ON JULIA DIRECTIONS OF ENTIRE FUNCTIONS OF SMALL ORDER

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Abstract

In this paper, the relationship between Julia direction and the growth of entire function of order $0 < \lambda \leq 1/2$ is discussed.

1. Introduction

A ray $\chi(\theta) = \{z = re^{i\theta} : 0 < r < +\infty\}$ in the complex plane is called a Julia direction of entire function $f(z)$, if $f(z)$ takes all finite complex numbers infinitely often in any sector containing $\chi(\theta)$, with at most one finite exceptional value. So in any sector containing Julia direction $\chi(\theta)$ of $f(z)$, the set of roots of equation $f(z) = \alpha$ is unbounded if α is not such a exceptional value, hence $f(z)$ can not tend to ∞ as $|z| \rightarrow +\infty$ in the sector. But if a ray $\chi(\theta)$ is not a Julia direction of $f(z)$, what will happen about $f(z)$? In [2] H. Yoshida conjectured: If $f(z)$ is an entire function of order less than $1/2$ and $\chi(\theta)$ is ray, then either $\chi(\theta)$ is a Julia direction of $f(z)$ or $f(z)$ tends ∞ as $|z| \rightarrow +\infty$ in some sector containing $\chi(\theta)$. In this paper we discuss this question and prove that H. Yoshida's conjecture is affirmative under some conditions. These conditions are different from the conditions which were discussed in [2] and we also give a estimate about the growth of this kind of functions in a sector that does not contain Julia directions. Our main result is:

THEOREM 1. *Let $f(z)$ be an entire function of order $0 < \lambda \leq 1/2$, and $f(z)$ be regular growth, i.e., its lower order is equal to its order, then either $\chi(\theta)$ is a Julia direction of $f(z)$ or in some sector G containing $\chi(\theta)$,*

$$\lim_{\substack{|z| \rightarrow +\infty \\ z \in G}} \frac{\log \log |f(z)|}{\log |z|} = \lambda$$

holds.

In Section 3 of this paper, an interesting corollary of this theorem will be discussed.

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2. Notations and lemmas

LEMMA 1 ([1, Theorem 3]). *Let $f(z)$ be an entire function of lower order $0 < \mu \leq 1/2$, then there exists an asymptotic path L tending to ∞ , such that*

$$\liminf_{\substack{|z| \rightarrow +\infty \\ z \in L}} \frac{\log \log |f(z)|}{\log |z|} = \mu.$$

LEMMA 2 ([5] or [6, Lemma 6.6]). *Let $f(z)$ be a meromorphic function on the disc $|z| \leq R$. Set*

$$N = n(R, f = 0) + n(R, f = 1) + n(R, f = \infty),$$

where $n(R, f = a)$ denotes the number of roots of the equation $f(z) = a$ on $|z| \leq R$ (Counting multiplicity). If the distance from $z = 0$ to these N points has a positive lower bound d , then there exists a constant C such that for any $0 < r < R$,

$$T(r, f) < \frac{CR(N+1)}{R-r} \log \frac{2R^2(N+1)}{d(R-r)} + \log^+ |f(0)|,$$

where $T(r, f)$ denotes the characteristic function of $f(z)$.

Denote $\Omega(\theta_1, \theta_2) = \{z = re^{i\theta} : \theta_1 \leq \theta \leq \theta_2, 0 \leq r < +\infty\}$ and $\Omega(\theta_1, \theta_2, r_1, r_2) = \{z = re^{i\theta} : \theta_1 \leq \theta \leq \theta_2, r_1 \leq r \leq r_2\}$.

LEMMA 3. *Let $f(z)$ be an entire function, which assumes distinct complex numbers α and β finite times in the sector $\Omega(-\sigma, \sigma)$ ($\sigma > 0$). If there exists a sequence $\{z_n\}$ ($|z_n| \rightarrow \infty$) in $\Omega(-\sigma/128, \sigma/128)$ such that*

$$|f(z_n)| < K(n),$$

then there exists a constant C which is independent of n and $K(n)$ such that for each n ,

$$\log |f(z)| \leq C(\log K(n) + 1)$$

holds on the set $E_n = \{z = |z_n|e^{i\theta} : \theta \in [-\sigma/128, \sigma/128]\}$.

Proof. It suffices to prove the conclusion holds when n is sufficiently large. Consider the disc $D_n = \{z : |z - z_n| \leq |z_n| \sin(\sigma/16)\}$. It is easy to see that $E_n \subset D_n$ and $D_n \subset \Omega(-\sigma, \sigma)$. Since $|z - z_n| \leq |z_n| \sin(\sigma/32)$ holds for any $z \in E_n$, from Possion-Jensen formula we have for any $z \in E_n$,

$$\begin{aligned} \log |f(z)| &\leq \frac{|z_n| \sin(\sigma/16) + |z_n| \sin(\sigma/32)}{|z_n| \sin(\sigma/16) - |z_n| \sin(\sigma/32)} m\left(|z_n| \sin \frac{\sigma}{16}, z_n, f\right) \\ &\leq CT\left(|z_n| \sin \frac{\sigma}{16}, z_n, f\right) \\ &\leq C\left\{T\left(|z_n| \sin \frac{\sigma}{16}, z_n, \frac{f - \alpha}{\beta - \alpha}\right) + 1\right\}, \end{aligned}$$

where $T(r, z_0, f)$ denotes the characteristic function of $f(z)$ on the disc $\{z : |z - z_0| \leq r\}$ (In this paper C is always a constant, although it maybe changes in each step).

Let N be the number of times that $f(z)$ assumes α and β in $\Omega(-\sigma, \sigma)$. When n is sufficiently large, the distance from z_n to these N points is larger than $|z_n| \sin(\sigma/16)$. Noticing that the disc $\{z : |z - z_n| \leq |z_n| \sin(\sigma/8)\}$ lies also in $\Omega(-\sigma, \sigma)$, from Lemma 2 we have

$$\begin{aligned} T\left(|z_n| \sin \frac{\sigma}{16}, z_n, \frac{f - \alpha}{\beta - \alpha}\right) &\leq C \left\{ \frac{|z_n| \sin(\sigma/8)(N+1)}{|z_n| \sin(\sigma/8) - |z_n| \sin(\sigma/16)} \right. \\ &\quad \times \log \frac{2(|z_n| \sin(\sigma/8))^2(N+1)}{(|z_n| \sin(\sigma/8) - |z_n| \sin(\sigma/16))|z_n| \sin(\sigma/16)} \\ &\quad \left. + \log^+ \left| \frac{f(z_n) - \alpha}{\beta - \alpha} \right| + 1 \right\} \\ &\leq C(\log K(n) + 1). \end{aligned}$$

Combining these two estimates the conclusion is obtained. $\square$

Denote $M(\Omega(-\theta_1, \theta_2, r_1, r_2), f) = \max_{z \in \Omega(-\theta_1, \theta_2, r_1, r_2)} |f(z)|$, and $M(r, f) = \max_{|z| \leq r} |f(z)|$.

LEMMA 4. *Let $f(z)$ be an entire function, which assumes distinct complex numbers α and β finite times in the sector $\Omega(-2\sigma, 2\sigma)$ ($\sigma > 0$). If there exists a sequence $\{z_n\}$ ($|z_n| \rightarrow \infty$) such that*

$$|f(|z_n|)| < K(n).$$

Then for any fixed η ($0 < \eta < 1$), there exists a constant C which is independent of n and $K(n)$ such that

$$\begin{aligned} \log M(\Omega(-\sigma, \sigma, |z_n|^{1-\eta}, |z_n|), f) \\ \leq C|z_n|^{(\pi/\sigma)\eta} \{(\log |z_n| + \log K(n) + 1)\}. \end{aligned}$$

Proof. It suffices to prove the conclusion holds when n is sufficiently large. Consider the transformation

$$\tau(z) = \frac{z^{\pi/(4\sigma)} - |z_n|^{\pi/(4\sigma)}}{z^{\pi/(4\sigma)} + |z_n|^{\pi/(4\sigma)}}.$$

It is easy to see that under this transformation $z = |z_n|$ is mapped to $\tau = 0$, and $\Omega(-2\sigma, 2\sigma)$ is mapped to the disc $|\tau| < 1$. Furthermore $\Omega(-\sigma, \sigma, |z_n|^{1-\eta}, |z_n|)$ is mapped into $|\tau| < t$, where $t = 1 - (1/4)|z_n|^{-(\pi/(2\sigma))\eta}$.

Since $f(z)$ assumes α and β only at finite number of points on $\Omega(-2\sigma, 2\sigma)$,

hence when n is large enough, these points are mapped outside of the disc $|\tau| < (1+t)/2$. Set

$$g(\tau) = \frac{f(z(\tau)) - \alpha}{\beta - \alpha},$$

where $z(\tau)$ is the inverse function of $\tau(z)$. Then $g(\tau)$ does not assume $0, 1, \infty$ on $|\tau| < (1+t)/2$. When taking $R = (3+t)/4$, $r = (1+t)/2$ and $d = (1+t)/2$ in Lemma 2 we deduce that

$$T\left(\frac{1+t}{2}, g\right) \leq \frac{C}{1-t} \left\{ \log \frac{1}{1-t} + \log^+ |g(0)| + 1 \right\},$$

hence

$$\begin{aligned} \log M(t, g) &\leq \frac{(1+t)/2 + t}{(1+t)/2 - t} T\left(\frac{1+t}{2}, g\right) \\ &\leq \frac{C}{(1-t)^2} \left\{ \log \frac{1}{1-t} + \log^+ |g(0)| + 1 \right\}. \end{aligned}$$

Since $1-t = (1/4)|z_n|^{-(\pi/(2\sigma))\eta}$ and $g(0) = (f(|z_n|) - \alpha)/(\beta - \alpha)$ we have

$$\begin{aligned} \log M(\Omega(-\sigma, \sigma, |z_n|^{1-\eta}, |z_n|), f) &\leq \log M(t, g) + C \\ &\leq C\{|z_n|^{(\pi/\sigma)\eta}(\log|z_n| + \log K(n) + 1)\}. \quad \square \end{aligned}$$

3. The proof of Theorem 1

The proof of Theorem 1.

Without loss of generality we assume that $\theta = 0$, i.e., we consider the ray $\chi(0) = \{z : \arg z = 0\}$. If $\chi(0)$ is not a Julia direction of $f(z)$, then there exists a sector $\Omega(-2\sigma, 2\sigma)$ and two distinct complex numbers α, β such that $f(z)$ assumes α and β only finite number of times. Now we prove that on the sector $\Omega(-\sigma/128, \sigma/128)$,

$$\lim_{\substack{|z| \rightarrow +\infty \\ z \in G}} \frac{\log \log |f(z)|}{\log |z|} = \lambda$$

holds.

If the conclusion of Theorem 1 were not true, then there exists a positive number ρ ($0 < \rho < \lambda$) and a sequence $\{z_n\}$ ($|z_n| \rightarrow \infty$) in $\Omega(-\sigma/128, \sigma/128)$ such that

$$\log |f(z_n)| \leq |z_n|^\rho.$$

Hence from Lemma 3 we have

$$\log |f(|z_n|)| \leq C\{|z_n|^\rho + 1\}.$$

Take a positive number η such that $(\pi/\sigma)\eta + \rho < (1 - 3\eta)\lambda$, from Lemma 4 we can deduce that

$$(1) \quad \begin{aligned} \log M(\Omega(-\sigma, \sigma, |z_n|^{1-\eta}, |z_n|), f) \\ \leq C\{|z_n|^{(\pi/\sigma)\eta}(\log|z_n| + |z_n|^\rho + 1)\} \\ \leq |z_n|^{\lambda(1-2\eta)} \end{aligned}$$

holds for n is sufficiently large.

By Lemma 1 there exists an asymptotic path L tending to ∞ , such that for any $\varepsilon > 0$ ($\varepsilon < \lambda - \lambda(1 - 2\eta)/(1 - \eta)$),

$$\log|f(z)| \geq |z|^{\lambda-\varepsilon}, \quad z \in L$$

holds for $|z|$ is sufficiently large. Hence

$$(2) \quad \log|f(z)| \geq |z_n|^{(\lambda-\varepsilon)(1-\eta)}$$

holds for $z \in L$ and $|z_n|^{1-\eta} \leq |z| \leq |z_n|$.

Obviously, there exists an positive number N' in the interval $[(1/4)|z_n|^{(1-\eta)(\lambda-\varepsilon)}, (1/2)|z_n|^{(1-\eta)(\lambda-\varepsilon)}]$ such that $f'(z)$ has no zero points on the level set $L_n = \{z : \log|f(z)| = N'\}$, hence L_n consists of analytic curves. Consider the set $F_n = \{z : \log|f(z)| > N' \text{ and } |z| < |z_n|\}$. Let z'_n be a point at which L intersects the circle $\{z : z = |z_n|^{1-\eta}\}$, and Ω_n be the connected component of F_n that contains z'_n . By maximum modulus principle, $\overline{\Omega_n} \cap \{z : z = |z_n|\} \neq \emptyset$. Let θ_t be the part of $\{z : |z| = t\}$ lying in Ω_n , $\theta(t)$ and $t\theta(t)$ be the angle measure and linear measure of θ_t respectively. Notice that

$$|z_n|^{(\lambda-\varepsilon)(1-\eta)} \geq |z_n|^{\lambda(1-2\eta)}.$$

Hence from (1) and (2) we have

$$(3) \quad 1 = \theta(t) \frac{1}{\theta(t)} \leq (2\pi - 2\sigma) \frac{1}{\theta(t)}, \quad |z_n|^{1-\eta} \leq t \leq |z_n|,$$

Let $\omega(z, \theta_t)$ be the harmonic measure of θ_t with respect to F_n . From [3, 111–117] we have the estimate

$$(4) \quad \omega(z'_n, \theta_t) \leq 9\sqrt{2} \exp\left\{-\pi \int_{2|z_n|^{1-\eta}}^{(1/2)|z_n|} \frac{dt}{t\theta(t)}\right\}.$$

Hence, by the maximum principle we have

$$\log|f(z'_n)| \leq N' + 9\sqrt{2} \exp\left\{-\pi \int_{2|z_n|^{1-\eta}}^{(1/2)|z_n|} \frac{dt}{t\theta(t)}\right\} \log M(|z_n|, f).$$

Noticing that $N' \in [(1/4)|z_n|^{(1-\eta)(\lambda-\varepsilon)}, (1/2)|z_n|^{(1-\eta)(\lambda-\varepsilon)}]$, from (2) we have

$$|z_n|^{(1-\eta)(\lambda-\varepsilon)} \leq 18\sqrt{2} \exp\left\{-\pi \int_{2|z_n|^{1-\eta}}^{(1/2)|z_n|} \frac{dt}{t\theta(t)}\right\} \log M(|z_n|, f).$$

Hence from (3) we have

$$\begin{aligned} \frac{\pi}{2\pi - 2\sigma} \int_{2|z_n|^{1-\eta}}^{(1/2)|z_n|} \frac{dt}{t} &\leq \pi \int_{2|z_n|^{1-\eta}}^{(1/2)|z_n|} \frac{dt}{t\theta(t)} \\ &\leq \log \log M(|z_n|, f) - \log|z_n|(1-\eta)(\lambda - \varepsilon) + \log(18\sqrt{2}), \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{\pi\eta}{2\pi - 2\sigma} \log|z_n| &\leq \log \log M(|z_n|, f) - \log|z_n|(1-\eta)(\lambda - \varepsilon) \\ &\quad + \log(18\sqrt{2}) + \frac{\pi}{2\pi - 2\sigma} \log 4. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\pi\eta}{2\pi - 2\sigma} &\leq \limsup_{n \rightarrow \infty} \frac{\log \log M(|z_n|, f)}{\log|z_n|} - (1-\eta)(\lambda - \varepsilon) \\ &\leq \lambda - (1-\eta)(\lambda - \varepsilon). \end{aligned}$$

Let $\varepsilon \rightarrow 0$ we have $\pi\eta/(2\pi - 2\sigma) \leq \lambda\eta$. Hence

$$\frac{1}{2} < \frac{\pi}{2\pi - 2\sigma} \leq \lambda.$$

This contradicts to that the order of $f(z)$ is less than or equal to $1/2$.

The proof of Theorem 1 is completed.

In [2] H. Yoshida proved that, if an entire function $f(z)$ satisfies $\log M(2r, f) \sim \log M(r, f)$, then the set of ray $\chi(\theta)$ for which θ is a limit point of the set

$$E(0, f) = \{\arg z_n : f(z_n) = 0\}$$

is precisely the set of Julia directions of $f(z)$. In this situation, the order $f(z)$ is 0 (see [4]). It is well known that this conclusion does not hold for general entire functions. But we have

COROLLARY 1. *Let $f(z)$ be an entire function of order $0 < \lambda \leq 1/2$, and $f(z)$ be regular growth. α is a complex number. Then for any limit point θ of the set*

$$E(\alpha, f) = \{\arg z_n : f(z_n) = \alpha\},$$

the ray $\chi(\theta)$ is a Julia direction of $f(z)$. Furthermore, if α and β are distinct complex numbers, then the set of ray $\chi(\theta)$ for which θ is the limit point of the set

$$E(\alpha, \beta, f) = \{\arg z_n : f(z_n) = \alpha \text{ or } f(z_n) = \beta\}$$

is precisely the set of Julia directions of $f(z)$.

Proof. Let θ be a limit point of the set $E(\alpha, f)$. If $\chi(\theta)$ is not a Julia direction of $f(z)$, then from Theorem 1 there exists a sector G containing $\chi(\theta)$

such that $|f(z)|$ tends to $+\infty$ as $|z| \rightarrow +\infty$ in G . Hence $f(z)$ can not assume α infinite number of times in the sector G . This contradicts to that θ is a limit point of the set $E(\alpha, f)$. Hence $\chi(\theta)$ is a Julia direction of $f(z)$.

Furthermore, if θ is a limit point of the set $E(\alpha, \beta, f)$, by the same discussion as above we can see that $\chi(\theta)$ is a Julia direction of $f(z)$. If θ is not a limit point of the set $E(\alpha, \beta, f)$, then there exists a sector G containing $\chi(\theta)$ such that $f(z)$ assumes α and β at most finite number of times in the sector G . Hence $\chi(\theta)$ is not a Julia direction of $f(z)$. $\square$

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