otimes i}$ is normally presented for $i \geq n$. Moreover when X is embedded by $\Gamma(L^{\otimes (n-1)})$, the ideal defining the image of X has generators of degree at most three.

In this paper we prove the followings.

Theorem 1. *Let L be an ample invertible sheaf on a projective toric variety X of dimension $n \geq 3$. Then $L^{\otimes (n-1)}$ is normally presented. In other words, the ideal defining X embedded by the global sections $H^0(X, L^{\otimes (n-1)})$ is generated by quadrics.*

We shall give a proof of Theorem 1 in Section 2.

Theorem 2. *Let $1 \leq t \leq n-1$ be an integer so that $\Gamma(L^i) \otimes \Gamma(L) \to \Gamma(L^{\otimes (i+1)})$ is surjective for all $i \geq t$. Then $L^{\otimes r}$ is normally presented if $r \geq \max\{t, [n/2]+1\}$.*

Theorem 2 will be proved in Section 3.

1. Preliminaries

Let M be a free $\boldsymbol{Z}$-module of rank n ($n \geq 3$) and let $M_{\boldsymbol{R}} := M \otimes_{\boldsymbol{Z}} \boldsymbol{R}$ the extension of the coefficients to the real numbers. We call the convex hull $\mathrm{Conv}\{u_0, u_1, \ldots, u_r\}$ in $M_{\boldsymbol{R}}$ of a finite subset $\{u_0, u_1, \ldots, u_r\} \subset M$ an integral convex polytope in $M_{\boldsymbol{R}}$. By the theory of toric varieties (see, for instance, Section 3.5 [Fl], or Section 2.2 [O]) an integral convex polytope P in $M_{\boldsymbol{R}}$ corresponds to a pair (X, L) consisting of a projective toric variety X and an ample invertible sheaf L on X. Let $T := \mathrm{Spec}\, \boldsymbol{C}[M]$ be an algebraic torus of dimension n. Then M is considered as the character group of T, i.e., $M =$

$\mathrm{Hom}_{gr}(M, \boldsymbol{C}^*)$. We denote an element $m \in M$ by $e(m)$ as a regular function on T, which is also a rational function on X. Then we have an isomorphism

$$H^0(X, L) \cong \bigoplus_{m \in P \cap M} \boldsymbol{C} e(m). \tag{1.1}$$

Let P, P_1 and P_2 be integral convex polytopes in $M_{\boldsymbol{R}}$. Then we can consider the Minkowski sum $P_1 + P_2 := \{u_1 + u_2 \in M_{\boldsymbol{R}}; u_i \in P_i \ (i = 1, 2)\}$ and the multiplication by scalars $rP := \{ru \in M_{\boldsymbol{R}}; u \in P\}$ for a positive real number r. If r is a natural number, then rP coincides with the r times sum of P, i.e., $rP = \{u_1 + \cdots + u_r \in M_{\boldsymbol{R}}; u_1, \dots, u_r \in P\}$. The i-fold tensor product $L^{\otimes i}$ corresponds to the convex polytope iP. Moreover the multiplication map

$$H^0(X, L^{\otimes i}) \otimes H^0(X, L) \to H^0(X, L^{\otimes (i+1)}) \tag{1.2}$$

transforms $e(u_1) \otimes e(u_2)$ for $u_1 \in iP \cap M$ and $u_2 \in P \cap M$ to $e(u_1 + u_2)$ through the isomorphism (1.1). Therefore the equality $iP \cap M + P \cap M = (i+1)P \cap M$ means the surjectivity of (1.2).

In [NO] we proved the following proposition.

Proposition 1.1 (Proposition 1.1 in [NO]). *Let P be an integral polytope of dimension n. Then*

$$iP \cap M + P \cap M = (i+1)P \cap M$$

for all $i \geq n - 1$.

In the following we denote $H^0(X, L)$ simply by $\Gamma(L)$.

Definition 1.1. Let F and G be coherent sheaves on a variety X. Define $R(F, G)$ to be the kernel of the canonical map

$$\Gamma(F) \otimes \Gamma(G) \to \Gamma(F \otimes G).$$

By using Proposition 1.1 and an analogous argument of Castelnuovo-Mumford's lemma (Theorem 4 in [M]) we proved in [NO] the following proposition.

Proposition 1.2 (Corollary 2.2 in [NO]). *Let L be an ample invertible sheaf on a projective toric variety X of dimension $n \geq 3$. Then the multiplication map*

$$\Gamma(L) \otimes R(L^{\otimes i}, L) \to R(L^{\otimes (i+1)}, L)$$

is surjective for all $i \geq n$.

As a corollary to Proposition 1.2 we proved in [NO] the following.

Corollary 1.3 (Proposition 3.2 in [NO]). *$L^{\otimes i}$ is normally presented for $i \geq n$. And the defining ideal of X embedded by the global sections $\Gamma(L^{\otimes (n-1)})$ is generated by elements of degree at most three.*

In this paper we shall prove that $L^{\otimes n-1}$ is also normally presented.

2. Normal presentation

In this section we give a proof of Theorem 1. In the following we denote $L^{\otimes i}$ simply by L^i and $\Gamma(L)^{\otimes i}$ by $\Gamma(L)^i$.

Definition 2.1. Let L_1, L_2 and L_3 be invertible sheaves on a variety X. Define $K(iL_1, L_2^j, kL_3)$ to be the kernel of the multiplication map

$$\Gamma(L_1)^i \otimes \Gamma(L_2^j) \otimes \Gamma(L_3)^k \to \Gamma(L_1^i \otimes L_2^j \otimes L_3^k).$$

When $i = 0$ or $k = 0$, we simply denote $K(L_2^j, kL_3)$ or $K(iL_1, L_2^j)$, respectively.

In this section we set $r = n - 1$. Consider the following diagram

$$\begin{array}{ccccc} & & \Gamma(L^r) \otimes R(L^r, L^r) & \longrightarrow & R(L^{2r}, L^r) \\ & & \downarrow & & \downarrow \\ R(L^r, L^r) \otimes \Gamma(L^r) & \longrightarrow & \Gamma(L^r)^3 & \longrightarrow & \Gamma(L^{2r}) \otimes \Gamma(L^r) \\ \downarrow & & \downarrow & & \downarrow \\ R(L^r, L^{2r}) & \longrightarrow & \Gamma(L^r) \otimes \Gamma(L^{2r}) & \longrightarrow & \Gamma(L^{3r}). \end{array}$$

If the multiplication map $\Gamma(L^r) \otimes R(L^r, L^r) \to R(L^{2r}, L^r)$ is surjective, then we would have

$$K(L^r, L^r, L^r) = \Gamma(L^r) \otimes R(L^r, L^r) + R(L^r, L^r) \otimes \Gamma(L^r). \tag{2.1}$$

Unfortunately, we cannot prove the surjectivity of the map $\Gamma(L^r) \otimes R(L^r, L^r) \to R(L^{2r}, L^r)$ for $r = n - 1$. For a proof of Theorem 1 we shall add one more term in the right hand side of (2.1), which is isomorphic to $R(L^r, L^r) \otimes \Gamma(L^r)$ after exchanging the second and the third factors in $\Gamma(L^r)^3$.

On the other hand, consider the graded ring $S = \bigoplus_{d \geq 0} S_d = \bigoplus_{d \geq 0} \Gamma(L^{dr})$. Since S is generated by $S_1 = \Gamma(L^r)$, it is isomorphic to the residue ring $\operatorname{Sym} \Gamma(L^r)/I(L^r)$. Eisenbud and Sturmfels [ES] showed that the homogeneous ideal $I(L^r)$ is generated by binomials. Here a binomial is a difference of two monomials. We are now interested whether the degree three part $I_3(L^r)$ is in $I_2(L^r)S_1$. Since a monomial in S_3 corresponds to a set of three elements in $rP \cap M$, a binomial in I_3 corresponds to a pair of two sets consisting of three elements with the same sum.

Definition 2.2. For $x \in (3rP) \cap M$ let me call the set of three elements $\{v_1, v_2, v_3\}$ in $rP \cap M$ with $x = v_1 + v_2 + v_3$ as a *path* to x in $rP \cap M$ with its length three. For two paths $T = \{v_1, v_2, v_3\}$ and $T' = \{v_1', v_2', v_3'\}$ in $rP \cap M$ to some $x \in (3rP) \cap M$, we define an element $B = e(v_1) \otimes e(v_2) \otimes e(v_3) - e(v_1') \otimes e(v_2') \otimes e(v_3')$ in $K(L^r, L^r, L^r)$. This defines a binomial in $I_3(L^r)$. By abuse of definition we call B a *binomial* in $K(L^r, L^r, L^r)$.

Lemma 2.3. *For* $r = n-1$, *a binomial in* $K(L^r, L^r, L^r)$ *can be written as a sum of an element in* $K(rL, L^r, rL)$ *and an element in* $\Gamma(L^r) \otimes R(L^r, L^r) + R(L^r, L^r) \otimes \Gamma(L^r)$.

Proof. A binomial in $K(L^r, L^r, L^r)$ corresponds to a pair of paths T, T' to some $x \in 3rP \cap M$. Let $T = \{v_1, v_2, v_3\}$ and $T' = \{v_1', v_2', v_3'\}$ with $v_i, v_i' \in rP \cap M$ and $x = v_1 + v_2 + v_3 = v_1' + v_2' + v_3'$. Then the binomial $B = e(v_1) \otimes e(v_2) \otimes e(v_3) - e(v_1') \otimes e(v_2') \otimes e(v_3')$ is in $K(L^r, L^r, L^r)$. Since $v_2 + v_3 \in 2rP \cap M$, from Proposition 1.1 we can choose $w \in rP \cap M$ and $x_1, \dots, x_r \in P \cap M$ such that $v_2 + v_3 = w + x_1 + \cdots + x_r$. Let $T_1 = \{v_1, w, x_1, \dots, x_r\}$. Then the pair T, T_1 defines the element

$$E_1 = e(v_1) \otimes \{e(v_2) \otimes e(v_3) - e(w) \otimes e(x_1 + \cdots + x_r)\}$$

in $\Gamma(L^r) \otimes R(L^r, L^r)$. In the same way we can choose $w' \in rP \cap M$ and $x_1', \dots, x_r' \in P \cap M$ such that $v_2' + v_3' = w' + x_1' + \cdots + x_r'$, and let $T_1' = \{v_1', w', x_1', \dots, x_r'\}$. Then the pair T', T_1' also defines the element

$$E_1' = e(v_1') \otimes \{e(v_2') \otimes e(v_3') - e(w') \otimes e(x_1' + \cdots + x_r')\}$$

in $\Gamma(L^r) \otimes R(L^r, L^r)$. On the other hand, the pair T_1, T_1' defines the element

$$e(v_1) \otimes e(w) \otimes e(x_1) \otimes \cdots \otimes e(x_r) - e(v_1') \otimes e(w') \otimes e(x_1') \otimes \cdots \otimes e(x_r')$$

in $K(L^r, L^r, rL)$, which is mapped to the binomial

$$B_1 = e(v_1) \otimes e(w) \otimes e(x_1 + \cdots + x_r) - e(v_1') \otimes e(w') \otimes e(x_1' + \cdots + x_r')$$

in $K(L^r, L^r, L^r)$. Thus we have $B = B_1 + E_1 - E_1'$ with B_1 in $K(L^r, L^r, L^r)$ and $E_1 - E_1'$ in $\Gamma(L^r) \otimes R(L^r, L^r)$. Here B_1 is coming from $K(L^r, L^r, rL)$.

Next we apply the same procedure to $v_1 + w$ and $v_1' + w'$ in $2rP \cap M$. Then we have $B_1 = B_2 + E_2 - E_2'$ such that B_2 is coming from $K(rL, L^r, rL)$ and that $E_2 - E_2'$ is in $R(L^r, L^r) \otimes \Gamma(L^r)$. This completes the proof.

Lemma 2.4.
(1) $K(L^{n-1}, (j+1)L) \to K(L^n, jL)$ *is surjective for* $j \geq 1$.
(2) $\Gamma(L) \otimes K(L^i, kL) \to K(L^{i+1}, kL)$ *is surjective for* $i \geq n$ *and* $k \geq 1$.

Proof. In order to prove (1) we consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K(L^{n-1}, (j+1)L) & \longrightarrow & \Gamma(L^{n-1}) \otimes \Gamma(L)^{j+1} & \longrightarrow & \Gamma(L^{n+j}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & K(L^n, jL) & \longrightarrow & \Gamma(L^n) \otimes \Gamma(L)^j & \longrightarrow & \Gamma(L^{n+j}) & \longrightarrow & 0 \end{array}$$

such that two horizontal sequences are exact. Since the middle vertical arrow is surjective, we obtain a proof of (1).

As for (2) we consider the diagram

$$\begin{array}{ccccc}
 & & \Gamma(L)\otimes K(L^i,kL) & \longrightarrow & K(L^{i+1},kL) \\
 & & \downarrow & & \downarrow \\
R(L,L^i)\otimes\Gamma(L)^k & \longrightarrow & \Gamma(L)\otimes\Gamma(L^i)\otimes\Gamma(L)^k & \longrightarrow & \Gamma(L^{i+1})\otimes\Gamma(L)^k \\
\downarrow{\scriptstyle\alpha} & & \downarrow & & \downarrow \\
R(L,L^{i+k}) & \longrightarrow & \Gamma(L)\otimes\Gamma(L^{i+k}) & \longrightarrow & \Gamma(L^{i+k+1}).
\end{array}$$

Since α is surjective for $i \geq n$ from Proposition 1.2, we obtain a proof of (2).

Proposition 2.5. *For $r = n-1(\geq 2)$ we have*

$$K(rL,L^r,rL) = \Gamma(L)^r\otimes K(L^r,rL) + K(rL,L^r)\otimes\Gamma(L)^r + \Gamma(L)\otimes K((r-1)L,L^r,L)\otimes\Gamma(L)^{r-1}.$$

Proof. Consider the diagram

$$\begin{array}{ccccc}
 & & \Gamma(L)^r\otimes K(L^r,rL) & \xrightarrow{\beta} & K(L^{2r+1},(r-1)L) \\
 & & \downarrow & & \downarrow \\
K(rL,L^r,L)\otimes\Gamma(L)^{r-1} & \longrightarrow & \Gamma(L)^r\otimes\Gamma(L^r)\otimes\Gamma(L)^r & \longrightarrow & \Gamma(L^{2r+1})\otimes\Gamma(L)^{r-1} \\
\downarrow & & \downarrow & & \downarrow \\
K(rL,L^{2r}) & \longrightarrow & \Gamma(L)^r\otimes\Gamma(L^{2r}) & \longrightarrow & \Gamma(L^{3r}).
\end{array}$$

The homomorphism β factors as $\Gamma(L)^r\otimes K(L^r,rL)\to\Gamma(L)^r\otimes K(L^{r+1},(r-1)L)\to K(L^{2r+1},(r-1)L)$. Thus β is surjective from Lemma 2.4. Hence we have

$$K(rL,L^r,rL) = \Gamma(L)^r\otimes K(L^r,rL) + K(rL,L^r,L)\otimes\Gamma(L)^{r-1}.$$

Next we consider the diagram

$$\begin{array}{ccccc}
 & & \Gamma(L)\otimes K((r-1)L,L^r,L) & \xrightarrow{\gamma} & R(L^{2r},L) \\
 & & \downarrow & & \downarrow \\
K(rL,L^r)\otimes\Gamma(L) & \longrightarrow & \Gamma(L)^r\otimes\Gamma(L^r)\otimes\Gamma(L) & \longrightarrow & \Gamma(L^{2r})\otimes\Gamma(L) \\
\downarrow & & \downarrow & & \downarrow \\
R(L,L^{2r}) & \longrightarrow & \Gamma(L)\otimes\Gamma(L^{2r}) & \longrightarrow & \Gamma(L^{2r+1}).
\end{array}$$

The homomorphism γ factors as $\Gamma(L) \otimes K((r-1)L, L^r, L) \to \Gamma(L) \otimes R(L^{2r-1}, L) \to R(L^{2r}, L)$. Since $2r - 1 = 2n - 3 \geq n$, the map γ is surjective from Lemma 2.4. Hence we have

$$K(rL, L^r, L) = \Gamma(L) \otimes K((r-1)L, L^r, L) + K(rL, L^r) \otimes \Gamma(L).$$

Proof of Theorem 1. From Lemma 2.3 we may consider binomials in $K(L^r, L^r, L^r)$ coming from $K(rL, L^r, rL)$. From Proposition 2.5 we may consider the elements in $K(L^r, L^r, L^r)$ coming from $\Gamma(L) \otimes K((r-1)L, L^r, L) \otimes \Gamma(L)^{r-1}$, because an element coming from $\Gamma(L)^r \otimes K(L^r, rL)$ or $K(rL, L^r) \otimes \Gamma(L)^r$ is mapped to an element in $\Gamma(L^r) \otimes R(L^r, L^r)$ or $R(L^r, L^r) \otimes \Gamma(L^r)$, respectively. It is easily seen that $K((r-1)L, L^r, L)$ is generated by elements of the form

$$e(y_1) \otimes \cdots \otimes e(y_{r-1}) \otimes e(w) \otimes e(z) - e(y_1') \otimes \cdots \otimes e(y_{r-1}') \otimes e(w') \otimes e(z'),$$

where y_i, y_i', z and z' are in $P \cap M$ and w and w' are in $rP \cap M$ with $y_1 + \cdots + y_{r-1} + w + z = y_1' + \cdots + y_{r-1}' + w' + z'$, by definition of $K((r-1)L, L^r, L)$.

Let

$$\begin{aligned} B = {} & e(x + y_1 + \cdots + y_{r-1}) \otimes e(w) \otimes e(z + x_1' + \cdots + x_{r-1}') \\ & - e(x + y_1' + \cdots + y_{r-1}') \otimes e(w') \otimes e(z' + x_1' + \cdots + x_{r-1}') \end{aligned}$$

be a binomial mapped from $\Gamma(L) \otimes K((r-1)L, L^r, L) \otimes \Gamma(L)^{r-1}$ to $K(L^r, L^r, L^r)$ such that $x, z, x_i', y_i, y_i' \in P \cap M$ and $w, w' \in rP \cap M$ with $y_1 + \cdots + y_{r-1} + w + z = y_1' + \cdots + y_{r-1}' + w' + z'$. Set

$$\begin{aligned} B' = {} & \{e(z + y_1 + \cdots + y_{r-1}) \otimes e(w) - e(z' + y_1' + \cdots + y_{r-1}') \otimes e(w')\} \\ & \otimes e(x + x_1' + \cdots + x_{r-1}'). \end{aligned}$$

Then B' is in $R(L^r, L^r) \otimes \Gamma(L^r)$. Consider the difference $B - B'$. The difference of the first terms in B and B' is

$$\begin{aligned} & e(x + y_1 + \cdots + y_{r-1}) \otimes e(w) \otimes e(z + x_1' + \cdots + x_{r-1}') \\ & \quad - e(z + y_1 + \cdots + y_{r-1}) \otimes e(w) \otimes e(x + x_1' + \cdots + x_{r-1}'). \end{aligned}$$

If we delete $e(w)$ from it, then we obtain an element in $R(L^r, L^r)$. Therefore $B - B'$ is an element in the image of $R(L^r, L^r) \otimes \Gamma(L^r)$ after exchanging the second and the third factors of $\Gamma(L^r)^3$.

3. Special cases

First we consider a special case that L is normally generated. In this case we can represent the graded ring $\bigoplus_{d \geq 0} \Gamma(L^d)$ as the residue ring $\operatorname{Sym} \Gamma(L)/I(L)$. Here $I(L)$ is the homogeneous ideal of $\operatorname{Sym} \Gamma(L)$ defining the image of X in $\boldsymbol{P}(\Gamma(L)^*)$. It is known that $I(L)$ has generators of degree at most $n + 1$ (see

Theorem 13.14 [S], or Theorem 0.3 [NO]) and that there exists an example whose generators need elements of degree $n+1$. In this section we want to obtain an estimate for an integer i_0 such that $L^{\otimes i}$ is normally presented, that is, the defining ideal $I(L^i)$ is generated by quadrics, for all $i \geq i_0$.

For example, we consider the case that $n = 5$ and L is normally generated. The image of X in $\boldsymbol{P}(\Gamma(L)^*)$ has generators of degree at most six. We may expect that the defining ideal of the image of X in $\boldsymbol{P}(\Gamma(L^3)^*)$ is generated by quadrics, because this embedding is a composition of the embedding $X \hookrightarrow \boldsymbol{P}(\Gamma(L)^*)$ and the Veronese embedding $\boldsymbol{P}(\Gamma(L)^*) \hookrightarrow \boldsymbol{P}(\Gamma(L^3)^*)$. In general, we may expect that $L^{\otimes i}$ is normally presented for $i > [n/2]$ when L is normally generated. We shall show in Proposition 3.1 that this is true. When $n = 3$, or $n = 4$, the equality $n - 1 = [n/2] + 1$ holds. Thus we may assume $n \geq 5$.

Example. Let $e_1, \ldots, e_5$ be a $\boldsymbol{Z}$-basis of $M \cong \boldsymbol{Z}^5$. Set $u_0 = 0$, $u_i = e_i$ $(i = 1, \ldots, 4)$ and $u_5 = e_1 + \cdots + e_4 + 3e_5$. Let $P = \mathrm{Conv}\{u_0, u_1, \ldots, u_5\}$. Then we easily see $4P \cap M = 3P \cap M + P \cap M$. If P corresponds to the polarized toric variety (X, L), then we have that $\Gamma(L^i) \otimes \Gamma(L) \to \Gamma(L^{i+1})$ are surjective for all $i \geq 3$. Thus we have that L^3 is normally generated. But L^2 is not very ample, because the lattice point $u_6 = e_1 + \cdots + e_5$ in $3P$ is not contained in $2P$. This implies that $\Gamma(L^2) \otimes \Gamma(L) \to \Gamma(L^3)$ is not surjective. From easy calculation we see that L^3 is normally presented.

The example suggests that we may weaken the condition on L from the normal generation to the surjectivity of the multiplication map $\Gamma(L^i) \otimes \Gamma(L) \to \Gamma(L^{i+1})$ for all $i > n/2$.

Assumption 3.1. Let $t = t(L)$ be the smallest positive integer such that the multiplication map $\Gamma(L^i) \otimes \Gamma(L) \to \Gamma(L^{i+1})$ is surjective for all $i \geq t$. From Proposition 1.1 we see that $1 \leq t \leq n - 1$. We assume that $n \geq 5$ and that $1 \leq t < n - 1$.

Proposition 3.1. *Let t be the integer in Assumption 3.1. Then L^r is normally presented for $r \geq \max\{t, [n/2] + 1\}$.*

In order to prove Proposition 3.1 we need the following lemma.

Lemma 3.2. *For $r \geq \max\{t, [n/2] + 1\}$, we have three equalities.*

(1) $K(rL, L^r, rL) = K(rL, L^r, iL) \otimes \Gamma(L)^{r-i} + \Gamma(L)^i \otimes K((r-i)L, L^r, rL)$ *for* $1 \leq i < r$.

(2) $K(rL, L^r, L) = K(rL, L^r) \otimes \Gamma(L) + \Gamma(L) \otimes K((r-1)L, L^r, L)$.

(3) $K(rL, L^r, iL) = K(rL, L^r, (i-1)L) \otimes \Gamma(L) + \Gamma(L)^i \otimes K((r-i)L, L^r, iL)$ *for* $2 \leq i < r$.

Proof. For $1 \le i < r$ and $0 \le s < k$, consider the diagram

$$\begin{array}{ccccc} \Gamma(L)^i \otimes K((r-i)L, L^r, kL) & \longrightarrow & \Gamma(L)^r \otimes \Gamma(L^r) \otimes \Gamma(L)^k & \longrightarrow & \Gamma(L)^i \otimes \Gamma(L^{2r+k-i}) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ K(L^{2r+s}, (k-s)L) & \longrightarrow & \Gamma(L^{2r+s}) \otimes \Gamma(L)^{k-s} & \longrightarrow & \Gamma(L^{2r+k}), \end{array}$$

where β and γ are surjective.

First set $k = r$ and $s = i$. Then we see that $\operatorname{Ker}\beta = K(rL, L^r, iL) \otimes \Gamma(L)^{r-i}$ and that α is surjective. This proves (1). Next set $k = i = 1$ and $s = 0$. Then we see that $\operatorname{Ker}\beta = K(rL, L^r) \otimes \Gamma(L)$ and that α is surjective. This proves (2). Finally if we set $k = i$ and $s = i-1$, then we see that $\operatorname{Ker}\beta = K(rL, L^r, (i-1)L) \otimes \Gamma(L)$ and that α is surjective. This proves (3).

Proof of Proposition 3.1. We note that the ideal defining the image of X embedded by $\Gamma(L^r)$ has generators of degree at most three from Proposition 3.2 in [NO]. Thus we may consider only $K(L^r, L^r, L^r)$. Furthermore we may consider binomials in $K(L^r, L^r, L^r)$ coming from $K(rL, L^r, rL)$ because in the proof of Lemma 2.3 we used only the condition $r \ge t$.

We shall prove the equality

$$K(rL, L^r, rL) = \sum_{i=0}^{r} \Gamma(L)^i \otimes K((r-i)L, L^r, iL) \otimes \Gamma(L)^{r-i}. \tag{3.3}$$

First apply Lemma 3.2 (1) for $i = r-1$. By applying (3) to the first term we obtain the sum in (3.3) from $i = 1$ to $i = r-1$ and the rest. Apply (2) to the rest.

The term of $i = 0$ or $i = r$ in the right hand side of (3.3) is mapped into $R(L^r, L^r) \otimes \Gamma(L^r)$ or $\Gamma(L^r) \otimes R(L^r, L^r)$, respectively. Let $1 \le i \le r-1$. By applying the same argument in the proof of Theorem 1, we consider a binomial

$$\begin{aligned} B = {} & e(x_1 + \cdots + x_i + y_1 + \cdots + y_{r-i}) \otimes e(w) \otimes e(z_1 + \cdots + z_i + x'_1 + \cdots + x'_{r-1}) \\ & - e(x_1 + \cdots + x_i + y'_1 + \cdots + y'_{r-i}) \otimes e(w') \otimes e(z'_1 + \cdots + z'_i + x'_1 + \cdots + x'_{r-1}) \end{aligned}$$

such that $x_j, x'_j, y_j, y'_j, z_j$ and z'_j are in $\Gamma(L)$ and w, w' are in $\Gamma(L^r)$ with $y_1 + \cdots + y_{r-1} + w + z_1 + \cdots + z_i = y'_1 + \cdots + y'_{r-1} + w' + z'_1 + \cdots + z'_i$. The binomial B is in $K(L^r, L^r, L^r)$ coming from $\Gamma(L)^i \otimes K((r-i)L, L^r, iL) \otimes \Gamma(L)^{r-i}$. Set

$$\begin{aligned} B' = {} & \{e(y_1 + \cdots + y_{r-i} + z_1 + \cdots + z_i) \otimes e(w) \\ & - e(y'_1 + \cdots + y'_{r-i} + z'_1 + \cdots + z'_i) \otimes e(w')\} \\ & \otimes e(x_1 + \cdots + x_i + x'_1 + \cdots + x'_{r-i}). \end{aligned}$$

Thus B' is in $R(L^r, L^r) \otimes \Gamma(L^r)$. The difference $B - B'$ is written as

$$\{e(x_1 + \cdots + x_i + y_1 + \cdots + y_{r-i}) \otimes e(w) \otimes e(z_1 + \cdots + z_i + x'_1 + \cdots + x'_{r-1})$$
$$- e(y_1 + \cdots + y_{r-i} + z_1 + \cdots + z_i) \otimes e(w) \otimes e(x_1 + \cdots + x_i + x'_1 + \cdots + x'_{r-i})\}$$
$$- \{e(x_1 + \cdots + x_i + y'_1 + \cdots + y'_{r-i}) \otimes e(w') \otimes e(z'_1 + \cdots + z'_i + x'_1 + \cdots + x'_{r-1})$$
$$- e(y'_1 + \cdots + y'_{r-i} + z'_1 + \cdots + z'_i) \otimes e(w') \otimes e(x_1 + \cdots + x_i + x'_1 + \cdots + x'_{r-i})\}.$$

Therefore $B - B'$ is in the image of $R(L^r, L^r) \otimes \Gamma(L^r)$ under the isomorphism of $\Gamma(L^r)^3$ defined by exchanging the second and the third factors. Since $K((r-i)L, L^r, iL)$ is generated by elements like binomials

$$e(y_1) \otimes \cdots e(y_{r-1}) \otimes e(w) \otimes e(z_1) \otimes \cdots \otimes e(z_i)$$
$$- e(y'_1) \otimes \cdots e(y'_{r-1}) \otimes e(w') \otimes e(z'_1) \cdots \otimes e(z'_i),$$

we obtain the proof.

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