T. SHINOHARA
KODAI MATH. J.
25 (2002), 15–42

DYNAMICS OF POLYNOMIAL MAPS ON C^2 WHOSE ALL UNBOUNDED ORBITS CONVERGE TO ONE POINT

TOMOKO SHINOHARA

Abstract

In this paper, we study a family of iteration of polynomial map on the 2-dimensional complex Euclidean space $\boldsymbol{C}^2$ whose all unbounded orbits converge to one point of the line at infinity in the 2-dimensional complex projective space $\boldsymbol{P}^2$. In particular, we show some sufficient condition for the Lebesgue measure of its Julia set to be equal to 0.

1. Introduction

Recently, several authors have researched Hénon maps $F_{a,c}$ which have the form $F_{a,c}(z,w) = (w, w^2 - az + c)$ for $(a,c) \in \boldsymbol{C}^* \times \boldsymbol{C}$. From the works of Bedford and Smillie, for instance [1], one can see that Hénon maps are the most fundamental and essential among all polynomial automorphisms of $\boldsymbol{C}^2$. One of the reasons why Hénon maps are studied so well may be that:

$(*)$ All unbounded orbits of them converge to one point of the line l_∞ at infinity in $\boldsymbol{P}^2$.

Therefore, their dynamics are very similar to those of polynomial maps in $\boldsymbol{C}$. On the other hand, it goes without saying that there are many other classes of holomorphic or meromorphic dynamics of several complex variables to be understood. In this paper, we focus our study on a family of polynomial maps F on $\boldsymbol{C}^2$ with the property $(*)$ above.

We assume that F has only one super attracting fixed point p_∞ on l_∞, and F sends all non-indeterminate points on l_∞ to p_∞.

In section 2, we first prove that F is conjugate to the map in Theorem 2.1. Let A_+ be the attracting basin of p_∞ for F and K_+ the set of points whose forward orbits are bounded in $\boldsymbol{C}^2$. Then, under some conditions, we can show that K_+ is the complement of A_+ in $\boldsymbol{C}^2$; in particular, F has the property $(*)$.

Now, we define the iteration $\{F^{\circ n}\}$ of F as usual, and denote by $g_{z_0}^n(w)$ the second component of $F^{\circ n}(z_0, w)$. Let J_{z_0} be the set of points in the extended complex plane $\hat{\boldsymbol{C}}$ where $\{g_{z_0}^n\}$ is not normal as a family of polynomials in one

Received February 22, 2001; revised November 26, 2001.

variable w. Then, in section 3, we verify that J_{z_0} is a non-empty compact subset of $\boldsymbol{C} \subset \hat{\boldsymbol{C}}$, and also we obtain some results on its connectivity.

In order to state our main results, we need a few preparations. Let J_+ be the set of points where $\{F^{\circ n}\}$ is not normal, and set $J_+(R_1) = J_+ \cap \{(z, w) \in \boldsymbol{C}^2 \mid |z| \leq R_1\}$ for a given $R_1 > 0$. We write the complex Jacobian matrix of $F^{\circ n}$ for $n = 1, 2, \dots,$ as

$$DF^{\circ n}(z, w) = \begin{pmatrix} a_n(z, w) & b_n(z, w) \\ c_n(z, w) & d_n(z, w) \end{pmatrix}.$$

Then, we can prove the following: (For the definitions of terminology and notation, see sections 3 and 4.)

Main result 1 (Theorem 4.6). *If F satisfies the condition $(\mathscr{F})$, then* $J_+(R_1) = \bigcup_{|z_0| \leq R}\{z_0\} \times J_{z_0}$.

Main result 2 (Theorem 4.10). *Let z_0 be an arbitrary point of $\boldsymbol{C}$ with $|z_0| < R_1$. We assume that the following three conditions are satisfied*:

(1) *There exist a constant $\delta > 0$ such that*

$$\inf_{w \in J_{z_0}} \inf_{w' \in \tilde{C}_n(z_0)} |w_n - w_n'| > \delta \quad \textit{for all } n,$$

where we have set $w_n = g_{z_0}^n(w), w_n' = g_{z_0}^n(w')$ for $w \in J_{z_0}, w' \in \tilde{C}_n(z_0)$, respectively.

(2) *F satisfies the condition $(\mathscr{F})$.*

(3) *Let $l_{w_\infty}^{n_j} = \{(z, w) \in \boldsymbol{C}^2 \mid w = \psi_{w_\infty}^{n_j}(z),\ z \in \Delta(R_1)\}$ be the leaf of S_{n_j} which converges to the leaf $l_{w_\infty} = \{(z, w) \in \boldsymbol{C}^2 \mid w = \psi_{w_\infty}(z),\ z \in \Delta(R_1)\}$ of $J_+(R_1)$ containing (z_∞, w_∞). Then there exist real numbers α, β with $0 < \alpha < 1 < \beta$ such that*

(i) $|d_{n+1}(z_0, w_0)| > \beta |d_n(z_0, w_0)|$,

(ii) $|c_{n_j}(z_\infty, \psi_{w_\infty}^{n_j}(z_\infty)) b_{n_j}(z_0, w_0)| / |d_{n_j}(z_\infty, \psi_{w_\infty}^{n_j}(z_\infty)) d_{n_j}(z_0, w_0)| < \alpha$

for all sufficiently large integers n, j and for any $w_0 \in J_{z_0}$.

Then the 2-dimensional Lebesgue measure of J_{z_0} is equal to 0. In particular, the 4-dimensional Lebesgue measure of $J_+(R_1)$ is equal to 0.

Here, we would like to remark that the conditions in Main results (except for (ii) of Main result 2) correspond to the expandingness in dynamical theory of polynomial maps in $\boldsymbol{C}$. In [4; Corollary 3.29] Fornæss and Sibony obtained the same result as in our Main result 2 for some Hénon map $F_{a,c}$ whose parameter $|a|$ is sufficiently small and c belongs to the set $M := \{c \in \boldsymbol{C} \mid P(z) := z^2 + c$ has an attracting periodic point$\}$. The main tool in their proof is the perturbation of dynamical systems of P. The perturbation is useful to discussion of a dynamical structure under small change of parameters of maps; however, it is not suitable for general maps. Therefore we prove Main result 2 without using the perturbation. In section 5, we give some concrete example of maps F_a which satisfy the assumption of Main result 2; and, their $F_{a,c}$ appears as a special one of our F_a. Therefore, our proof of Main result 2 provides an alternative proof of [4; Corollary 3.29].

2. Normal form of a map whose all unbounded orbits converge to one point

Let us fix an affine coordinate system (z, w) of $\boldsymbol{C}^2$ and a homogeneous coordinate system $[z : w : t]$ of $\boldsymbol{P}^2$. Sometimes we identify $\boldsymbol{C}^2$ with $\{[z : w : t] \in \boldsymbol{P}^2 \mid t \neq 0\}$.

We consider a polynomial map $F(z, w) = (f_0, f_1)$ on $\boldsymbol{C}^2$ with degree d, where f_0, f_1 are polynomials of z, w and $d := \max\{\deg f_0, \deg f_1\}$. As usual, the iteration $F^{\circ n}$ of F is defined by setting $F^{\circ 1} = F$ and $F^{\circ n} = F \circ F^{\circ (n-1)}$ for $n \geq 2$. Also, we put $F^{\circ 0} = \mathrm{id}$, the identity map. We extend F to a self-map of $\boldsymbol{P}^2$ by setting

$$F[z : w : t] = [t^d f_0(z/t, w/t) : t^d f_1(z/t, w/t) : t^d].$$

Clearly, F is a rational map of $\boldsymbol{P}^2$. Set $\tilde{f_0} = t^d f_0(z/t, w/t), \tilde{f_1} = t^d f_1(z/t, w/t)$ and $l_\infty = \{[z : w : t] \in \boldsymbol{P}^2 \mid t = 0\}$. A point p is called a *super attracting fixed point* of F if $F(p) = p$ and the eigenvalues of the differential dF_p of F at p are 0 and a with $|a| < 1$. Define the map $\tilde{F} : \boldsymbol{C}^3 \to \boldsymbol{C}^3$ by $(z, w, t) \mapsto (\tilde{f_0}, \tilde{f_1}, t^d)$. Then we have $\pi \circ \tilde{F} = F \circ \pi$ on $\boldsymbol{C}^3$ except some analytic sets, where $\pi : \boldsymbol{C}^3 - \{0\} \to \boldsymbol{P}^2$ denotes the canonical projection. A point $p \in \boldsymbol{P}^2$ is said to be an *indeterminate point* of F if $\tilde{F}(\tilde{p}) = 0$ for some point $\tilde{p} \in \pi^{-1}(p)$. In general, if p is an indeterminate point of F, then $\bigcap_{N_p} F(N_p)$ is not a singleton, where the intersection is taken over all open neighbourhoods N_p of p. Hence, F is not continuous at such a point p.

THEOREM 2.1. *Assume that F has only one fixed point p_∞ of the line l_∞ at infinity and that all non-indeterminate points on l_∞ are mapped to p_∞ by F. Then, up to a suitable conjugation of projective linear transformation of $\boldsymbol{P}^2$, F can be written in the form $F[z : w : t] = [t\bar{f_0} : \bar{f_1} : t^d]$, where $\bar{f_0}$ is a homogeneous polynomial of degree $d - 1$ and $\bar{f_1}$ has the form $\bar{f_1} = w^d + O(w^{d-1})$ with no term of z^d. In particular, $[0 : 1 : 0]$ is a super attracting fixed point of F and $[1 : 0 : 0]$ is an indeterminate point of F.*

Proof. By a suitable change of the coordinates, we can assume that $p_\infty = [0 : 1 : 0]$. As a result, $\tilde{f_0}$ does not have the term of w^d and $\tilde{f_1}(0, w, 0) = w^d$. By the assumption that all points of l_∞ except indeterminate points are mapped to p_∞ by F, we have $\tilde{f_0} = t\bar{f_0}$, where $\bar{f_0}$ is a homogeneous polynomial with degree $d - 1$. On the other hand, there exist roots α_i of the equation $\tilde{f_1}(1, w, 0) = 0$. Then, $[1 : \alpha_i : 0]$, $i = 1, \ldots, d$, are indeterminate points of F. By a change of the coordinates which fixes $[0 : 1 : 0]$, we can further assume that $[1 : \alpha_1 : 0] = [1 : 0 : 0]$. Then we see that $\tilde{f_1}$ does not have the term of z^d, as required. By a direct calculation, one can check that eigenvalues of the differential of F at $[0 : 1 : 0]$ are 0 and 0. Therefore, $[0 : 1 : 0]$ is a super attracting fixed point of F. $\square$

Notice that Hénon maps $F_{a,c}$ belong to the category of maps in Theorem 2.1. In general, one knows that, for any non-indeterminate point p of l_∞, there is an open neighbourhood N_p of p with $F^{\circ n}(N_p) \to p_\infty$ as $n \to \infty$. (See [7; §6.2].) Hence, to see the dynamical structure near l_∞, it suffices to consider the behaviour of $\{F^{\circ n}\}$ near each indeterminate point on l_∞. In case of Hénon maps $F_{a,c}$, it is known [1] that any unbounded orbits converge to p_∞. However, some F in Theorem 2.1 has an unbounded orbit which converges to an indeterminate point on l_∞. For an example of such a map, we have $F(z,w) = (w, w^2 - azw + c)$ with $|a| > 1$. (For further detail, see [11].) These illustrate that the kinds of dynamical structure of F given by Theorem 2.1 are not unique. We would like to choose a map F in Theorem 2.1 whose all unbounded orbits converge to p_∞.

Throughout this paper, we always assume that F has the form

$$F(z,w) = (w^m, w^d + \Sigma_{mn_1+n_2\le d, n_2<d} a_{n_1n_2} z^{n_1} w^{n_2}), \quad a_{n_1n_2} \in \boldsymbol{C}, \tag{2.1}$$

where m is a fixed integer with $1 \le m < d$ and n_1, n_2 are non-negative integers. Under some additional conditions, it will be shown that all unbounded orbits of F converge to p_∞. (See the remark at the end of this section.)

Before proceeding, we need to introduce some notation and terminology. For positive constants $R_1, R_2 > 0$ and $\varepsilon_0 > 0$ with $R_2^m = R_1 - \varepsilon_0$, we define the sets V_-, D and V_+ by

$$\begin{aligned} V_- &= \{(z,w) \in \boldsymbol{C}^2 \mid |z| > R_1,\ |z| > |w|^m + \varepsilon_0\}, \\ D &= \{(z,w) \in \boldsymbol{C}^2 \mid |z| < R_1,\ |w| < R_2\}, \\ V_+ &= \{(z,w) \in \boldsymbol{C}^2 \mid |w| > R_2,\ |w|^m > |z| - \varepsilon_0\}. \end{aligned} \tag{2.2}$$

Let S be a subset of a given set X. Then we denote by $S^c, \partial S, \mathrm{int}(S)$ and $\bar{S}$ the complement, the boundary, the interior and the closure of the set S in X, respectively. Finally, for a given point $(z,w) \in \boldsymbol{C}^2$, we put

$$(z_n, w_n) = F^{\circ n}(z,w) \quad \text{for } n = 0, 1, 2, \ldots.$$

Then, by our assumption on F, we see that $z_n = (w_{n-1})^m$ for $n \ge 1$.

Proposition 2.2. *Assume that $\Sigma_{mn_1+n_2=d}|a_{n_1n_2}| < 1$. Then, for sufficiently large (resp. small) positive constants R_1 and R_2 (resp. ε_0), we have the following*:

(1) *There exists $\rho > 1$ such that $(z_1, w_1) \in V_+, |w_1| > \rho|w|$ for any $(z,w) \in \overline{V_+}$.*

(2) *For each $(z,w) \in V_-$, we have that $|z_1| < |z| - \varepsilon_0$; and hence $(z_{n_0}, w_{n_0}) \in \overline{V_+ \cup D}$ for some n_0.*

(3) *For every $(z,w) \in \bar{D}$ and for every $n \ge 1$, we have that $F^{\circ n}(z,w) \in \overline{V_+ \cup D}$.*

Proof. (1) For a given point $(z,w) \in \overline{V_+}$,

$$\begin{aligned} |w_1| &= |w^d + \Sigma a_{n_1n_2} z^{n_1} w^{n_2}| \\ &\ge |w|^d - \Sigma|a_{n_1n_2} z^{n_1} w^{n_2}| \\ &\ge |w|^d - \Sigma|a_{n_1n_2}|(|w|^m + \varepsilon_0)^{n_1}|w|^{n_2} \\ &= |w|^d - \Sigma_{mn_1+n_2=d}|a_{n_1n_2} w^{mn_1+n_2}| - \Sigma_{mn_1+n_2<d}|\tilde{a}_{n_1n_2} w^{mn_1+n_2}| \\ &= |w|^d(1 - \Sigma_{mn_1+n_2=d}|a_{n_1n_2}| - \Sigma_{mn_1+n_2<d}|\tilde{a}_{n_1n_2} w^{mn_1+n_2-d}|). \end{aligned}$$

Set $\beta = 1 - \Sigma_{mn_1+n_2=d}|a_{n_1n_2}| > 0$ and $\rho = R_2^{d-1}\beta/2$. By rechoosing a large R_2 and a small ε_0, if necessary, we can assume that $\rho > 1$ and $\Sigma_{mn_1+n_2<d}|\tilde{a}_{n_1n_2}w^{mn_1+n_2-d}| < \beta/2$. Hence, $|w_1| \geq |w|^d\beta/2 \geq |w|R_2^{d-1}\beta/2 \geq \rho|w|$. Moreover, since $|w_1|^m > \rho^m|w|^m = \rho^m|z_1|$,

$$|w_1|^m - |z_1| + \varepsilon_0 \geq \rho^m|z_1| - |z_1| + \varepsilon_0 = (\rho^m - 1)|z_1| + \varepsilon_0 > 0,$$

and so $|w_1|^m > |z_1| - \varepsilon_0$. Combining this with $|w_1| > \rho|w| > \rho R_2 > R_2$, we have that $(z_1, w_1) \in V_+$.

(2) For a given point $(z, w) \in V_-$, we have $|z_1| = |w|^m$ and $|z_1| < |z| - \varepsilon_0$. Similarly, if $(z_l, w_l) \in V_-$ for each $l = 1, \dots, k-1$, then $|z_k| < |z| - k\varepsilon_0$. As a result, for an arbitrary point $(z, w) \in V_-$, there exists a positive integer n_0 such that $(z_{n_0}, w_{n_0}) \in \overline{V_+ \cup D}$.

(3) For a given point $(z, w) \in \bar{D}$, $|z_1| = |w|^m \leq R_2^m = R_1 - \varepsilon_0$ and so $(z_1, w_1) \in \overline{V_+ \cup D}$. Hence, (3) follows immediately from (1). □

In the reminder of this paper, we always assume that $\Sigma_{mn_1+n_2=d}|a_{n_1n_2}| < 1$ *and the constants* R_1, R_2, ε_0 *and* ρ *are chosen as in Proposition* 2.2.

Here we recall quickly the definition of a normal family of maps. Let M be a complex manifold. A sequence of maps $f_\nu : M \to M$, $\nu = 1, 2, \dots$, *diverges locally uniformly to infinity in* M if for any compact subsets K and $\tilde{K}$ of M there is an integer ν_0 such that $f_\nu(K) \cap \tilde{K} = \emptyset$ for $\nu \geq \nu_0$. A collection Γ of self-maps of M is said to be *normal* if every infinite sequence of maps chosen from Γ contains either a subsequence which converges locally uniformly or a subsequence which diverges locally uniformly to infinity in M. Now we can define the *Fatou set* N_+ and the *Julia set* J_+ of F as follows:

$$N_+ = \{x \in \boldsymbol{C}^2 \mid \{F^{\circ n}\} \text{ is a normal family in a neighbourhood of } x\},$$
$$J_+ = \boldsymbol{C}^2 \backslash N_+.$$

As an immediate consequence of the definition, we have the following:

Proposition 2.3. *We have that* $F^{-1}(N_+) \subset N_+$ *and* $F(J_+) \subset J_+$.

Remark. Since not all holomorphic maps in $\boldsymbol{C}^2$ are open, the reverse inclusions do not hold in Proposition 2.3, in general.

In order to characterise N_+ and J_+ we define some sets. Denoting by $\|\cdot\|$ the Euclidean norm on $\boldsymbol{C}^2$, we define

$$A_+ = \{(z, w) \in \boldsymbol{C}^2 \mid \|F^{\circ n}(z, w)\| \to \infty \text{ as } n \to \infty\},$$
$$K_+ = \{(z, w) \in \boldsymbol{C}^2 \mid \{F^{\circ n}(z, w)\}_{n \geq 0} \text{ is bounded}\}.$$

We call A_+ *the set of escaping points* of F and K_+ *the set of non-escaping points* of F.

THEOREM 2.4. *We have the following*:
(1) $A_+ = \bigcup_{n=0}^{\infty} F^{-n}(V_+)$ *and* A_+ *is an open subset of* $\boldsymbol{C}^2$.
(2) $K_+ = \bigcap_{n=0}^{\infty} F^{-n}(\boldsymbol{C}^2 \backslash \overline{V_+})$.
(3) $\boldsymbol{C}^2 = A_+ \cup K_+$ *and* K_+ *is a closed subset of* $\boldsymbol{C}^2$.
(4) $\partial K_+ = J_+$.

Proof. By Proposition 2.2, for an arbitrary point $(z,w) \in \boldsymbol{C}^2$, there exists a positive integer n_0 such that $F^{\circ n}(z,w) \in \overline{V_+ \cup D}$ for $n \geq n_0$. Together with (1) of Proposition 2.2, this implies that either $\|F^{\circ n}(z,w)\| \to \infty$ as $n \to \infty$ or $\{F^{\circ n}(z,w)\}$ is bounded. From these facts we have (1), (2) and (3).

(4) For an arbitrary point $x_0 \in \operatorname{int}(K_+)$, one can choose a small open ball $B_\varepsilon(x_0) := \{x \in \boldsymbol{C}^2 \mid \|x - x_0\| < \varepsilon\}$ in such a way that $B_\varepsilon(x_0) \subset \operatorname{int}(K_+)$. By (2) of Proposition 2.2, there exists some integer n_0 such that $F^{\circ n}(B_\varepsilon(x_0)) \subset D$ for all $n \geq n_0$. Consequently, $\{F^{\circ n}\}$ is a normal family in $\operatorname{int}(K_+)$. On the other hand, it is clear that $\{F^{\circ n}\}$ diverges to infinity on A_+ and $A_+ \subset N_+$. As a result, $\partial K_+ \supset J_+$. The reverse inclusion is clear. □

Remark. By the proof of Proposition 2.2, for $(z,w) \in A_+$, we see that $|w_{n+1}| \geq |w_n|^d \beta/2$ for all sufficiently large n. Hence, $|w_{n+1}|/|z_{n+1}| = |w_{n+1}|/|w_n|^m \geq |w_n|^{d-m}\beta/2 \to \infty$ as $n \to \infty$, by $d > m$. As a result,

$$A_+ = \{(z,w) \in \boldsymbol{C}^2 \mid F^{\circ n}(z,w) = F^{\circ n}([z:w:1]) \to p_\infty = [0:1:0] \text{ as } n \to \infty\}.$$

In particular, it is reasonable that one calls A_+ the *attracting basin of* p_∞ of F. Moreover, we see that $N_+ = \tilde{N}_+ \cap \{[z:w:t] \in \boldsymbol{P}^2 \mid t \neq 0\}$, where we have set

$$\tilde{N}_+ = \{p \in \boldsymbol{P}^2 \mid \{F^{\circ n}\} \text{ is a normal family in a neighbourhood of } p\}.$$

3. The slice of Julia set for F

We define the functions $f_n(z,w), g_n(z,w)$ and $g_{z_0}^n(w)$ for fixed $z_0 \in \boldsymbol{C}$, by setting

$$(f_n(z,w), g_n(z,w)) = F^{\circ n}(z,w), \quad g_{z_0}^n(w) = g_n(z_0,w) \quad \text{for } n = 0,1,2,\ldots.$$

Since F has the form as in (2.1), it follows that $f_n(z,w) = (g_{n-1}(z,w))^m$ for $n \geq 1$, and $g_{z_0}^n(w)$ is a polynomial in w of degree d^n. The proof of the following proposition is similar to that of Proposition 2.2 and hence is left to the reader:

PROPOSITION 3.1. *Let* $z_0 \in \boldsymbol{C}$ *be an arbitrary point with* $|z_0| \leq R_1$. *Assume that there exist some point* w *and a non-negative integer* n_0 *such that* $|g_{z_0}^n(w)| < R_2$ *for every* n, $0 \leq n < n_0$, *and* $|g_{z_0}^{n_0}(w)| \geq R_2$. *Then* $|g_{z_0}^{l+1}(w)| > \rho |g_{z_0}^l(w)|$ *for all* $l \geq n_0$.

Now let us set

$$A_{z_0} = \{w \in \hat{\boldsymbol{C}} \mid |g_{z_0}^n(w)| \to \infty \text{ as } n \to \infty\}, \quad K_{z_0} = \{w \in \hat{\boldsymbol{C}} \mid \{g_{z_0}^n(w)\} \text{ is bounded}\},$$

$$N_{z_0} = \{w \in \hat{\boldsymbol{C}} \mid \{g_{z_0}^n(w)\} \text{ is a normal family in a neighbourhood of } w\},$$

$$J_{z_0} = \hat{\boldsymbol{C}} \backslash N_{z_0}.$$

We call N_{z_0} and J_{z_0} the *Fatou set* and the *Julia set* of $\{g_{z_0}^n\}$, respectively. Note that N_{z_0} is open and J_{z_0} is closed in $\hat{\boldsymbol{C}}$, respectively. In the proofs of Proposition 3.2, Theorems 3.4 and 3.5, we will set $E = \{w \in \hat{\boldsymbol{C}} \mid |w| > R_2\}$.

Proposition 3.2. *Assume that* $|z_0| \le R_1$. *Then we have the following*:
(1) $A_{z_0} \cup K_{z_0} = \hat{\boldsymbol{C}}$ *and* $A_{z_0} \cap K_{z_0} = \emptyset$.
(2) A_{z_0} *is an open connected subset of* $\hat{\boldsymbol{C}}$ *and* K_{z_0} *is a non-empty compact subset of* $\boldsymbol{C} \subset \hat{\boldsymbol{C}}$.
(3) $\partial K_{z_0} = J_{z_0}$.
(4) *The connected components of the Fatou set* N_{z_0} *except* A_{z_0} *are simply connected.*

Proof. Using Proposition 3.1, we can check the assertion (1).

(2) We have now $A_{z_0} = \bigcup_{n=1}^{\infty} (g_{z_0}^n)^{-1}(E)$, A_{z_0} is open and K_{z_0} is a compact subset of $\hat{\boldsymbol{C}}$ contained in $\overline{\Delta(R_2)}$. Here we assert that $(g_{z_0}^n)^{-1}(E)$ is connected. Indeed, we assume that $(g_{z_0}^n)^{-1}(E)$ have a connected component which does not contain E. Notice that this component is a bounded set. On the other hand, $g_{z_0}^n$ is a holomorphic map from each connected component of $(g_{z_0}^n)^{-1}(E)$ onto E. Then, by the maximum modulus principle we obtain a contradiction, proving our assertion. Together with the fact that $\{(g_{z_0}^n)^{-1}(E)\}$ is an increasing sequence, A_{z_0} is connected. Next, we assume that K_{z_0} is empty. Then $\hat{\boldsymbol{C}} = A_{z_0}$; and hence, there exists n such that $g_{z_0}^n(E^c) \subset E$ and $g_{z_0}^n(E) \subset E$, which means that $|g_{z_0}^n(w)| \ge R_2$ on $\boldsymbol{C}$. This contradicts the fact that $g_{z_0}^n(w)$ is a polynomial with degree d^n (and so it has a zero in $\boldsymbol{C}$).

The proof of (3) is the same as that of (4) of Theorem 2.4.

(4) Since A_{z_0} is connected by (2), so is $A_{z_0} \cup J_{z_0}$. Therefore, each connected component of $(A_{z_0} \cup J_{z_0})^c$ is simply connected. □

As an immediate consequence of Proposition 3.2, we have the following:

Corollary 3.3. *For an arbitrarily given point* $z_0 \in \boldsymbol{C}$, *it follows that*

$$K_+ \cap \{(z, w) \in \boldsymbol{C}^2 \mid z = z_0\} = \{z_0\} \times K_{z_0} \quad \textit{and} \quad K_{z_0} \ne \emptyset.$$

Using the notation $g'(w) = dg(w)/dw$ for a given holomorphic function $g(w)$, we set

$$\tilde{C}_n(z_0) = \{w \in \boldsymbol{C} \mid (g_{z_0}^n)'(w) = 0\}, \quad \tilde{C}(z_0) = \bigcup_{n=1}^{\infty} \tilde{C}_n(z_0), \quad \tilde{C} = \bigcup_{|z_0| \le R_1} \tilde{C}(z_0).$$

Next, we discuss the connectivity of J_{z_0} in Theorems 3.4 and 3.5. Especially, one can see that J_{z_0} is just like Julia set of a polynomial maps in $\boldsymbol{C}$.

Theorem 3.4. *If* $\tilde{C}(z_0) \subset K_{z_0}$, *then* A_{z_0} *is simply connected and* J_{z_0} *is connected.*

Proof. Since $\tilde{C}(z_0) \subset K_{z_0}$, we see that $\{(g_{z_0}^n)^{-1}(E)\}$ is an increasing sequence of simply connected sets; and hence, $A_{z_0} = \bigcup_{n=1}^{\infty}(g_{z_0}^n)^{-1}(E)$ is simply connected. Moreover, since $\partial A_{z_0} = J_{z_0}$, J_{z_0} is connected. □

From now on, we set $\Delta(w_0, r) = \{w \in \boldsymbol{C} \mid |w - w_0| < r\}$ and $\Delta(r) = \Delta(0, r)$.

THEOREM 3.5. *If $\overline{\tilde{C}(z_0)} \subset A_{z_0}$, then J_{z_0} is a totally disconnected set.*

Proof. Set $\tilde{E}_n = (g_{z_0}^n)^{-1}(E)$ and $U_n = (g_{z_0}^n)^{-1}(\overline{\Delta(R_2)})$. Since $\{\tilde{E}_n\}$ is an increasing open covering of A_{z_0} and since $\overline{\tilde{C}(z_0)}$ is a compact subset of $\hat{\boldsymbol{C}}$ contained in A_{z_0}, there is an integer n_0 such that, for any $n \geq n_0$,

$\tilde{E}_n \supset \tilde{C}(z_0)$ and U_n has d^n connected components which are all simply connected.

We denote the connected components of U_n by $U_n^{i_n}$, $i_n = 1, 2, \ldots, d^n$. By $g_{z_0}^n(\tilde{C}(z_0)) \subset E$, there are holomorphic maps $h_n^{i_n} : \overline{\Delta(R_2)} \to U_n^{i_n}$, $i_n = 1, \ldots, d^n$, that invert $g_{z_0}^n$ and $h_n^{i_n}(\overline{\Delta(R_2)}) = U_n^{i_n}$. Using the assertion of Proposition 3.1, we see that $h_{n+1}^{i_{n+1}}(\overline{\Delta(R_2)}) = U_{n+1}^{i_{n+1}} \subset U_n$. Here, let us consider the set Γ consisting of all sequences $\{i_n\} = \{i_1, i_2, \ldots\}$ with $i_n \in \{1, \ldots, d^n\}$, $n = 1, 2, \ldots$. Then

$$K_{z_0} = \bigcap_{n=n_0}^{\infty} (g_{z_0}^n)^{-1}(\overline{\Delta(R_2)}) \subset \bigcup_{\{i_n\} \in \Gamma} \bigcap_{n=n_0}^{\infty} h_n^{i_n}(\overline{\Delta(R_2)}) = \bigcup_{\{i_n\} \in \Gamma} \bigcap_{n=n_0}^{\infty} U_n^{i_n}.$$

Observe that, for each sequence $\{i_n\} \in \Gamma$, there are two possibilities as follows:

$$U_n^{i_n} \supset U_{n+1}^{i_{n+1}} \text{ for all } n \geq n_0; \quad \text{or} \tag{3.1}$$

$$U_k^{i_k} \cap U_{k+1}^{i_{k+1}} = \emptyset \text{ for some integer } k \geq n_0 \text{ and so } \bigcap_{n=n_0}^{\infty} U_n^{i_n} = \emptyset.$$

Thus, in order to prove the theorem, it is enough to show that $\operatorname{diam}(U_n^{i_n})$, the diameter of $U_n^{i_n}$, converges to 0 as $n \to \infty$. To this end, we first assert that for $n \geq n_0$

(3.2) there is a positive constant R_2' such that $R_2' < R_2$ and $g_{z_0}^n(U_{n+1}^{i_{n+1}}) \subset \Delta(R_2')$.

Indeed, assume the contrary. Then, passing to a subsequence if necessary, one can find a sequence $\{y_n\}$ of points $y_n \in U_{n+1}^{i_{n+1}}$ and a sequence $\{R_{y_n}\}$ of positive constants such that $R_{y_n} \uparrow R_2$ and $|g_{z_0}^n(y_n)| \geq R_{y_n}$. On the other hand, choose a positive constants R_2' (resp. ρ') sufficiently close to the constants R_2 (resp. ρ) such that $R_2' < R_2$ and $R_2 < \rho' R_2'$. Then, in exactly the same way as in the proof of Proposition 2.2 (1), it can be shown that, if $|g_{z_0}^n(w)| \geq R_2'$ for some point w, then $|g_{z_0}^{n+1}(w)| \geq \rho' |g_{z_0}^n(w)|$. Since R_{y_n} converges to R_2, we have now some y_n and R_{y_n} with $R_2' < R_{y_n} < R_2$, $|g_{z_0}^n(y_n)| \geq R_{y_n}$. Then $|g_{z_0}^{n+1}(y_n)| \geq \rho' |g_{z_0}^n(y_n)| \geq \rho' R_2' > R_2$. This contradicts the fact that $y_n \in U_{n+1}^{i_{n+1}}$, proving (3.2). Here, we fix arbitrary sequences $\{i_n\}, \{U_n^{i_n}\}$ and $\{h_n^{i_n}\}$ satisfying (3.1). Then, by (3.2), it follows that $U_{n+1}^{i_{n+1}} \subset h_n^{i_n}(\overline{\Delta(R_2')})$ for all $n \geq n_0$. We are now in position to apply the following lemma, by setting $K_n = U_n^{i_n}$, $\Phi_n = h_n^{i_n}$, $V = \Delta(R_2)$ and $L = \overline{\Delta(R_2')}$:

Lemma 3.6 ([7; Lemma 6.3.7]). *Let $\{K_n\}$ be a decreasing sequence of compact subsets of $\boldsymbol{C}$. Suppose that there exist a domain $V \subset \boldsymbol{C}$, a compact set $L \subset V$ and a sequence of holomorphic maps $\Phi_n : V \to \boldsymbol{C}$ such that $K_n \supset \Phi_n(V)$ and $\Phi_n(L) \supset K_{n+1}$ for all n. Then* $\operatorname{diam}(K_n) \to 0$ *as $n \to \infty$ and $\bigcap_{n=1}^{\infty} K_n$ consists of a single point.*

Therefore, we conclude that $\operatorname{diam}(U_n^{i_n}) \to 0$ as $n \to \infty$, and hence the proof of the theorem is completed. □

We set $J_+(R_1) = J_+ \cap \{(z, w) \in \boldsymbol{C}^2 \mid |z| \leq R_1\}$. Then it is easy to see the following theorem, which states the relation between J_+ and J_{z_0}.

Theorem 3.7. $J_+(R_1) \supset \overline{\bigcup_{|z_0| \leq R_1} \{z_0\} \times J_{z_0}}$.

4. The Lebesgue measure of Julia set

We start with the following:

Definition 4.1. The set X *is foliated by the leaves* $\{l_c\}_{c \in C}$ if
(1) $X = \bigcup_{c \in C} l_c$; and
(2) $l_c \cap l_{c'} = \emptyset$ for any $c, c' \in C$ with $c \neq c'$.

In the following, we wish to show that $J_+(R_1)$ can be foliated by the graphs of holomorphic functions. To this end, we need the following:

Definition 4.2. We say that F *satisfies the condition* $(\mathscr{F})$ if the following holds:

$(\mathscr{F})$ There exist a constant $\tilde{R}_2 > R_2$ and a sequence $\{n_j\}$ of positive integers such that $|g_{z_0}^{n_j}(w)| \neq \tilde{R}_2$ on $\tilde{C}(z_0)$ for each n_j and $|z_0| \leq R_1'$, where R_1' is an arbitrary constant with $R_1 < R_1' < \tilde{R}_1 := \tilde{R}_2^m + \varepsilon_0$.

In the following part of this paper, we always denote by $R_1', \tilde{R}_1, \tilde{R}_2$ and $\{n_j\}$ the same objects as in Definition 4.2.

Now, in the paper [4], Fornæss and Sibony proved that if a Hénon map $F_{a,c}$ satisfies some conditions as well as the condition $(\mathscr{F})$, then its Julia set is foliated by complex submanifolds described as the graphs of holomorphic functions. By improving their method, one can obtain more general results. In fact, just under the condition $(\mathscr{F})$, we can show that $J_+(R_1)$ can be foliated by the graphs of holomorphic functions:

THEOREM 4.3. *Let z_0 be an arbitrary point of $\boldsymbol{C}$ with $|z_0| \le R_1$. Assume that F satisfies the condition $(\mathscr{F})$. Then we have the following*:

(1) *$J_+(R_1)$ is foliated by the leaves $\{l_{w_0}\}_{w_0 \in J_{z_0}}$, where each leaf l_{w_0} can be expressed as $l_{w_0} = \{(z,w) \in \boldsymbol{C}^2 \mid w = \psi_{w_0}(z), |z| \le R_1\}$ by a holomorphic function ψ_{w_0} on $\overline{\Delta(R_1)}$ with $w_0 = \psi_{w_0}(z_0)$. In particular, the leaf which contains (z_0, w_0) is uniquely determined.*

(2) *$\{\psi_{w_0}\}_{w_0 \in J_{z_0}}$ is equicontinuous on $\overline{\Delta(R_1)}$ and, for every $\varepsilon > 0$, a point $w_0 \in J_{z_0}$, there is an open neighbourhood U_{w_0} of w_0 such that $|\psi_{w_0}(z) - \psi_{\tilde{w}}(z)| < \varepsilon$ for all $\tilde{w} \in U_{w_0} \cap J_{z_0}$ and for all $z \in \overline{\Delta(R_1)}$.*

The proof of this theorem will be preceded by several lemmas. First, for a given c with $|c| = \tilde{R}_2$ we set

$$S_n = \{(z,w) \in \boldsymbol{C}^2 \mid |g_n(z,w)| = \tilde{R}_2,\ |z| \le R_1\};$$
$$\tilde{S}_n = \{(z,w) \in \boldsymbol{C}^2 \mid |g_n(z,w)| = \tilde{R}_2,\ |z| < R_1'\};$$
$$l_c^n = \{(z,w) \in \boldsymbol{C}^2 \mid g_n(z,w) = c,\ |z| \le R_1\};$$
$$\tilde{l}_c^n = \{(z,w) \in \boldsymbol{C}^2 \mid g_n(z,w) = c,\ |z| < R_1'\}.$$

LEMMA 4.4. *S_{n_j} and $\tilde{S}_{n_j}$ have the structure of foliation.*

Proof. From the condition $(\mathscr{F})$ we see that $|(g_{z_0}^{n_j})'(w_0)| \ne 0$ for every $(z_0, w_0) \in \tilde{S}_{n_j}$. Hence, by the implicit function theorem it follows that there are holomorphic functions $\psi_{c_k}^{n_j}$, $k = 1, \dots, d^{n_j}$, defined on $\Delta(R_1')$ for every c with $|c| = \tilde{R}_2$ such that

$$l_c^{n_j} = \bigcup_{k=1}^{d^{n_j}} \{(z,w) \in \boldsymbol{C}^2 \mid w = \psi_{c_k}^{n_j}(z),\ z \in \overline{\Delta(R_1)}\};$$

$$\tilde{l}_c^{n_j} = \bigcup_{k=1}^{d^{n_j}} \{(z,w) \in \boldsymbol{C}^2 \mid w = \psi_{c_k}^{n_j}(z),\ z \in \Delta(R_1')\}.$$

Moreover, for $k = 1, \dots, d^{n_j}$ setting

$$l_{c_k}^{n_j} = \{(z,w) \in \boldsymbol{C}^2 \mid w = \psi_{c_k}^{n_j}(z),\ z \in \overline{\Delta(R_1)}\},$$
$$\tilde{l}_{c_k}^{n_j} = \{(z,w) \in \boldsymbol{C}^2 \mid w = \psi_{c_k}^{n_j}(z),\ z \in \Delta(R_1')\},$$

we see that $l_c^{n_j}$ (resp. $\tilde{l}_c^{n_j}$) has the structure of foliation with leaves $\{l_{c_k}^{n_j}\}_{k=1}^{d^{n_j}}$ (resp. $\{\tilde{l}_{c_k}^{n_j}\}_{k=1}^{d^{n_j}}$). Therefore, $S_{n_j} = \bigcup_{|c|=\tilde{R}_2} l_c^{n_j}$ (resp. $\tilde{S}_{n_j} = \bigcup_{|c|=\tilde{R}_2} \tilde{l}_c^{n_j}$) can be foliated by the leaves $l_{c_k}^{n_j}$ (resp. $\tilde{l}_{c_k}^{n_j}$). □

Next, let us recall the Hausdorff metric. Let X be a complete metric space and $H(X)$ the space of non-empty compact subsets of X. Then $H(X)$ is a complete metric space with respect to the Hausdorff metric d_H defined as follows:

$$d_H(A,B) = \max\left\{\sup_{x \in A} \inf_{y \in B} d(x,y),\ \sup_{y \in B} \inf_{x \in A} d(x,y)\right\} \quad \text{for } A, B \in H(X),$$

where $d(\cdot,\cdot)$ denotes the metric on X. Considering the special case of $X = \boldsymbol{C}^2$ with the Euclidean distance, we have now the following:

LEMMA 4.5. $J_+(R_1) = \lim_{j\to\infty} S_{n_j}$.

Proof. For the given constants $\tilde{R}_1$ and $\tilde{R}_2$ as in Definition 4.2, we define the sets $\tilde{V}_-, \tilde{D}$ and $\tilde{V}_+$ by replacing R_1, R_2 by $\tilde{R}_1, \tilde{R}_2$ in (2.2). Then, all the results obtained in section 2 hold for $\tilde{V}_-, \tilde{D}$ and $\tilde{V}_+$. In particular, we have that

$$l_c^n \subset S_n \subset \tilde{S}_n \subset \overline{\Delta(\tilde{R}_1)} \times \overline{\Delta(\tilde{R}_2)} \quad \text{for all } n$$

by the proof of Proposition 2.2 and

$$\text{(4.1)} \quad A_+ = \bigcup_{n=0}^{\infty} F^{-n}(\tilde{V}_+) \text{ and } \{F^{-n}(\tilde{V}_+)\} \text{ is an increasing open covering of } A_+.$$

First, we assert that: For any small $\varepsilon > 0$, there exists an integer n_0 such that

$$\text{(4.2)} \qquad \sup_{y\in S_n} \inf_{x\in J_+(R_1)} \|x-y\| < \varepsilon \quad \text{for } n \geq n_0.$$

Indeed, consider the open covering $\{B_{\varepsilon/2}(x)\}_{x\in J_+(R_1)}$ of $J_+(R_1)$ and set $U = \bigcup_{x\in J_+(R_1)} B_{\varepsilon/2}(x)$, where $B_{\varepsilon/2}(x)$ stands for the open $\varepsilon/2$-ball with centered at x. Then U is an open neighbourhood of $J_+(R_1)$ and, without loss of generality, we may assume that $U \subset \{(z,w)\in \boldsymbol{C}^2 \mid |w| \leq \tilde{R}_2\}$, because $J_+(R_1) \subset \{(z,w)\in \boldsymbol{C}^2 \mid |w| \leq R_2\}$ and $R_2 < \tilde{R}_2$. Since $J_+ = \partial A_+$ by Theorem 2.4, there are compact subsets V_1, V_2 of $\boldsymbol{C}^2$ such that $\{(z,w)\in \boldsymbol{C}^2 \mid |z| \leq R_1,\ |w| \leq \tilde{R}_2\} \cap U^c = V_1 \cup V_2$, $V_1 \subset A_+$ and $V_2 \subset \operatorname{int}(K_+)$. Therefore, $A_+ \cap \{(z,w)\in \boldsymbol{C}^2 \mid |z| \leq R_1, |w| \leq \tilde{R}_2\} \cap U^c = V_1$ is a compact subset of A_+. Thus, it follows from (4.1) that there is an integer n_0 such that

$$F^{-n}(\tilde{V}_+) \supset A_+ \cap \{(z,w)\in \boldsymbol{C}^2 \mid |z| \leq R_1,\ |w| \leq \tilde{R}_2\} \cap U^c \quad \text{for } n \geq n_0.$$

Moreover, by (4.1) we see that

$$\{(z,w)\in \boldsymbol{C}^2 \mid |z| \leq R_1,\ |w| \geq \tilde{R}_2\} \subset \tilde{V}_+ \subset F^{-n}(\tilde{V}_+).$$

Thus

$$\partial[F^{-n}(\tilde{V}_+)] \cap \{(z,w)\in \boldsymbol{C}^2 \mid |z| \leq R_1\} \subset [A_+ \cap \{(z,w)\in \boldsymbol{C}^2 \mid |z| \leq R_1\} \cap U^c]^c,$$

which implies that

$$\partial[F^{-n}(\tilde{V}_+)] \cap \{(z,w)\in \boldsymbol{C}^2 \mid |z| \leq R_1\} \subset U \quad \text{for every } n \geq n_0.$$

Here, we assert that

$$\text{(4.3)} \qquad \begin{aligned} &\partial[F^{-n}(\tilde{V}_+)] \cap \{(z,w)\in \boldsymbol{C}^2 \mid |z| < \tilde{R}_1\} \\ &\quad = \{(z,w)\in \boldsymbol{C}^2 \mid |w_n| = \tilde{R}_2, |z_n| < \tilde{R}_1,\ |z| < \tilde{R}_1\}. \end{aligned}$$

Indeed, it is clear from the continuity of $F^{\circ n}$ that $\partial F^{-n}(\tilde{V}_+) \subset F^{-n}(\partial \tilde{V}_+)$. By (3) of Proposition 2.2, we know that for $(z_0, w_0) \in \tilde{D}$, if there exists some integer n_0

such that $(z_n, w_n) \in \tilde{D}$ $(1 \leq n < n_0)$ and $(z_{n_0}, w_{n_0}) \in \overline{\tilde{V}_+}$, then $|z_{n_0}| < \tilde{R}_1, |w_{n_0}| = \tilde{R}_2$ and $(z_l, w_l) \in V_+$ for all $l > n_0$. Therefore,

$$F^{-n}(\partial \tilde{V}_+) \cap \{(z,w) \in \boldsymbol{C}^2 \mid |z| < \tilde{R}_1\} \subset \{(z,w) \in \boldsymbol{C}^2 \mid |w_n| = \tilde{R}_2, |z_n| < \tilde{R}_1, |z| < \tilde{R}_1\};$$

$$\partial[F^{-n}(\tilde{V}_+)] \cap \{(z,w) \in \boldsymbol{C}^2 \mid |z| < \tilde{R}_1\} \subset \{(z,w) \in \boldsymbol{C}^2 \mid |w_n| = \tilde{R}_2, |z_n| < \tilde{R}_1, |z| < \tilde{R}_1\}.$$

To show the reverse inclusion we assume that there is a point $(z_0, w_0) \notin \{\partial[F^{-n}(\tilde{V}_+)] \cap \{(z,w) \in \boldsymbol{C}^2 \mid |z| < \tilde{R}_1\}\}$ such that $|w_n| = \tilde{R}_2$, $|z_n| < \tilde{R}_1$, $|z_0| < \tilde{R}_1$. Then, it follows that $(z_0, w_0) \in \{\overline{F^{-n}(\tilde{V}_+)}\}^c \cap \{(z,w) \in \boldsymbol{C}^2 \mid |z| < \tilde{R}_1\}$; and hence, there exists some open neighbourhood U_0 of (z_0, w_0) such that $U_0 \subset \tilde{D}$, $F^{\circ n}(U_0) \cap \tilde{V}_+ = \emptyset$. Moreover, for every $(z,w) \in U_0$ we have that $|f_n(z,w)| < \tilde{R}_1$, $|g_n(z,w)| \leq \tilde{R}_2$. Choose here a disk $\Delta(w_0, \delta)$ in $\boldsymbol{C}$ in such a way that $\{z_0\} \times \Delta(w_0, \delta) \subset U_0$ and we have that $|g_{z_0}^n(w)| \leq \tilde{R}_2$ on $\Delta(w_0, \delta)$ and $|g_{z_0}^n(w_0)| = \tilde{R}_2$. By the maximum modulus principle, we obtain a contradiction, proving (4.3). Hence, we obtain that

$$\begin{aligned} S_n &= \{(z,w) \in \boldsymbol{C}^2 \mid |g_n(z,w)| = |w_n| = \tilde{R}_2,\ |z| \leq R_1\} \\ &= \{(z,w) \in \boldsymbol{C}^2 \mid |w_n| = \tilde{R}_2, |z_n| < \tilde{R}_1,\ |z| \leq R_1\} \\ &= \partial[F^{-n}(\tilde{V}_+)] \cap \{(z,w) \in \boldsymbol{C}^2 \mid |z| \leq R_1\} \subset U. \end{aligned}$$

This completes the proof of (4.2).

Next, we show the following: For any small $\varepsilon > 0$, there is an integer j_0 such that

$$\sup_{x \in J_+(R_1)} \inf_{y \in S_{n_j}} \|x - y\| < \varepsilon \quad \text{for } j \geq j_0. \tag{4.4}$$

Assume the assertion were false. Then, there would be a positive constant ε', subsequences $\{x_{n_j}\}$, $\{y_{n_j}\}$ with $x_{n_j} \in J_+(R_1)$, $y_{n_j} \in S_{n_j}$ and points $x_\infty \in J_+(R_1)$, $y_\infty \in \boldsymbol{C}^2$ such that

$$\begin{gathered} x_{n_j} \to x_\infty,\ y_{n_j} \to y_\infty \ \ (j \to \infty) \quad \text{and} \\ \inf_{y \in S_{n_j}} \|x_{n_j} - y\| = \|x_{n_j} - y_{n_j}\| > \varepsilon' \ \text{ for every } j, \end{gathered} \tag{4.5}$$

since $J_+(R_1)$ is compact and $\{y_{n_j}\}$ is bounded. On the other hand, since $x_\infty \in J_+(R_1) \subset \partial A_+$, we see that $B(x_\infty, \varepsilon'/k) \cap A_+ \neq \emptyset$ whenever k is a positive constant. It follows then from (4.1) that

$$B(x_\infty, \varepsilon'/k) \cap F^{-n}(\tilde{V}_+) \neq \emptyset,$$

$$B(x_\infty, \varepsilon'/k) \cap \partial[F^{-n}(\tilde{V}_+)] \cap \{(z,w) \in \boldsymbol{C}^2 \mid |z| \leq R_1'\} \neq \emptyset,$$

$$B(x_\infty, \varepsilon'/k) \cap \tilde{S}_{n_j} \neq \emptyset \quad \text{for all sufficiently large } n \text{ and } j.$$

In particular, for an arbitrarily given sequence $\{k_{n_j}\} \subset \boldsymbol{N}$ with $R_1 + \varepsilon'/k_{n_j} < R_1'$ and $k_{n_j} \uparrow \infty$, there exist points $\tilde{y}_{n_j} \in B(x_\infty, \varepsilon'/k_{n_j}) \cap \tilde{S}_{n_j}$ for all sufficiently large j.

Let us denote by $\tilde{l}_{n_j} = \{(z,w) \in \boldsymbol{C}^2 \mid w = \psi_{n_j}(z),\ z \in \Delta(R_1')\}$ the leaf of $\tilde{S}_{n_j}$ passing through the point $\tilde{y}_{n_j}$ as defined in the proof of Lemma 4.4, and write $\tilde{y}_{n_j} = (\tilde{y}^1_{n_j}, \tilde{y}^2_{n_j})$, $x_\infty = (x^1_\infty, x^2_\infty)$ with coordinates. Then $\{\psi_{n_j}\}$ is uniformly bounded and equicontinuous on $\Delta(R_1')$, because $\tilde{S}_{n_j} \subset \overline{\Delta(R_1')} \times \overline{\Delta(\tilde{R}_2)}$. Now we assert the following:

$$S_{n_j} \cap B(x_\infty, \varepsilon'/2) \neq \emptyset \quad \text{for all sufficiently large } j. \tag{4.6}$$

Indeed, if $|x^1_\infty| < R_1$, then $|\tilde{y}^1_{n_j}| < R_1$ for large j. Thus $\tilde{y}_{n_j} \in S_{n_j} \cap B(x_\infty, \varepsilon'/2)$. So we have only to consider the case of $|x^1_\infty| = R_1$. Assume the contrary that (4.6) were false. Then we have

$$(x^1_\infty, \psi_{n_j}(x^1_\infty)) \notin B(x_\infty, \varepsilon'/2) \quad \text{and} \quad (\tilde{y}^1_{n_j}, \psi_{n_j}(\tilde{y}^1_{n_j})) = \tilde{y}_{n_j} \in B(x_\infty, \varepsilon'/k_{n_j})$$

for all large j with $R_1 < |\tilde{y}^1_{n_j}| < R_1'$. This contradicts the equicontinuity of $\{\psi_{n_j}\}$ at $z = x^1_\infty$, proving (4.6). On the other hand, it is clear that (4.6) contradicts (4.5). Therefore, we have shown the assertion (4.4); and hence the proof of Lemma 4.5 is completed. □

Proof of (1) *of Theorem* 4.3. By Lemma 4.5, for an arbitrarily given point $(z_0, w_0) \in J_+(R_1)$, there is a sequence $\{(x_{n_j}, y_{n_j})\}$ such that $(x_{n_j}, y_{n_j}) \in S_{n_j}$ and $(x_{n_j}, y_{n_j}) \to (z_0, w_0)$ as $j \to \infty$. We denote the leaf of S_{n_j} containing the point (x_{n_j}, y_{n_j}) by $l_{n_j} = \{(z,w) \in \boldsymbol{C}^2 \mid w = \psi_{n_j}(z), z \in \overline{\Delta(R_1)}\}$, where ψ_{n_j} is a holomorphic function on $\Delta(R_1')$ with $y_{n_j} = \psi_{n_j}(x_{n_j})$ as in the proof of Lemma 4.5. Then $\{\psi_{n_j}\}$ is normal on $\Delta(R_1')$, since it is bounded uniformly on it. Hence, we may assume that some subsequence $\{\psi_{n_{j_k}}\}$ of $\{\psi_{n_j}\}$ converges to a holomorphic function ψ_{w_0} on $\Delta(R_1')$ uniformly on $\overline{\Delta(R_1)}$. Setting

$$l_{w_0} = \{(z,w) \in \boldsymbol{C}^2 \mid w = \psi_{w_0}(z),\ z \in \overline{\Delta(R_1)}\},$$

we see that $(z_0, w_0) \in l_{w_0} \subset J_+(R_1)$ by Lemma 4.5. Here, we claim that the leaf l_{w_0} containing the point (z_0, w_0) is unique. Consider another sequence of points $(\tilde{x}_{n_j}, \tilde{y}_{n_j}) \in S_{n_j}$ with $(\tilde{x}_{n_j}, \tilde{y}_{n_j}) \to (z_0, w_0)$ as $j \to \infty$ and denote the leaf of S_{n_j} containing $(\tilde{x}_{n_j}, \tilde{y}_{n_j})$ by $\tilde{l}_{n_j} = \{(z,w) \in \boldsymbol{C}^2 \mid w = \tilde{\psi}_{n_j}(z),\ z \in \overline{\Delta(R_1)}\}$. By the same reasoning as above, one can assume that there is a subsequence $\{\tilde{\psi}_{n_{j_k}}\}$ of $\{\tilde{\psi}_{n_j}\}$ which converges to some holomorphic function $\tilde{\psi}_{w_0}$ on $\Delta(R_1')$ uniformly on $\overline{\Delta(R_1)}$. Once it is shown that $\tilde{\psi}_{w_0} = \psi_{w_0}$ on $\overline{\Delta(R_1)}$, then the function ψ_{w_0} is independent of the choice of a sequence $\{(x_{n_j}, y_{n_j})\}$ converging to (z_0, w_0); and hence the leaf l_{w_0} is unique. Therefore, we have only to prove that $\tilde{\psi}_{w_0} = \psi_{w_0}$. To this end, let us set $\psi_k = \tilde{\psi}_{n_{j_k}} - \psi_{n_{j_k}}$. Then $\{\psi_k\}$ converges to the function $\psi := \tilde{\psi}_{w_0} - \psi_{w_0}$ uniformly on $\overline{\Delta(R_1)}$. If there are infinitely many integers k such that $\psi_k \equiv 0$, then $\psi(z) \equiv 0$. So we may assume that all ψ_k are nowhere vanishing on $\overline{\Delta(R_1)}$ by Lemma 4.4. Since $\psi(z_0) = \tilde{\psi}_{w_0}(z_0) - \psi_{w_0}(z_0) = w_0 - w_0 = 0$, Hurwitz's theorem implies that $\psi(z) \equiv 0$, as desired.

Proof of (2) *of Theorem* 4.3. For given the point z_0 with $|z_0| < R_1$, we consider the family of holomorphic functions $\{\psi_{w_0}\}_{w_0 \in J_{z_0}}$ on $\Delta(R_1')$ which define the leaves $\{l_{w_0}\}_{w_0 \in J_{z_0}}$ of $J_+(R_1)$. Since $\{\psi_{w_0}\}_{w_0 \in J_{z_0}}$ is bounded uniformly on $\Delta(R_1')$, we know that $\{\psi_{w_0}\}_{w_0 \in J_{z_0}}$ is normal and equicontinuous on $\Delta(R_1')$. In particular, there exists a subsequence $\{\psi_{\tilde{w}_k}\}_{\tilde{w}_k \in J_{z_0}}$ of $\{\psi_{w_0}\}_{w_0 \in J_{z_0}}$ such that $\psi_{\tilde{w}_k}$ converges locally uniformly on $\Delta(R_1')$ to some holomorphic function ψ as $\tilde{w}_k \to w_0$. Moreover, by the definition of foliation we know that every $\psi_{\tilde{w}_k} - \psi_{w_0}$ is nowhere vanishing on $\overline{\Delta(R_1)}$, and $\lim_{k\to\infty}(\psi_{\tilde{w}_k}(z_0) - \psi_{w_0}(z_0)) = \lim_{k\to\infty}(\tilde{w}_k - w_0) = 0$. Hence, Hurwitz's theorem implies that $\psi = \psi_{w_0}$ on $\Delta(R_1')$ and so $\psi_{\tilde{w}}$ converges to ψ_{w_0} uniformly on $\overline{\Delta(R_1)}$ as $\tilde{w} \to w_0$ within J_{z_0}. Therefore, for every $\varepsilon > 0$ and $w_0 \in J_{z_0}$ there exists an open neighbourhood U_{w_0} of w_0 such that $|\psi_{w_0}(z) - \psi_{\tilde{w}}(z)| < \varepsilon$ for all $\tilde{w} \in U_{w_0} \cap J_{z_0}$ and for all $z \in \overline{\Delta(R_1)}$. We have completed the proof of (2). $\square$

THEOREM 4.6. *If F satisfies the condition $(\mathscr{F})$, then* $J_+(R_1) = \bigcup_{|z| \le R_1} \{z\} \times J_z$.

Proof. To prove the theorem, we assume the contrary. Then there exists a point $(z_0, w_0) \in J_+(R_1)$ with $w_0 \notin J_{z_0}$. Since $J_{z_0} = \partial A_{z_0} = \partial K_{z_0}$ by Proposition 3.2, it follows that $w_0 \in \operatorname{int}(K_{z_0})$. According to Lemma 4.5, this means that there exists a constant $\varepsilon' > 0$ such that

$$\operatorname{dist}((z_0, w_0), S_{n_j} \cap (\{z_0\} \times \boldsymbol{C})) \ge \varepsilon' \quad \text{for all } j,$$

where $\operatorname{dist}(\cdot,\cdot)$ stands for the Euclidean distance on $\boldsymbol{C}^2$. On the other hand, Lemma 4.5 also guarantees the existence of points $(z_{n_j}, w_{n_j}) \in S_{n_j}$ converging to (z_0, w_0). Then, just with the same argument as in the proof of (4.6), one may obtain a contradiction, proving the theorem. $\square$

From now on, we study quasi-conformal geometry of slices $J_{z_0}, |z_0| < R_1$. Recall that a homeomorphism f of $\boldsymbol{C}$ onto itself is *quasi-conformal* if and only if f has derivatives in $L^2_{loc}(\boldsymbol{C})$ and $\partial f/\partial \bar{z} = \mu(\partial f/\partial z)$, where $\mu \in L^\infty(\boldsymbol{C})$ and $\|\mu\|_\infty < 1$. Let X be a subset of $\boldsymbol{C}$ and let $T \subset \boldsymbol{C}$ be an open disc containing 0.

DEFINITION 4.7. A map $f : T \times X \to \boldsymbol{C}$ is said to be a *holomorphic motion* of X in $\boldsymbol{C}$, if

(1) for any fixed $x \in X$, $f_x(\cdot) := f(\cdot, x)$ is a holomorphic map on T;
(2) for any fixed $t \in T$, $f_t(\cdot) := f(t, \cdot)$ is injective on X; and
(3) $f_0(x) = x$ on X.

The following result is proved in [12] and it is appeared in [4; Theorem 3.27].

THEOREM 4.8. *A holomorphic motion $f : T \times X \to \boldsymbol{C}$ of a set $X \subset \boldsymbol{C}$ can be extended to a holomorphic motion $f : T \times \boldsymbol{C} \to \boldsymbol{C}$ of $\boldsymbol{C}$, and for each fixed $t \in T$ the map f_t is a quasi-conformal homeomorphism of $\boldsymbol{C}$ onto $\boldsymbol{C}$. Moreover, the map $f : T \times \boldsymbol{C} \to \boldsymbol{C}$ is continuous.*

The proof of the following result is similar to that of [4; Theorem 3.28, Corollary 3.29]:

THEOREM 4.9. *Assume that F satisfies the condition* ($\mathscr{F}$). *Then all J_{z_0}, $|z_0| < R_1$, are mutually quasi-conformally equivalent. In particular, if some J_{z_0} is of Lebesgue measure* 0, *so is J_{z_1} for every $|z_1| < R_1$.*

Proof. By Theorem 4.3, we know that $J_+(R_1)$ has the structure of foliation whose leaves $l_{\tilde{w}}$ $(\tilde{w} \in J_0)$, are given by the graphs of holomorphic functions $\psi_{\tilde{w}}$ on $\Delta(R_1')$ with $\psi_{\tilde{w}}(0) = \tilde{w}$. It is now an easy matter to check that the map

$$\Psi : \Delta(R_1) \times J_0 \to \boldsymbol{C} \quad \text{defined by} \quad \Psi(z, \tilde{w}) = \psi_{\tilde{w}}(z)$$

is a holomorphic motion of J_0. Consequently, Ψ extends to a holomorphic motion $\Psi : \Delta(R_1) \times \boldsymbol{C} \to \boldsymbol{C}$ of $\boldsymbol{C}$ as in Theorem 4.8. Then, we have the first statement of the theorem. Since the image of a set of Lebesgue measure 0 under a quasi-conformal homeomorphism is of measure 0 (cf. [6; p150]), we have the latter half. □

In the rest of this paper, we wish to give some sufficient condition for the Lebesgue measure of $J_+(R_1)$ to be equal to 0. To this end, we use a similar argument as in [7; Theorem 1.4.6] for polynomial maps with expandingness on its Julia set. Assume that F satisfies the condition ($\mathscr{F}$). From Theorems 4.6 and 4.9, it is enough to show that for some z_0 with $|z_0| < R_1$ the Lebesgue measure of J_{z_0} is 0. For a given point $z_0 \in \boldsymbol{C}$ with $|z_0| < R_1$ and a point $w_0 \in J_{z_0}$, we have $(z_n, w_n) \in J_+(R_1)$ for $n = 0, 1, 2, \dots$, by (3) of Proposition 2.2, Proposition 2.3 and Theorem 4.6. Since $J_+(R_1)$ is compact in $\boldsymbol{C}^2$, we may assume that some subsequence $\{(z_{n_j}, w_{n_j})\}$ of $\{(z_n, w_n)\}$ converges to a point $(z_\infty, w_\infty) \in J_+(R_1)$. In this situation, we can prove the following:

THEOREM 4.10. *Let z_0 be an arbitrary point of $\boldsymbol{C}$ with $|z_0| < R_1$. We assume that the following three conditions are satisfied*:

(1) *There exists a constant $\delta > 0$ such that*

$$\inf_{w \in J_{z_0}} \inf_{w' \in \tilde{C}_n(z_0)} |w_n - w_n'| > \delta \quad \textit{for all } n,$$

where we set $w_n = g_{z_0}^n(w)$, $w_n' = g_{z_0}^n(w')$ for $w \in J_{z_0}$, $w' \in \tilde{C}_n(z_0)$, respectively.

(2) *F satisfies the condition* ($\mathscr{F}$).

(3) *Let $l_{w_\infty}^{n_j} = \{(z, w) \in \boldsymbol{C}^2 \mid w = \psi_{w_\infty}^{n_j}(z),\ z \in \Delta(R_1)\}$ be the leaf of S_{n_j} which converges to the leaf $l_{w_\infty} = \{(z, w) \in \boldsymbol{C}^2 \mid w = \psi_{w_\infty}(z),\ z \in \Delta(R_1)\}$ of $J_+(R_1)$ containing (z_∞, w_∞). Then there exist real numbers α, β with $0 < \alpha < 1 < \beta$ such that*

(i) $|d_{n+1}(z_0, w_0)| > \beta |d_n(z_0, w_0)|$,

(ii) $|c_{n_j}(z_\infty, \psi_{w_\infty}^{n_j}(z_\infty)) b_{n_j}(z_0, w_0)| / |d_{n_j}(z_\infty, \psi_{w_\infty}^{n_j}(z_\infty)) d_{n_j}(z_0, w_0)| < \alpha$

for all sufficiently large integers n, j and for any $w_0 \in J_{z_0}$.

Then the 2*-dimensional Lebesgue measure of J_{z_0} is equal to* 0. *In particular, the* 4*-dimensional Lebesgue measure of $J_+(R_1)$ is equal to* 0.

Proof. We divide the proof into several steps. We fix an arbitrary point $(z_0, w_0) \in J_+(R_1)$ with $w_0 \in J_{z_0}$. By the assumption (1), one can obtain holomorphic functions $h_{z_0}^{n_j}$ on $\Delta(w_{n_j}, \delta)$ such that $h_{z_0}^{n_j}(w_{n_j}) = w_0$ and $g_{z_0}^{n_j} \circ h_{z_0}^{n_j} = \mathrm{id}$ on

$\Delta(w_{n_j},\delta)$ for all j. By the proof of (3) of Proposition 2.2, we see that $|z_\infty| \le R_1 - \varepsilon_0$. We may now assume that there are positive constants δ', δ'' with $\delta'' < \delta' < \delta$ such that

$$\Delta(w_{n_j},\delta) \supset \Delta(w_\infty,\delta') \supset \Delta(w_{n_j},\delta'') \quad \text{for all } n_j. \tag{4.7}$$

It should be remarked here the following: Since we always consider subsequences of the given $\{n_j\}$ in our argument below, these constants δ', δ'' and δ may be chosen as small as we wish, without worrying about various subsequences taken from $\{n_j\}$.

First, we want to define a new graph associated with F. For this purpose, setting $\lambda_n = (g_{z_0}^n)'(w_0)$, we consider the holomorphic functions $\tilde{h}_n : \Delta(1) \to \boldsymbol{C}$ defined by

$$\tilde{h}_n(w) = \lambda_n[h_{z_0}^n(\delta w + w_n) - w_0]/\delta, \quad w \in \Delta(1), \quad \text{for } n = 1, 2, \ldots.$$

Then, each $\tilde{h}_n$ is injective on $\Delta(1)$, $\tilde{h}_n(0) = 0$ and $\tilde{h}_n'(0) = 1$. Here, by the Koebe distortion theorem we have

$$\frac{\delta r}{(1+r)^2|\lambda_n|} \le |\tilde{h}_n(\delta w + w_n) - w_0| \le \frac{\delta r}{(1-r)^2|\lambda_n|}$$

for all w with $|w| = r < 1$. Now, fix a point $r_0 \in (0,1)$ satisfying $(1+r_0)^2/(1-r_0)^2 < \beta$, and recall that

$$|\lambda_n|/|\lambda_{n-1}| = |d_n(z_0,w_0)|/|d_{n-1}(z_0,w_0)| > \beta \quad \text{for all large } n$$

by our assumption (i) of (3). Then

$$\frac{\delta r_0}{(1-r_0)^2|\lambda_n|} < \frac{\delta r_0}{(1+r_0)^2|\lambda_{n-1}|} \quad \text{for all large } n,$$

which implies that

$$\overline{h_{z_0}^n(\Delta(w_n,\delta r_0))} \subset h_{z_0}^{n-1}(\Delta(w_{n-1},\delta r_0)) \quad \text{for all large } n.$$

Therefore, replacing δr_0 by δ again, we have

$$\overline{h_{z_0}^n(\Delta(w_n,\delta))} \subset h_{z_0}^{n-1}(\Delta(w_{n-1},\delta)) \quad \text{and so} \quad |g_{z_0}^{n-1}(w) - w_{n-1}| < \delta \text{ on } h_{z_0}^n(\Delta(w_n,\delta))$$

for all large n. Here, as stated above, the constant δ may be rechoosen so small that

$$|(g_{z_0}^{n-1}(w))^m - z_n| = |(g_{z_0}^{n-1}(w))^m - (w_{n-1})^m| < \varepsilon_0/4 \quad \text{on } h_{z_0}^n(\Delta(w_n,\delta))$$

for all large n. Thus, together with the fact $z_{n_j} \to z_\infty$, we can assume that

$$|(g_{z_0}^{n_j-1}(w))^m - z_\infty| \le |(g_{z_0}^{n_j-1}(w))^m - z_{n_j}| + |z_{n_j} - z_\infty| < \varepsilon_0/2 \quad \text{on } h_{z_0}^{n_j}(\Delta(w_{n_j},\delta)) \tag{4.8}$$

for all large j. This, combined with the fact $|z_\infty| \le R_1 - \varepsilon_0$, guarantees that

$$|(g_{z_0}^{n_j-1}(w))^m| \le R_1 \quad \text{on } h_{z_0}^{n_j}(\Delta(w_{n_j},\delta)) \text{ for all large } j. \tag{4.9}$$

Recall that $g_{z_0}^{n_j} \circ h_{z_0}^{n_j} = \mathrm{id}$ on $\Delta(w_\infty, \delta')$. Then, setting $\phi_{z_0}^{n_j}(w) = f_{n_j}(z_0, h_{z_0}^{n_j}(w))$, we have that

(4.10) $F^{\circ n_j}(z_0, h_{z_0}^{n_j}(\Delta(w_\infty, \delta'))) = \{(z, w) \in \boldsymbol{C}^2 \mid z = \phi_{z_0}^{n_j}(w),\ w \in \Delta(w_\infty, \delta')\}$.

By (4.7) and (4.9), it follows that

$$|\phi_{z_0}^{n_j}(w)| = |f_{n_j}(z_0, h_{z_0}^{n_j}(w))| = |\{g_{z_0}^{n_j-1}(h_{z_0}^{n_j}(w))\}^m| \le R_1 \quad \text{on } \Delta(w_\infty, \delta').$$

Thus, $\{\phi_{z_0}^{n_j}\}$ is normal on $\Delta(w_\infty, \delta')$; so that one can assume that $\{\phi_{z_0}^{n_j}\}$ converges locally uniformly to a holomorphic function ϕ_{z_0} on $\Delta(w_\infty, \delta')$. Since $z_{n_j} = \phi_{z_0}^{n_j}(w_{n_j})$ and $(z_{n_j}, w_{n_j}) \to (z_\infty, w_\infty)$, we see that $z_\infty = \phi_{z_0}(w_\infty)$. Let us set

$$l = \{(z, w) \in \boldsymbol{C}^2 \mid z = \phi_{z_0}(w),\ w \in \Delta(w_\infty, \delta')\}; \quad \text{and}$$
$$l_{\tilde{w}} = \{(z, w) \in \boldsymbol{C}^2 \mid w = \psi_{\tilde{w}}(z),\ z \in \overline{\Delta(R_1)}\}, \quad \tilde{w} = \psi_{\tilde{w}}(z_\infty),$$

where $\{\psi_{\tilde{w}}\}$ are holomorphic functions on $\Delta(R_1')$ defining the leaves of foliation $\{l_{\tilde{w}}\}_{\tilde{w} \in J_{z_\infty}}$ of $J_+(R_1)$. We have now two cases to consider.

Case 1. ϕ_{z_0} is a non-constant map on $\Delta(w_\infty, \delta')$.

For some small $\varepsilon_1 > 0$ and each $\tilde{w} \in J_{z_\infty}$, we define new holomorphic maps

$$\tilde{\phi}_{z_0} : \Delta(1+\varepsilon_1) \times \Delta(w_\infty, \delta') \to \boldsymbol{C} \text{ by } (t, w) \mapsto z = t\phi_{z_0}(w) + (1-t)z_\infty; \quad \text{and}$$
$$\tilde{\Phi}_{\tilde{w}} : \Delta(1+\varepsilon_1) \times \Delta(w_\infty, \delta') \to \boldsymbol{C} \text{ by } (t, w) \mapsto w = \psi_{\tilde{w}} \circ \tilde{\phi}_{z_0}(t, w).$$

For each fixed $t \in \Delta(1+\varepsilon_1)$, the maps $\tilde{\phi}_{z_0}(t, w)$ and $\tilde{\Phi}_{\tilde{w}}(t, w)$ of one variable w will be denoted by $\phi_{z_0}^t(w)$ and $\Phi_{\tilde{w}}^t(w)$, respectively. Then $\phi_{z_0}(w) = \phi_{z_0}^1(w)$ and by (4.8)

$$|\phi_{z_0}^t(w) - z_\infty| = |t|\,|\phi_{z_0}(w) - z_\infty| \le (1+\varepsilon_1)\varepsilon_0/2 \quad \text{on } \Delta(w_\infty, \delta')$$

for all $t \in \Delta(1+\varepsilon_1)$. Combining this with $|z_\infty| \le R_1 - \varepsilon_0$, we see that $|\phi_{z_0}^t(w)| \le R_1$ on $\Delta(w_\infty, \delta')$ and the composition $\psi_{\tilde{w}} \circ \phi_{z_0}^t$ can be defined for each $\tilde{w} \in J_{z_\infty}$, by replacing ε_1 small. Setting

$$l_t = \{(z, w) \in \boldsymbol{C}^2 \mid z = \phi_{z_0}^t(w), w \in \Delta(w_\infty, \delta')\},$$

we next study the slice of $J_+(R_1)$ by l_t. To this end, since $J_+(R_1)$ is foliated as $J_+(R_1) = \bigcup_{\tilde{w} \in J_{z_\infty}} l_{\tilde{w}}$, it is enough to consider the intersection $l_t \cap l_{\tilde{w}} = \{(\phi_{z_0}^t(w), w) \in \boldsymbol{C}^2 \mid \Phi_{\tilde{w}}^t(w) = w,\ w \in \Delta(w_\infty, \delta')\}$ for each $\tilde{w} \in J_{z_\infty}$.

Lemma 4.11. *There exist an open neighbourhood U_{w_∞} of w_∞, positive constants ε_1 with $\alpha(1+\varepsilon_1) < 1$ and δ' satisfying* (4.7) *such that the intersection $l_t \cap l_{\tilde{w}}$ consists of a unique point for each $\tilde{w} \in U_{w_\infty} \cap J_{z_\infty}$ and for each $t \in \Delta(1+\varepsilon_1)$.*

Proof. Without loss of generality, we may assume that the positive constant ε_1 satisfies the inequality $\alpha(1+\varepsilon_1) < 1$. To prove the lemma it is enough to show that the map $\Phi_{\tilde{w}}^t$ has a unique fixed point in $\Delta(w_\infty, \delta')$ for any given points $\tilde{w}$ and t contained in some open neighbourhoods of w_∞ and $\overline{\Delta(1)}$, respectively. Since $\psi_{w_\infty}(z_\infty) = w_\infty$, it is easy to see that $\Phi^t_{w_\infty}(w_\infty) = w_\infty$ for all t. First, we assert that:

(4.11) For arbitrarily given $t \in \Delta(1+\varepsilon_1)$, there is a constant δ_t, $0 < \delta_t < \delta'$, such that $\Phi^t_{w_\infty}$ has a unique fixed point in $\overline{\Delta(w_\infty, \delta_t)}$.

To see this, we have only to show that the set of the roots of $\Phi^t_{w_\infty}(w) = w$ does not accumulate at $w = w_\infty$ for every $t \in \Delta(1+\varepsilon_1)$. Assume the contrary. Then, since $\Phi^0_{w_\infty}(w) = w_\infty$, there exists some non-zero $t \in \Delta(1+\varepsilon_1)$ such that $\Phi^t_{w_\infty}(w) = \psi_{w_\infty} \circ \phi^t_{z_0}(w) \equiv w$ by the identity theorem. Therefore,

$$(\psi_{w_\infty})'(\phi^t_{z_0}(w))(t\phi_{z_0} + (1-t)z_\infty)'(w) = (\psi_{w_\infty})'(\phi^t_{z_0}(w))(\phi_{z_0})'(w)t \equiv 1. \tag{4.12}$$

Here, since $w_{n_j} \to w_\infty$ and $\phi^{n_j}_{z_0}$ converges to ϕ_{z_0} locally uniformly on $\Delta(w_\infty, \delta')$, we see $\lim_{j\to\infty}(\phi^{n_j}_{z_0})'(w_\infty) = \lim_{j\to\infty}(\phi^{n_j}_{z_0})'(w_{n_j})$. Taking this into account, we set $w = w_\infty$ in (4.12). Then, using the sequence $\{\psi^{n_j}_{w_\infty}\}$ converging to ψ_{w_∞} as in the proof of Theorem 4.3, we have that

$$\lim_{j\to\infty} (\psi^{n_j}_{w_\infty})'(z_\infty)(\phi^{n_j}_{z_0})'(w_{n_j})t = 1.$$

Here, recall that $g_{n_j}(z, \psi^{n_j}_{w_\infty}(z)) = c$, $\phi^{n_j}_{z_0}(w) = f_{n_j}(z_0, h^{n_j}_{z_0}(w))$ and $g^{n_j}_{z_0} \circ h^{n_j}_{z_0} = \mathrm{id}$. Then

$$(\psi^{n_j}_{w_\infty})'(z) = -\frac{\partial g_{n_j}(z, \psi^{n_j}_{w_\infty}(z))}{\partial z} \Big/ \frac{\partial g_{n_j}(z, \psi^{n_j}_{w_\infty}(z))}{\partial w} = -c_{n_j}(z, \psi^{n_j}_{w_\infty}(z))/d_{n_j}(z, \psi^{n_j}_{w_\infty}(z))$$

and

$$\begin{aligned}(\phi^{n_j}_{z_0})'(w) &= [f_{n_j}(z_0, h^{n_j}_{z_0}(w))]' = [(g_{n_j-1}(z_0, h^{n_j}_{z_0}(w))^m]' \\ &= m[g_{n_j-1}(z_0, h^{n_j}_{z_0}(w))]^{m-1}[\partial g_{n_j-1}(z_0, h^{n_j}_{z_0}(w))/\partial w]/(g^{n_j}_{z_0})'(h^{n_j}_{z_0}(w)).\end{aligned}$$

On the other hand, since $F^{\circ n_j}(z_0, w) = ((g^{n_j-1}_{z_0}(w))^m, g^{n_j}_{z_0}(w))$, we have

$$b_{n_j}(z, w) = \partial[(g_{n_j-1}(z, w))^m]/\partial w, \quad (\phi^{n_j}_{z_0})'(w) = b_{n_j}(z_0, h^{n_j}_{z_0}(w))/d_{n_j}(z_0, h^{n_j}_{z_0}(w)).$$

Consequently, since $0 < |t| < 1 + \varepsilon_1$ and $\alpha(1+\varepsilon_1) < 1$, we have

$$\lim_{j\to\infty}\left|\frac{c_{n_j}(z_\infty, \psi^{n_j}_{w_\infty}(z_\infty))b_{n_j}(z_0, w_0)}{d_{n_j}(z_\infty, \psi^{n_j}_{w_\infty}(z_\infty))d_{n_j}(z_0, w_0)}\right| = \lim_{j\to\infty}|(\psi^{n_j}_{w_\infty})'(z_\infty)(\phi^{n_j}_{z_0})'(w_{n_j})| = \frac{1}{|t|} > \alpha. \tag{4.13}$$

Thus, (4.13) contradicts the assumption (ii) of (3) of the theorem, proving (4.11).

Next, we claim that

(4.14) There are positive constants $\tilde{\delta}$, $\tilde{\varepsilon}_1$ with $0 < \tilde{\delta} < \delta'$, $1 < \tilde{\varepsilon}_1 < \varepsilon_1$, such that $\Phi^t_{w_\infty}(w) - w = 0$ has a unique solution w_∞ in $\overline{\Delta(w_\infty, \tilde{\delta})}$ for all $t \in \Delta(1+\tilde{\varepsilon}_1)$.

To prove our claim, we set $m_t = \min\{|\Phi^t_{w_\infty}(w) - w| \mid |w - w_\infty| = \delta_t\} > 0$ by using the constant δ_t in (4.11). For t and $\tilde{t} \in \Delta(1+\varepsilon_1)$, it is easy to see that

$\{\Phi_{w_\infty}^{\tilde{t}}(w) - w\}$ converges locally uniformly to $\Phi_{w_\infty}^{t}(w) - w$ as $\tilde{t} \to t$ and there is a positive constant γ_t such that

$$|(\Phi_{w_\infty}^{\tilde{t}}(w) - w) - (\Phi_{w_\infty}^{t}(w) - w)| < m_t/2 \quad \text{for all } \tilde{t} \in \Delta(t, \gamma_t), \quad w \in \overline{\Delta(w_\infty, \delta_t)}.$$

Then, Hurwitz's theorem guarantees that the equations $\Phi_{w_\infty}^{\tilde{t}}(w) - w = 0$ and $\Phi_{w_\infty}^{t}(w) - w = 0$ have the same number of zeros in $\Delta(w_\infty, \delta_t)$; and consequently, w_∞ is a unique solution of $\Phi_{w_\infty}^{\tilde{t}}(w) - w = 0$ in $\Delta(w_\infty, \delta_t)$. On the other hand, one can choose some numbers $\tilde{\varepsilon}_1$ and a finite sequence $\{t_k\}_{k=1}^{l}$ such that $0 < \tilde{\varepsilon}_1 < \varepsilon_1$, $t_k \in \Delta(1 + \tilde{\varepsilon}_1)$ and $\overline{\Delta(1 + \tilde{\varepsilon}_1)} \subset \bigcup_{k=1}^{l} \Delta(t_k, \gamma_{t_k})$. Set $\tilde{\delta} = \min_{1 \le k \le l} \delta_{t_k}/2$. Then, it is easily seen that these $\tilde{\varepsilon}_1$ and $\tilde{\delta}$ satisfy the requirements of (4.14).

Now, we set $\tilde{m}_t = \min\{|\Phi_{w_\infty}^{t}(w) - w| \mid |w - w_\infty| = \tilde{\delta}\} > 0$ for each $t \in \Delta(1 + \tilde{\varepsilon}_1)$. Then, by (2) of Theorem 4.3 and by the uniformly continuity of ψ_{w_∞}, there exist a constant $\tilde{\gamma}_t$ and an open neighbourhood $U_{w_\infty}^{t}$ of w_∞ such that

$$\begin{aligned} |\Phi_{\tilde{w}}^{\tilde{t}}(w) - \Phi_{w_\infty}^{t}(w)| &\le |\psi_{\tilde{w}} \circ \phi_{z_0}^{\tilde{t}}(w) - \psi_{w_\infty} \circ \phi_{z_0}^{\tilde{t}}(w)| \\ &\quad + |\psi_{w_\infty} \circ \phi_{z_0}^{\tilde{t}}(w) - \psi_{w_\infty} \circ \phi_{z_0}^{t}(w)| < \tilde{m}_t/2 \end{aligned}$$

for $\tilde{t} \in \Delta(t, \tilde{\gamma}_t)$, $\tilde{w} \in U_{w_\infty}^{t} \cap J_{z_\infty}$ and $w \in \overline{\Delta(w_\infty, \tilde{\delta})}$. Just as in the proof of (4.14), taking some ε_2 with $0 < \varepsilon_2 < \tilde{\varepsilon}_1$, we consider a finite covering $\{\Delta(\tilde{t}_k, \tilde{\gamma}_{\tilde{t}_k})\}_{k=1}^{\tilde{l}}$ of $\overline{\Delta(1 + \varepsilon_2)}$ and set $U_{w_\infty} = \bigcap_{k=1}^{\tilde{l}} U_{w_\infty}^{\tilde{t}_k}$. Then, for arbitrarily given points $\tilde{w} \in U_{w_\infty} \cap J_{z_\infty}$ and $t \in \Delta(1 + \varepsilon_2)$, there is a point $\tilde{t}_k \in \Delta(1 + \varepsilon_2)$ such that for $w \in \overline{\Delta(w_\infty, \tilde{\delta})}$

$$|(\Phi_{\tilde{w}}^{t}(w) - w) - (\Phi_{w_\infty}^{\tilde{t}_k}(w) - w)| = |\Phi_{\tilde{w}}^{t}(w) - \Phi_{w_\infty}^{\tilde{t}_k}(w)| < \tilde{m}_{\tilde{t}_k}/2.$$

Thus, applying again Hurwitz's theorem, we can see by (4.14) that $\Phi_{\tilde{w}}^{t}(w) - w = 0$ has a unique solution in $\overline{\Delta(w_\infty, \tilde{\delta})}$. Therefore, for convenience, denoting such $\tilde{\delta}$, ε_2 by δ', ε_1 again, we complete the proof of Lemma 4.11. □

Thanks to Lemma 4.11, one can now define a map

$$\Psi_{\tilde{w}} : \Delta(1 + \varepsilon_1) \to \boldsymbol{C}, \quad t \mapsto \Psi_{\tilde{w}}(t) \quad \text{for each } \tilde{w} \in U_{w_\infty} \cap J_{z_\infty}$$

by requiring the condition

$$l_t \cap l_{\tilde{w}} = \{(\phi_{z_0}^{t} \circ \Psi_{\tilde{w}}(t), \Psi_{\tilde{w}}(t))\} \quad \text{for all } t \in \Delta(1 + \varepsilon_1).$$

So we obtain a map $\Psi : \Delta(1 + \varepsilon_1) \times (U_{w_\infty} \cap J_{z_\infty}) \to \boldsymbol{C}$ given by $\Psi(t, \tilde{w}) = \Psi_{\tilde{w}}(t)$.

Lemma 4.12. *$\Psi : \Delta(1 + \varepsilon_1) \times (U_{w_\infty} \cap J_{z_\infty}) \to \boldsymbol{C}$ is a holomorphic motion of $U_{w_\infty} \cap J_{z_\infty}$ in $\boldsymbol{C}$.*

Proof. From the proof of Lemma 4.11, one knows that

$$\{(t, w) \in \Delta(1 + \varepsilon_1) \times \Delta(w_\infty, \delta') \mid \tilde{\Phi}_{\tilde{w}}(t, w) = w\} = \{(t, w) \mid t \in \Delta(1 + \varepsilon_1), w = \Psi_{\tilde{w}}(t)\}.$$

Hence, $\Psi_{\tilde{w}}$ is a holomorphic function on $\Delta(1 + \varepsilon_1)$ (cf. [9; Theorem 4.4.1]).

To show that $\Psi_t : U_{w_\infty} \cap J_{z_\infty} \to \boldsymbol{C}$ is injective for each fixed $t \in \Delta(1+\varepsilon_1)$, we assume that there are two distinct points w', $w'' \in U_{w_\infty} \cap J_{z_\infty}$ such that $\Psi_t(w') = \Psi_t(w'')$, or equivalently, the two equations $\psi_{w'} \circ \phi_{z_0}^t(w) = w$ and $\psi_{w''} \circ \phi_{z_0}^t(w) = w$ have the same solution, say w_*, in $\Delta(w_\infty, \delta')$. Then, setting $z_* = \phi_{z_0}^t(w_*)$, we have $\psi_{w'}(z_*) = w_* = \psi_{w''}(z_*)$, and hence $w' = w''$ by our construction of the foliation of $J_+(R_1)$. This is a contradiction, as desired. Moreover, since $\tilde{\Phi}_{\tilde{w}}(0, \tilde{w}) = \psi_{\tilde{w}}(z_\infty) = \tilde{w}$ for all $\tilde{w} \in U_{w_\infty} \cap J_{z_\infty}$, it is easily checked that $\Psi(0, \cdot) = \mathrm{id}$ on $U_{w_\infty} \cap J_{z_\infty}$. Therefore, all the conditions of Definition 4.7 are fulfilled for Ψ. $\square$

By a direct application of Theorem 4.8, Ψ extends to a map from $\Delta(1+\varepsilon_1) \times \boldsymbol{C}$ to $\boldsymbol{C}$, and $\Psi_1 : \boldsymbol{C} \to \boldsymbol{C}$, $\tilde{w} \mapsto \Psi_1(\tilde{w})$, is a quasi-conformal homeomorphism.

Before proceeding, we need to introduce some notation and terminology from the measure theory. We refer the reader to books [2] or [13; §6]. Let V be a bounded measurable set in the n-dimensional Euclidean space Ω and set

$$r(V) = \sup_{V \subset L} m(V)/m(L),$$

where the supremum is taken over all cubes L whose boundaries are parallel to the coordinates of Ω, and $m(\cdot)$ is the n-dimensional Lebesgue measure. Let $\{V_k\}$ be a sequence of measurable sets in Ω. Then, $\{V_k\}$ is called *regular at a point* $p \in \Omega$ if $p \in V_k$ for all k, $V_k \to \{p\}$ as $k \to \infty$ and there is a constant c such that $r(V_k) \geq c > 0$ for all k. Moreover, for a given regular sequence $\{V_k\}$ of closed measurable sets at p, we define the constant

$$l_{\{V_k\}} = \lim_{k \to \infty} m(V_k \cap V)/m(V_k) \text{ if the limit on the right exists;}$$

$$\text{and set } \underline{v}(p) = \inf l_{\{V_k\}} \text{ and } \overline{v}(p) = \sup l_{\{V_k\}},$$

where the infimum and the supremum are taking over all regular sequences $\{V_k\}$ of closed measurable sets at a point p. If $\underline{v}(p) = \overline{v}(p)$, we denote this number by $v(p)$ and call it the *density of* V *at* $p \in \Omega$. For later use, we shall recall the following:

THEOREM 4.13 (Lebesgue density theorem). *Let E be a measurable set. Then $v(p) = 1$ for almost every $p \in E$. In particular, if $v(p) < 1$ for all $p \in V$, then V has the Lebesgue measure* 0.

Now, in order to prove Theorem 4.10, it suffices to show that $v(w_0) < 1$ for every $w_0 \in J_{z_0}$. To this end, we introduce a regular sequence $\{V_{n_j}\}$ of closed measurable sets as follows:

$$V_{n_j} = \{w' \in \boldsymbol{C} \mid w' = h_{z_0}^{n_j}(w),\ w \in \overline{\Delta(w_{n_j}, \eta)}\} \quad \text{for some } \eta \text{ with } 0 < \eta < \delta''.$$

Note that $w_0 \in V_{n_j}$ for all j, since $w_0 = h_{z_0}^{n_j}(w_{n_j})$ for all j. To see that $\{V_{n_j}\}$ is, in fact, a regular sequence of measurable sets at w_0, we need the following estimate.

LEMMA 4.14. *For all sufficiently large j, there are constants $M > 0$, $0 < \kappa < 1$ and δ'' satisfying* (4.7) *such that*: *For each w, $\tilde{w} \in \Delta(w_{n_j}, \delta'')$, we have*

(1) $|(h_{z_0}^{n_j})'(w)/(h_{z_0}^{n_j})'(\tilde{w}) - 1| \leq M|w - \tilde{w}|$;

(2) $(1-\kappa)|(h_{z_0}^{n_j})'(\tilde{w})(w-\tilde{w})| \leq |h_{z_0}^{n_j}(w) - h_{z_0}^{n_j}(\tilde{w})| \leq (1+\kappa)|(h_{z_0}^{n_j})'(\tilde{w})(w-\tilde{w})|$.

Proof. Put $\Delta^2(w_\infty, \delta') = \Delta(w_\infty, \delta') \times \Delta(w_\infty, \delta')$ and consider the functions

$$H_{n_j}(w, \tilde{w}) = (h_{z_0}^{n_j})'(w)/(h_{z_0}^{n_j})'(\tilde{w}), \quad (w, \tilde{w}) \in \Delta^2(w_\infty, \delta'), \quad \text{for } j = 1, 2, \ldots.$$

Then, applying the Koebe distortion theorem to the maps $\tilde{h}_{n_j} : \Delta(1) \to \boldsymbol{C}$ defined by $w \mapsto \lambda_{n_j}[h_{z_0}^{n_j}(\delta w + w_{n_j}) - w_0]/\delta$, one can check that $\{H_{n_j}\}$ is bounded uniformly on $\Delta^2(w_\infty, \delta')$ and so it is a normal family on it. Therefore, we can assume that $\{H_{n_j}\}$ converges locally uniformly to some holomorphic function H on $\Delta^2(w_\infty, \delta')$. Moreover, since $H_{n_j}(w, w) = H(w, w) = 1$ for all $w \in \Delta(w_\infty, \delta')$,

$$G_{n_j}(w, \tilde{w}) = (H_{n_j}(w, \tilde{w}) - 1)/(w - \tilde{w}) \quad \text{and} \quad G(w, \tilde{w}) = (H(w, \tilde{w}) - 1)/(w - \tilde{w})$$

are well-defined holomorphic functions on the whole space $\Delta^2(w_\infty, \delta')$. (See, for instance, [10; Corollary 6.26].) Here, we assert that there are positive constants M, δ'' with $\delta'' < \delta'$ such that

(4.15) $$|G_{n_j}(w, \tilde{w})| \leq M \quad \text{on } \Delta^2(w_{n_j}, \delta'') \text{ for all sufficiently large } j,$$

which shows the inequality (1) of the lemma. Indeed, since G is a holomorphic function on $\Delta^2(w_\infty, \delta')$, there are positive constants $\tilde{M}$ and δ_0, δ_1 with $0 < \delta_0 < \delta_1 < \delta'$ such that

(4.16) $$|G(w, \tilde{w})| < \tilde{M} \quad \text{on } \overline{\Delta(w_\infty, \delta_1) \times \Delta(w_\infty, \delta_0)}.$$

On the other hand, considering the Silov boundary of $\Delta(w_\infty, \delta_1) \times \Delta(w_\infty, \delta_0)$, we have

(4.17) $$|G_{n_j}(w, \tilde{w})| \leq \sup\{|G_{n_j}(w, \tilde{w})| \mid |w| = \delta_1, |\tilde{w}| = \delta_0\} \quad \text{on} \quad \Delta(w_\infty, \delta_1) \times \Delta(w_\infty, \delta_0)$$

for every j. Since $G_{n_j} \to G$ locally uniformly on $[\Delta(w_\infty, \delta_1) \times \Delta(w_\infty, \delta_0)] \setminus \{(w, \tilde{w}) \in \boldsymbol{C}^2 \mid w = \tilde{w}\}$, it follows then from (4.16), (4.17) that there is a constant $M > 0$ such that

$$|G_{n_j}(w, \tilde{w})| \leq M \quad \text{on } \Delta(w_\infty, \delta_1) \times \Delta(w_\infty, \delta_0) \text{ for all } j.$$

As a result, by choosing a positive constant δ'', $0 < \delta'' < \delta_0$, as in (4.7), we obtain (4.15), as desired.

In order to prove the second inequality of the lemma, we first claim that:

(4.18) $h_{z_0}^{n_j}(\Delta(w_{n_j}, \delta''))$ is a geometrically convex subset of $\boldsymbol{C}^2$ for all sufficiently large j.

To this end, we set $\tilde{h}_{z_0}^{n_j}(w) := \lambda_{n_j}[h_{z_0}^{n_j}(\delta'' w + w_{n_j}) - w_0]/\delta''$ and denote by $(\tilde{h}_{z_0}^{n_j})''$ the second derivative of $\tilde{h}_{z_0}^{n_j}$ and $\mathrm{Re}(\cdot)$ the real part. Once it is shown that

$$1 + \mathrm{Re}(w(\tilde{h}_{z_0}^{n_j})''(w)/(\tilde{h}_{z_0}^{n_j})'(w)) > 0 \quad \text{on } \Delta(1),$$

then it is well-known that $\tilde{h}_{z_0}^{n_j}(\Delta(1))$ is convex and, so is $h_{z_0}^{n_j}(\Delta(w_{n_j}, \delta''))$. By (1) of Lemma 4.14, we have now

$$\left|\frac{(\tilde{h}_{z_0}^{n_j})'(w) - (\tilde{h}_{z_0}^{n_j})'(\tilde{w})}{w - \tilde{w}}\right| \le M\delta'' |(\tilde{h}_{z_0}^{n_j})'(\tilde{w})|, \quad |(\tilde{h}_{z_0}^{n_j})''(\tilde{w})| \le M\delta'' |(\tilde{h}_{z_0}^{n_j})'(\tilde{w})| \text{ on } \Delta(1).$$

Therefore, by rechoosing δ'' with $M\delta'' < 1$, if necessary, we can assume that $|w(\tilde{h}_{z_0}^{n_j})''(w)/(\tilde{h}_{z_0}^{n_j})'(w)| \le M\delta'' < 1$ on $\Delta(1)$, proving (4.18).

By (4.15), for each j and for each $w, \tilde{w} \in \Delta(w_{n_j}, \delta'')$, there is a constant $M_{n_j}(w, \tilde{w}) \in \boldsymbol{C}$ depending on $(w, \tilde{w})$ such that $|M_{n_j}(w, \tilde{w})| \le M$ and

$$(h_{z_0}^{n_j})'(w) = (h_{z_0}^{n_j})'(\tilde{w}) + M_{n_j}(w, \tilde{w})(w - \tilde{w})(h_{z_0}^{n_j})'(\tilde{w}). \tag{4.19}$$

On the other hand, integrating $(h_{z_0}^{n_j})'(w)$ along the line segment $w(s) = \tilde{w} + s(w - \tilde{w})$, $s \in [0, 1]$, we have

$$h_{z_0}^{n_j}(w) - h_{z_0}^{n_j}(\tilde{w}) = \int_0^1 (h_{z_0}^{n_j})'(\tilde{w} + s(w - \tilde{w}))(w - \tilde{w})\, ds.$$

This combined with (4.19) yields that

$$|h_{z_0}^{n_j}(w) - h_{z_0}^{n_j}(\tilde{w})| \le |(h_{z_0}^{n_j})'(\tilde{w})(w - \tilde{w})|(1 + |w - \tilde{w}|M).$$

Since M depends neither on w, $\tilde{w}$ nor on j, there are constants $0 < \kappa < 1$ and $\delta'' > 0$ satisfying (4.7) and (4.15) such that

$$|w - \tilde{w}|M < 2\delta'' M < \kappa \quad \text{and} \quad |h_{z_0}^{n_j}(w) - h_{z_0}^{n_j}(\tilde{w})| \le (1 + \kappa)|(h_{z_0}^{n_j})'(\tilde{w})(w - \tilde{w})|$$

for all sufficiently large j and for all $w, \tilde{w} \in \Delta(w_{n_j}, \delta'')$.

To complete the proof of the lemma, let us fix j and w, $\tilde{w} \in \Delta(w_{n_j}, \delta'')$ arbitrarily, and consider the curves $\tilde{L}$, L with parameter $s \in [0, 1]$:

$$\tilde{L} : s \mapsto \tilde{u}(s) = s h_{z_0}^{n_j}(\tilde{w}) + (1 - s) h_{z_0}^{n_j}(w), \quad L : s \mapsto u(s) = (h_{z_0}^{n_j})^{-1} \circ \tilde{u}(s).$$

Since $h_{z_0}^{n_j}(\Delta(w_{n_j}, \delta''))$ is convex by (4.18) and since $(h_{z_0}^{n_j})^{-1}$ is a well-defined holomorphic function on $h_{z_0}^{n_j}(\Delta(w_{n_j}, \delta''))$, $\tilde{L}$ is a line segment in $h_{z_0}^{n_j}(\Delta(w_{n_j}, \delta''))$ and L is a curve in $\Delta(w_{n_j}, \delta'')$. Then, we see

$$\begin{aligned} |h_{z_0}^{n_j}(w) - h_{z_0}^{n_j}(\tilde{w})| &= \int_{\tilde{L}} |dw| = \int_{h_{z_0}^{n_j}(L)} |dw| = \int_0^1 |(h_{z_0}^{n_j})'(u(s)) u'(s)|\, ds \\ &\ge |(h_{z_0}^{n_j})'(u(s_0))| \int_0^1 |u'(s)|\, ds \ge |(h_{z_0}^{n_j})'(u(s_0))(w - \tilde{w})|, \end{aligned}$$

where $s_0 \in [0,1]$ is a point at which the continuous function $|(h_{z_0}^{n_j})'(u(s))|$ on $[0,1]$ attains its minimum. Since $(h_{z_0}^{n_j})'(u(s_0)) = (h_{z_0}^{n_j})'(\tilde{w}) + M_{n_j}(u(s_0),\tilde{w})(u(s_0)-\tilde{w}) \cdot (h_{z_0}^{n_j})'(\tilde{w})$ by (4.19),

$$\begin{aligned}|h_{z_0}^{n_j}(w) - h_{z_0}^{n_j}(\tilde{w})| &\geq |(h_{z_0}^{n_j})'(\tilde{w})(w-\tilde{w})|\{1 - |M_{n_j}(u(s_0),\tilde{w})(u(s_0)-\tilde{w})|\} \\ &\geq |(h_{z_0}^{n_j})'(\tilde{w})(w-\tilde{w})|(1-2M\delta'') \geq |(h_{z_0}^{n_j})'(\tilde{w})(w-\tilde{w})|(1-\kappa),\end{aligned}$$

which implies (2) of the lemma. We have completed the proof of the lemma. $\square$

By (2) of Lemma 4.14, we have

$$\overline{\Delta(w_0,(1-\kappa)\eta|(h_{z_0}^{n_j})'(w_{n_j})|)} \subset V_{n_j} \subset \overline{\Delta(w_0,(1+\kappa)\eta|(h_{z_0}^{n_j})'(w_{n_j})|)}$$

for all sufficiently large j. Moreover, by our assumption (i) of (3), it follows that

$$|(h_{z_0}^{n_j})'(w_{n_j})| = 1/|(g_{z_0}^{n_j})'(w_0)| = 1/|d_{n_j}(z_0,w_0)| \to 0 \quad \text{as } j \to \infty.$$

Therefore, we see that $\{V_{n_j}\}$ is a regular sequence of closed measurable sets at w_0.

Let us set, for a given small constant $0 < \eta < \delta''$,

$$\tilde{\boldsymbol{B}} = \{w \in \Delta(w_\infty,\eta) \mid (\phi_{z_0}(w),w) \in J_+(R_1)\}, \quad \text{and}$$
$$\tilde{\boldsymbol{B}}_{n_j} = \{w \in \Delta(w_{n_j},\eta) \mid (\phi_{z_0}^{n_j}(w),w) \in J_+(R_1)\} \quad \text{for } j = 1,2,\ldots.$$

Here, we assert that

$$\text{(4.20)} \qquad \Psi_1(U_{w_\infty} \cap J_{z_\infty}) \supset \tilde{\boldsymbol{B}} = \{w \in \Delta(w_\infty,\eta) \mid (\phi_{z_0}(w),w) \in J_+(R_1)\},$$

after rechoosing η small enough, if necessary. To show this assertion, we have only to check the following: If $l = l_1$ intersects $l_{\tilde{w}}$ at $(\phi_{z_0}(w),w)$ for $\tilde{w} \in J_{z_\infty}$, $w \in \Delta(w_\infty,\eta)$, then $\tilde{w} \in U_{w_\infty} \cap J_{z_\infty}$. Assume the contrary. Then we may choose sequences $\{\eta_{n_k}\} \subset \boldsymbol{R}$, $\{w'_{n_k}\} \subset \Delta(w_\infty,\eta_{n_k})$ and $\{\tilde{w}_{n_k}\} \subset J_{z_\infty}$ such that $\eta_{n_k} \downarrow 0$ as $k \to \infty$, $(z'_{n_k},w'_{n_k}) := (\phi_{z_0}(w'_{n_k}),w'_{n_k}) \in l \cap l_{\tilde{w}_{n_k}}$ and $\tilde{w}_{n_k} \notin J_{z_\infty} \cap U_{w_\infty}$ for all k. We take some disk $\Delta(w_\infty,\varepsilon') \subset U_{w_\infty}$ with $\tilde{w}_{n_k} \notin \Delta(w_\infty,\varepsilon')$ for every k. Then, for large k we have that

$$\begin{aligned}|\psi_{\tilde{w}_{n_k}}(z_\infty) - \psi_{\tilde{w}_{n_k}}(z'_{n_k})| &= |\tilde{w}_{n_k} - \psi_{\tilde{w}_{n_k}}(z'_{n_k})| \geq |w_\infty - \tilde{w}_{n_k}| - |w_\infty - \psi_{\tilde{w}_{n_k}}(z'_{n_k})| \\ &= |w_\infty - \tilde{w}_{n_k}| - |w_\infty - w'_{n_k}| \geq \varepsilon' - \eta_{n_k} > \varepsilon'/2.\end{aligned}$$

Moreover, by the continuity of ϕ_{z_0}, $z'_{n_k} = \phi_{z_0}(w'_{n_k}) \to \phi_{z_0}(w_\infty) = z_\infty$ as $k \to \infty$. This contradicts the fact that $\{\psi_{\tilde{w}_{n_k}}\}_{\tilde{w}_{n_k} \in J_{z_\infty}}$ is equicontinuous at $z = z_\infty$ by (2) of Theorem 4.3, proving (4.20).

In the reminder of the proof, we fix a constant $\eta > 0$ as in (4.20). Since $w_\infty \in J_{z_\infty} = \partial A_{z_\infty}$, $U_{w_\infty} \cap N_{z_\infty}$ contains a non-empty open set. In particular, by the fact that Ψ_1 is a homeomorphism on $\boldsymbol{C}$, one can find a non-empty open set W with $W \subset U_{w_\infty} \cap N_{z_\infty}$ and $\Psi_1(W) \subset \Delta(w_\infty,\eta)$. Then it follows from (4.20) that $(\phi_{z_0}(w),w) \in N_+$ for every $w \in \Psi_1(W)$. Hence, there exists a positive constant $\tilde{\gamma}$ with $m(\tilde{\boldsymbol{B}}) \leq \pi\eta^2 - \tilde{\gamma}$. Moreover, since $\phi_{z_0}^{n_j} \to \phi_{z_0}$ locally uniformly on $\Delta(w_\infty,\delta')$ and N_+ is open in $\boldsymbol{C}^2$, there are a sequence $\{w'_{n_j}\} \subset \Delta(w_{n_j},\eta)$ and a positive constant $\mu < \eta$, which does not depend on n_j, such that $(\phi_{z_0}^{n_j}(w), w) \in N_+$ for all $w \in \Delta(w'_{n_j}, \mu)$ and for all n_j. So, without loss of generality, we can assume that $m(\tilde{B}_{n_j}) \leq \pi\eta^2 - \gamma$ for all j and some positive constant γ.

Here, recall that $(z_0, h_{z_0}^{n_j}(w)) \in (F^{\circ n_j})^{-1}(\phi_{z_0}^{n_j}(w), w)$ for $w \in \Delta(w_\infty, \delta')$ by (4.10), $(F^{\circ n_j})^{-1}(N_+) \subset N_+$ by Proposition 2.3 and $\{z_0\} \times N_{z_0} = N_+ \cap \{(z, w) \in \boldsymbol{C}^2 \mid z = z_0\}$ for $|z_0| < R_1$ by Theorem 4.6. Then, if $(\phi_{z_0}^{n_j}(w), w) \in N_+$ for some $w \in \Delta(w_{n_j}, \eta)$, we have $(z_0, h_{z_0}^{n_j}(w)) \in \{z_0\} \times N_{z_0} \subset N_+$. Therefore, $(z_0, h_{z_0}^{n_j}(w)) \in N_+$ for all $w \in \Delta(w'_{n_j}, \mu)$; and

$$h_{z_0}^{n_j}(\Delta(w'_{n_j}, \mu)) \subset V_{n_j} \cap J_{z_0}^c \quad \text{for all } j. \tag{4.21}$$

Lemma 4.15. *For all sufficiently large j, there is a positive constant $\tilde{\gamma}$ such that*

$$m(V_{n_j} \cap J_{z_0})/m(V_{n_j}) \leq 1 - \tilde{\gamma}.$$

Proof. From the estimate (2) in Lemma 4.14, it follows that

$$m(h_{z_0}^{n_j}(\Delta(w'_{n_j}, \mu))) \geq \pi\{\mu|(h_{z_0}^{n_j})'(w'_{n_j})|(1-\kappa)\}^2, \quad m(V_{n_j}) \leq \pi\{\eta|(h_{z_0}^{n_j})'(w_{n_j})|(1+\kappa)\}^2.$$

On the other hand, by (1) of Lemma 4.14 and $M\eta < M\delta'' < 1$ as in the proof of (4.18) we have that

$$|(h_{z_0}^{n_j})'(w'_{n_j})/(h_{z_0}^{n_j})'(w_{n_j}) - 1| \leq M|w'_{n_j} - w_{n_j}| \leq M\eta;$$
$$1 - M\eta \leq |(h_{z_0}^{n_j})'(w'_{n_j})/(h_{z_0}^{n_j})'(w_{n_j})|$$

for all sufficiently large j. These combined with (4.21) yield that

$$\begin{aligned}\frac{m(V_{n_j} \cap J_{z_0})}{m(V_{n_j})} &\leq \frac{m(V_{n_j}) - m(h_{z_0}^{n_j}(\Delta(w'_{n_j}, \mu)))}{m(V_{n_j})} \\ &\leq 1 - \frac{\mu^2|(h_{z_0}^{n_j})'(w'_{n_j})|^2(1-\kappa)^2}{\eta^2|(h_{z_0}^{n_j})'(w_{n_j})|^2(1+\kappa)^2} \leq 1 - \frac{\mu^2(1-M\eta)^2(1-\kappa)^2}{\eta^2(1+\kappa)^2} < 1\end{aligned}$$

for all sufficiently large j. □

By the lemma above, we conclude that $v(w_0) < 1$ for every $w_0 \in J_{z_0}$. Therefore, we have shown that 2-dimensional Lebesgue measure of J_{z_0} is equal to 0 in Case 1.

Case 2. ϕ_{z_0} is a constant map on $\Delta(w_\infty, \delta')$.

Since $\phi_{z_0}(w_\infty) = z_\infty$, we have $\phi_{z_0}(w) = z_\infty$ on $\Delta(w_\infty, \delta')$ in this case; and consequently, $l = \{z_\infty\} \times \Delta(w_\infty, \delta)$ and $J_+(R_1) \cap l = \{z_\infty\} \times J_{z_\infty}$. Therefore, without using the notion of holomorphic motion, one can find a positive constant $\tilde{\gamma}$ with $m(\tilde{B}) \leq \pi\eta^2 - \tilde{\gamma}$. Then, repeating exactly the same argument as in Case 1, we can show that Lemma 4.15 also holds in Case 2; so that $v(w_0) < 1$. Hence, the 2-

dimensional Lebesgue measure of J_{z_0} equals to 0, completing the proof of Theorem 4.10 in Case 2. □

Remark. We used the condition (ii) of (3) of Theorem 4.10 only to show the assertion (4.11). Clearly this is a very artificial condition, and so we would like to remove it. However, we do not know at this moment whether it is really needed or not.

By (2) of Proposition 2.2 and Theorem 4.10, we have the following:

COROLLARY 4.16. *Assume that F satisfies all the conditions in Theorem* 4.10 *and further assume that the critical values of F are not contained in J_+. Then the Lebesgue measure of J_+ is equal to* 0. *In particular, the Lebesgue measure of J_+ of Hénon maps are equal to* 0.

5. An example

For an arbitrary constant $a \in \boldsymbol{C}^*$, we consider a polynomial map

$$\tilde{F}_a(z, w) = (aw^m, P(w) + aQ(z, w)) \quad \text{for } (z, w) \in \boldsymbol{C}^2,$$

of degree $d \geq 2$ and P, Q are polynomials of the form

$$P(w) = w^d + O(w^{d-1}), \quad Q(z, w) = \Sigma_{mn_1+n_2 \leq d, n_2 < d} a_{n_1 n_2} z^{n_1} w^{n_2}, \quad a_{n_1 n_2} \in \boldsymbol{C}, \quad m < d$$

Let us denote by $c_1, \ldots, c_{d-1}$ the critical points of P and J_P, K_P the Julia set, the filled-in Julia set of P, respectively. Throughout this section, we always assume that:

(5.1) Each c_i belongs to the immediate basin of some attracting periodic point p_i of P with period k_i.

Notice that the Hénon map $F_{a,c} = (aw, w^2 - az + c)$ considered in [4; Theorem 3.9, Corollary 3.29] is a typical example of such a map F_a with $d = 2$. Also, consider a polynomial $P(w) = w^d + c$ with $d \geq 3$ and assume that $P(w)$ has one attracting fixed point. Then

$$\tilde{F}_a(z, w) = (aw^m, w^d + c - a\Sigma_{mn_1+n_2 \leq d, n_2 < d} z^{n_1} w^{n_2})$$

is an example of maps that satisfy all the conditions required above. Indeed, it is a result due to Fatou that in this case the only one critical point 0 of P is in the immediate basin of attrating fixed point.

Let $\phi(z, w) = (az, w)$ and define the map F_a by

$$F_a(z, w) = \phi^{-1} \circ \tilde{F}_a \circ \phi(z, w) = (w^m, P(w) + aQ(az, w)).$$

The main purpose of this section is to prove that all the conditions of Main result 2 are satisfied for our F_a if $|a|$ is sufficiently small. For such a F_a, there exist constants $R_2 < R_2''$ which are chosen as in Proposition 2.2. Set $R_1 = R_2^m + \varepsilon_0$ and $R_1'' = (R_2'')^m + \varepsilon_0$. In particular, since P is a polynomial of degree

$d \geq 2$, we can assume that $P^{\circ n}(w) \to \infty$ for $w \notin \Delta(R_2)$. Since each c_i belongs to some attractive immediate basin, P is expanding on J_P and there are a Riemannian metric on $\boldsymbol{C}$, an open neighbourhood U of J_P and a constant γ such that $|P'(w)| > \gamma > 1$ on U. Let $\{p_i^k\}_{k=1}^{k_i} = \{P^{\circ(k-1)}(p_i)\}_{k=1}^{k_i}$ be the orbit of p_i and $\{U_i^k\}_{k=1}^{k_i}$ the immediate basin of $\{p_i^k\}_{k=1}^{k_i}$ with $p_i^k \in U_i^k$ for $1 \leq i \leq d-1$. Then, from the results due to Fatou and Sullivan, P has no other non-repelling cycles and any other component V of $K_P \backslash J_P$ is preperiodic to $\{U_i^k\}_{k=1}^{k_i}$ for $1 \leq i \leq d-1$; this means that there are some integer $l \geq 1$ and U_i^k such that $P^{\circ l} : V \to U_i^k$ is surjective. On the other hand, we know that

(5.2) the number of components of $K_P \backslash J_P$ is $0, 1, 2$ or ∞ [7; Theorem 4.2.16];

(5.3) J_P is connected and locally connected [7; Theorem 4.4.5].

Thus it follows from [7; Proposition 4.4.6] that for any constant $\varepsilon > 0$ the number of components of $K_P \backslash J_P$ whose diameters exceed ε is finite. Together with the fact that any boundary of Fatou components are contained in J_P, one can see that only finitely many components of $K_P \backslash J_P$ are not contained in U; except U_i^k, we say them U_j for $1 \leq j \leq j_1$. We can now choose domains $\tilde{V}_i^k$ for $1 \leq i \leq d-1$ and $\tilde{V}_j$ for $1 \leq j \leq j_1$ with the following properties:

(i) $p_i^k \in \overline{\tilde{V}_i^k} \subset U_i^k$, $\overline{\tilde{V}_j} \subset U_j$;

(ii) $\overline{P^{\circ k_i}(\tilde{V}_i^k)} \subset \tilde{V}_i^k$ and $\overline{P^{\circ l_j}(\tilde{V}_j)} \subset \tilde{V}_i^k$ for some integers l_j, i and k;

(iii) $\hat{\boldsymbol{C}} \backslash \{(\bigcup_{i=1}^{d-1} \bigcup_{k=1}^{k_i} \tilde{V}_i^k) \cup (\bigcup_{j=1}^{j_1} \tilde{V}_j) \cup A_P\} \subset U$,

where $A_P = \bigcup_{n \geq 1}^{\infty} (P^{\circ n})^{-1}(\Delta(R_2'')^c)$ is the set of escaping points of P.

Let $\tilde{F}_a^{\circ n}(z,w) = (\tilde{f}_n(z,w), \tilde{g}_n(z,w))$, $\quad \tilde{g}_{z_0}^n(w) = \tilde{g}_n(z_0,w)$, $\quad \tilde{F}_a^n(z_0,w_0) = (\tilde{z}_n, \tilde{w}_n)$.

From a direct calculation, we can see that $\tilde{g}_n(z,w) = P^{\circ n}(w) + Q_n(z,w)$ and all the coefficients of $\tilde{f}_n$ and Q_n contain positive power of a. Under this situation, we can prove the following lemmas.

Lemma 5.1. *There exists a constant $a_0 > 0$ such that, for $0 < |a| < a_0$,*

(1) $\tilde{F}_a$ has attractive cycles $\{\tilde{p}_i^k\}_{k=1}^{k_i}$ of order k_i;

(2) $\{z \in \boldsymbol{C} \mid |z| < |a|R_1''\} \times \tilde{V}_i^k$ is contained in the immediate basin of $\tilde{p}_i^k$ for $1 \leq i \leq d-1$;

(3) $\{z \in \boldsymbol{C} \mid |z| < |a|R_1''\} \times \tilde{V}_j$ is mapped into some $\{z \in \boldsymbol{C} \mid |z| < |a|R_1''\} \times \tilde{V}_i^k$ by $\tilde{F}_a^{\circ l_j}$ for $1 \leq j \leq j_1$.

Proof. Since the proofs of (1) and (2) are similar to those of [4; Lemma 3.10], we omit it. Since $\overline{P^{\circ l_j}(\tilde{V}_j)} \subset \tilde{V}_i^k$, we can see that if $|a|$ is small enough, then

$$\tilde{F}_a^{l_j}(\{z \in \boldsymbol{C} \mid |z| < |a|R_1''\} \times \tilde{V}_j) \subset \{z \in \boldsymbol{C} \mid |z| < |a|R_1''\} \times \tilde{V}_i^k,$$

for $j = 1, \ldots, j_1$, proving (3). □

Since U is an open neighbourhood of J_P, there is an integer N with $\partial(P^{\circ N})^{-1}(\Delta(R_2''))\subset U$. Put $\tilde{S}_n^a=\phi(\tilde{S}_n)$, where $\tilde{S}_n$ is the subset of $\boldsymbol{C}^2$ defined as in the proof of Theorem 4.3 for F_a. Then we have the following:

Lemma 5.2. *There exists a constant $a_0>0$ such that, for $0<|a|<a_0$ and for $(z_0,w_0)\in\tilde{S}_n^a$, if $\tilde{w}_1,\dots,\tilde{w}_l\in U$ and $\tilde{w}_{l+1},\dots,\tilde{w}_n\in\Delta(R_2'')$, then $n-l\le N$.*

Proof. For any constant $\varepsilon>0$ and $|z_0|<|a|R_1''$, we can assume that $|\tilde{g}_{z_0}^{N+1}(w)-P^{\circ(N+1)}(w)|<\varepsilon$ on $\Delta(R_2'')$ by rechoosing a_0 small, if necessary. Therefore, we can assume that $(\tilde{g}_{z_0}^{N+1})^{-1}(\Delta(R_2''))\subset(P^{\circ N})^{-1}(\Delta(R_2''))$, proving our assertion. □

We set

$$\tilde{U}=\left\{z\in\boldsymbol{C}\,\middle|\,|z|<|a|R_1''\right\}\times\left\{\Delta(\tilde{R}_2)\Big\backslash\left(\bigcup_{i=1}^{d-1}\bigcup_{k=1}^{k_i}\overline{\tilde{V}_i^k}\cup\bigcup_{j=1}^{j_1}\overline{\tilde{V}_j}\right)\right\},$$

$$(x_l,y_l)=D\tilde{F}_a^{\circ l}(z_0,w_0)(\alpha,1)\quad\text{for }\alpha\in\boldsymbol{C}\text{ with }|\alpha|<R_2''\text{ and }1\le l\le n.$$

Lemma 5.3. *Assume that (z_0,w_0), $(\tilde{z}_l,\tilde{w}_l)\in\tilde{U}$ for $1\le l\le n$. Then there exist constants $a_0>0$, $C>0$ not depending on n such that, if $0<|a|<a_0$, then*
(i) $|x_l|\le C|a|\,|y_l|<|y_l|$,
(ii) $|y_l|>C\lambda^l$, *where we set* $\lambda=(\gamma+1)/2>1$.

Proof. The lemma is proved by Lemma 5.2 and similar discussion in [4; Lemma 3.5], and hence we omit it. □

The proof of the following lemma is similar to that of [4; Proposition 3.4], and hence is left to the reader:

Lemma 5.4. *There exist positive constants $a_0>0$, $C>1$, $\lambda>1$ such that if (z_0,w_0), $(\tilde{z}_l,\tilde{w}_l)\in\tilde{U}$ for $1\le l\le n$ and $D\tilde{F}_a^{\circ n}(z_0,w_0)=\begin{pmatrix}a_n & b_n\\ c_n & d_n\end{pmatrix}$, then $|d_n|\ge C\lambda^n$, $|c_n|\le|d_n|/\tilde{R}_2$, $|a_n|\le C|d_n|$, $|b_n|\le C|a|\,|d_n|$.*

Finally, we show that F_a satisfies all the conditions of Main result 2, if $|a|$ is sufficiently small. By Lemmas 5.1 and 5.4, the set $\tilde{C}_n(z_0)$ of critical values of $\tilde{g}_{z_0}^n$ is not contained $\Delta(R_2'')\backslash\{(\bigcup_{i=1}^{d-1}\bigcup_{k=1}^{k_i}\overline{\tilde{V}_i^k})\cup(\bigcup_{j=1}^{j_1}\overline{\tilde{V}_j})\}$ for all $|z_0|<|a|R_1''$. By taking suitable domains V_i^k, V_j with $U_i^k\supset V_i^k\supset\overline{\tilde{V}_i^k}$, $U_j\supset V_j\supset\overline{\tilde{V}_j}$ and repeating the same discussion as in Lemma 5.1 to V_i^k and V_j, we can see that

$$\tilde{J}_+(|a|R_1'')\subset\left\{z\in\boldsymbol{C}\,\middle|\,|z|<|a|R_1''\right\}\times\left[\Delta(R_2)\Big\backslash\left\{\left(\bigcup_{i=1}^{d-1}\bigcup_{k=1}^{k_i}\overline{V_i^k}\right)\cup\left(\bigcup_{j=1}^{j_1}\overline{V_j}\right)\right\}\right];$$

$$\tilde{C}_n(z_0) \notin \Delta(R_2'') \Bigg\backslash \left\{ \left(\bigcup_{i=1}^{d-1} \bigcup_{k=1}^{k_i} \overline{\tilde{V}_i^k} \right) \cup \left(\bigcup_{j=1}^{j_1} \overline{\tilde{V}_j} \right) \right\} \text{ for } |z_0| < |a| R_1'',$$

after rechoosing a_0 small enough, if necessary, where we set $\tilde{J}_+(|a|\tilde{R}_1) = \phi(J_+(\tilde{R}_1))$. It shows that $\tilde{F}_a$ satisfies the condition (1). For $\tilde{R}_2$ with $R_2 < \tilde{R}_2 < R_2''$ and $R_1' = \tilde{R}_2^m + \varepsilon_0$ we can see that $\tilde{F}_a$ satisfies the condition ($\mathscr{F}$). By Lemma 5.4 and the proof of Lemma 5.3, $\tilde{F}_a$ also satisfies the condition (3). Therefore we have shown that all the conditions of Main result 2 are fulfilled for F_a.

References

[1] E. Bedford and J. Smillie, Polynomial diffeomorphisms of $\boldsymbol{C}^2$: currents, equilibrium measure and hyperbolicity, Invent. Math., **103** (1991), 69–99.

[2] D. L. Cohn, Measure Theory, Birkhäuser, Boston, 1980.

[3] J. Diller, Dynamics of birational maps of $\boldsymbol{P}^2$, Indiana Univ. Math. J., **45** (1996), 721–772.

[4] J. E. Fornæss and N. Sibony, Complex Hénon mappings in $\boldsymbol{C}^2$ and Fatou-Bieberbach domains, Duke Math. J., **65** (1992), 345–380.

[5] M. Jonsson, Dynamics of polynomial skew products on $\boldsymbol{C}^2$: exponents, connectedness and expansion, Ph.D. Thesis: Dynamical studies in several complex variables, Royal Institute of Technology, 1997.

[6] O. Lehto und K. I. Virtanen, Quasikonforme Abbildungen, Grundlehren Math. Wiss. 126, Springer-Verlag, Berlin, 1965.

[7] S. Morosawa, Y. Nishimura, M. Taniguchi, and T. Ueda, Holomorphic Dynamics, Cambridge Stud. Adv. Math. 66, Cambridge Univ. Press, Cambridge, 2000.

[8] Y. Nishimura, Iteration of some birational polynomial quadratic maps of $\boldsymbol{P}^2$, Sūrikaisekikenkyūsho Kōkyūroku, **959** (1996), 152–167.

[9] J. Noguchi and T. Ochiai, Geometric Function Theory in Several Complex Variables, Transl. Math. Monographs 80, AMS, Providence, 1990.

[10] R. M. Range, Holomorphic Functions and Integral Representations in Several Complex Variables, Graduate Texts in Math. 108, Springer-Verlarg, New York, 1986.

[11] T. Shinohara, Iteration of some birational polynomial maps in $\boldsymbol{P}^2$, Proceedings of the Second ISAAC Congress Vol. 2 (H. G. W. Begehr, R. P. Gilbert and J. Kajiwara eds.), Kluwer Academic Publishers, Dordrecht, 2000, 1063–1070.

[12] D. P. Sullivan and W. P. Thurston, Extending holomorphic motions, Acta Math., **157** (1986), 243–257.

[13] M. Tsuji, Jitsu Kansuuron, Maki Shoten, Tokyo, 1962 (in Japanese).

Department of Mathematics
Faculty of Science
Kanazawa University
Kanazawa 920-1192
Japan
e-mail: sinohara@kappa.s.kanazawa-u.ac.jp