UNICITY THEOREMS FOR ENTIRE FUNCTIONS CONCERNING FOUR SMALL FUNCTIONS

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Abstract

This paper studies the problem of uniqueness of entire functions concerning four small functions and shows that if two entire functions f and g satisfy $\bar{E}(a_j, k, f) = \bar{E}(a_j, k, g)$ for $j = 1, 2, 3, 4$, where a_j are four distinct small functions with respect to f and g , and k is a positive integer or ∞ with $k \geq 8$, then $f \equiv g$.

1. Introduction and main result

In this paper, by meromorphic function we shall always mean a meromorphic function in the complex plane $\mathbb{C}$. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [1]. For any non-constant meromorphic function $f(z)$, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty$ except possibly for a set of r of finite linear measure. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r, a) = S(r, f)$. Let $S(f)$ be the set of meromorphic functions in the complex plane $\mathbb{C}$ which are small functions with respect to f . Note that $\mathbb{C} \in S(f)$ and $S(f)$ is a field (see [2]).

Let $f(z)$ be a nonconstant entire function, $a(z) \in S(f)$, and let k be a positive integer or ∞ . We denote by $\bar{E}(a, k, f)$ the set of distinct zeros of $f(z) - a(z)$ with multiplicities $\leq k$ (see [3]). In particular, we denote by $\bar{E}(a, \infty, f)$ the set of distinct zeros of $f(z) - a(z)$.

Let $f(z)$ and $g(z)$ be nonconstant entire functions and let $a(z) \in S(f) \cap S(g)$. We denote by $\bar{N}_0(r, a, f, g)$ the counting function of common zeros of $f(z) - a(z) = 0$ and $g(z) - a(z) = 0$ (ignoring multiplicities), each point counted only once. Let

$$(1.1) \quad \bar{N}_{12}(r, a, f, g) := \bar{N}(r, a, f) + \bar{N}(r, a, g) - 2\bar{N}_0(r, a, f, g),$$

then $\bar{N}_{12}(r, a, f, g)$ denotes the counting function of different solutions to $f(z) - a(z) = 0$ and $g(z) - a(z) = 0$. Set

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$$(1.2) \quad \lambda(a, f, g) := 1 - \limsup_{r \rightarrow \infty} \frac{\bar{N}_{12}(r, a, f, g)}{T(r, f) + T(r, g)}.$$

Obviously, $0 \leq \lambda(a, f, g) \leq 1$ (see [4]).

If

$$\bar{N}(r, a, f) - \bar{N}_0(r, a, f, g) = 0 \quad \text{and} \quad \bar{N}(r, a, g) - \bar{N}_0(r, a, f, g) = 0,$$

we say $f(z)$ and $g(z)$ share $a(z)$ IM. If

$$\bar{N}(r, a, f) - \bar{N}_0(r, a, f, g) = S(r, f) \quad \text{and} \quad \bar{N}(r, a, g) - \bar{N}_0(r, a, f, g) = S(r, g),$$

we say $f(z)$ and $g(z)$ share $a(z)$ “IM” (see [5]). It is obvious that if $f(z)$ and $g(z)$ share $a(z)$ IM, then $\bar{E}(a, \infty, f) = \bar{E}(a, \infty, g)$ and $\bar{N}_{12}(r, a, f, g) = 0$; if $f(z)$ and $g(z)$ share $a(z)$ “IM”, then $\bar{N}_{12}(r, a, f, g) = S(r, f) + S(r, g)$.

Recently, B. Li [6] and independently Y. Li [7] proved the following theorem:

THEOREM A. *If f and g are nonconstant entire functions sharing a_j IM for $j = 1, 2, 3, 4$, where a_1, a_2, a_3, a_4 are four distinct elements in $S(f) \cap S(g)$, then $f \equiv g$.*

In this paper, we deal with the problem of uniqueness of entire functions concerning four small functions. We shall deal with the problem of uniqueness of meromorphic functions concerning five small functions in another paper. In this paper, we prove the following results which are improvements of Theorem A:

THEOREM 1.1. *Let f and g be nonconstant entire functions and let a_j ($j = 1, 2, 3, 4$) be four distinct elements in $S(f) \cap S(g)$. If*

$$(1.3) \quad \bar{E}(a_j, k, f) = \bar{E}(a_j, k, g) \quad (j = 1, 2, 3, 4),$$

where k is a positive integer or ∞ with $k \geq 8$, then $f \equiv g$.

Remark. In Theorem 1.1, we do not suppose that f and g always share a_j ($j = 1, 2, 3, 4$) IM. In Theorem A, we suppose that f and g share a_j ($j = 1, 2, 3, 4$) IM, and hence

$$\bar{E}(a_j, k, f) = \bar{E}(a_j, k, g) \quad (j = 1, 2, 3, 4),$$

where $k = \infty$. In Theorem 1.1, $k \geq 8$, from this we know that Theorem 1.1 is an improvement of Theorem A.

THEOREM 1.2. *Let f and g be nonconstant entire functions and let a_j ($j = 1, 2, 3, 4$) be four distinct elements in $S(f) \cap S(g)$. If f and g share a_j ($j = 1, 2, 3$) “IM”, and*

$$(1.4) \quad \bar{E}(a_4, k, f) = \bar{E}(a_4, k, g),$$

where k is a positive integer or ∞ with $k \geq 3$, then $f \equiv g$.

THEOREM 1.3. *Let f and g be nonconstant entire functions and let a_j ($j = 1, 2, 3, 4$) be four distinct elements in $S(f) \cap S(g)$. If f and g share a_j ($j = 1, 2, 3$) "IM", and*

$$(1.5) \quad \lambda(a_4, f, g) > \frac{2}{3},$$

then $f \equiv g$.

Remark. In Theorem 1.3, we do not suppose that f and g always share a_4 IM. In Theorem A, we suppose that f and g share a_4 IM, and hence $\lambda(a_4, f, g) = 1$, from this we know that Theorem 1.3 is an improvement of Theorem A.

2. Some lemmas

LEMMA 2.1. *Let h be a nonconstant entire function, and let $a \in S(h)$ and $a \neq 0$. Then*

$$a'h - ah' \neq 0,$$

and

$$(2.1) \quad m\left(r, \frac{a'h - ah'}{h - a}\right) = S(r, h),$$

$$(2.2) \quad m\left(r, \frac{a'h - ah'}{h(h - a)}\right) = S(r, h).$$

Proof. Suppose that

$$a'h - ah' \equiv 0.$$

By integration we have

$$a \equiv ch,$$

where c is a nonzero constant. From this we get

$$T(r, a) = T(r, h) + O(1),$$

which contradicts $a \in S(h)$. Thus

$$a'h - ah' \neq 0.$$

From the following equalities

$$(2.3) \quad \frac{a'h - ah'}{h - a} = a' - \frac{a(h' - a')}{h - a},$$

$$(2.4) \quad \frac{a'h - ah'}{h(h - a)} = \frac{h'}{h} - \frac{h' - a'}{h - a},$$

we obtain (2.1) and (2.2).

LEMMA 2.2. *Let f and g be nonconstant entire functions and let a_1, a_2, a_3, a_4 be four distinct elements in $S(f) \cap S(g)$. If $f \not\equiv g$, then*

$$(2.5) \quad \bar{N}_0(r, a_4, f, g) \leq \sum_{j=1}^3 \bar{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g),$$

$$(2.6) \quad \bar{N}(r, a_4, f) + \bar{N}(r, a_4, g) \leq 2 \sum_{j=1}^4 \bar{N}_{12}(r, a_j, f, g) - \bar{N}_{12}(r, a_4, f, g) \\ + S(r, f) + S(r, g).$$

The proof of Lemma 2.2 is given in §3.

LEMMA 2.3 (see [1]). *Let $f(z)$ be a nonconstant meromorphic function and let a_1, a_2, a_3 be three distinct elements in $S(f) \cup \{\infty\}$. Then*

$$T(r, f) \leq \sum_{j=1}^3 \bar{N}(r, a_j, f) + S(r, f).$$

Let $f(z)$ be a nonconstant entire function, $a(z) \in S(f)$, and k be a positive integer. We denote by $\bar{N}_k(r, a, f)$ the counting function of distinct zeros of $f(z) - a(z)$ with multiplicities $\leq k$, by $\bar{N}_{(k+1)}(r, a, f)$ the counting function of distinct zeros of $f(z) - a(z)$ with multiplicities $\geq k+1$, each point in these counting functions is counted only once (see [3]).

LEMMA 2.4. *Let $f(z)$ be a nonconstant entire function and let a_1, a_2 be two distinct elements in $S(f)$, and k be a positive integer. Then*

$$(2.7) \quad T(r, f) \leq \sum_{j=1}^2 \bar{N}(r, a_j, f) + S(r, f),$$

$$(2.8) \quad \sum_{j=1}^2 \bar{N}_{(k+1)}(r, a_j, f) \leq \frac{1}{k} T(r, f) + S(r, f).$$

Proof. Noting that $f(z)$ is entire function, from Lemma 2.3, we can obtain (2.7). Noting that for $j = 1, 2$

$$k\bar{N}_{(k+1)}(r, a_j, f) + \bar{N}(r, a_j, f) \leq N(r, a_j, f) \leq T(r, f) + S(r, f),$$

we have from (2.7)

$$k \sum_{j=1}^2 \bar{N}_{(k+1)}(r, a_j, f) + T(r, f) \leq 2T(r, f) + S(r, f).$$

From this we get (2.8).

LEMMA 2.5. *Let $f(z)$ be a nonconstant entire function and let a_1, a_2, a_3 be three distinct elements in $S(f)$, and k be a positive integer. Then*

$$(2.9) \quad \frac{3}{2}T(r, f) \leq \sum_{j=1}^3 \bar{N}(r, a_j, f) + S(r, f),$$

$$(2.10) \quad \sum_{j=1}^3 \bar{N}_{(k+1)}(r, a_j, f) \leq \frac{3}{2k}T(r, f) + S(r, f).$$

Proof. By Lemma 2.4, we have

$$T(r, f) \leq \bar{N}(r, a_1, f) + \bar{N}(r, a_2, f) + S(r, f),$$

$$T(r, f) \leq \bar{N}(r, a_2, f) + \bar{N}(r, a_3, f) + S(r, f),$$

$$T(r, f) \leq \bar{N}(r, a_3, f) + \bar{N}(r, a_1, f) + S(r, f).$$

From the above we get (2.9). Again by Lemma 2.4, we have

$$\bar{N}_{(k+1)}(r, a_1, f) + \bar{N}_{(k+1)}(r, a_2, f) \leq \frac{1}{k}T(r, f) + S(r, f),$$

$$\bar{N}_{(k+1)}(r, a_2, f) + \bar{N}_{(k+1)}(r, a_3, f) \leq \frac{1}{k}T(r, f) + S(r, f),$$

$$\bar{N}_{(k+1)}(r, a_3, f) + \bar{N}_{(k+1)}(r, a_1, f) \leq \frac{1}{k}T(r, f) + S(r, f).$$

From the above we get (2.10).

LEMMA 2.6. *Let $f(z)$ be a nonconstant entire function and let a_1, a_2, a_3, a_4 be four distinct elements in $S(f)$, and k be a positive integer. Then*

$$(2.11) \quad 2T(r, f) \leq \sum_{j=1}^4 \bar{N}(r, a_j, f) + S(r, f),$$

$$(2.12) \quad \sum_{j=1}^4 \bar{N}_{(k+1)}(r, a_j, f) \leq \frac{2}{k}T(r, f) + S(r, f).$$

Proof. By Lemma 2.4, we have

$$T(r, f) \leq \bar{N}(r, a_1, f) + \bar{N}(r, a_2, f) + S(r, f),$$

$$T(r, f) \leq \bar{N}(r, a_3, f) + \bar{N}(r, a_4, f) + S(r, f).$$

From the above we get (2.11). Again by Lemma 2.4, we have

$$\begin{aligned}\bar{N}_{(k+1)}(r, a_1, f) + \bar{N}_{(k+1)}(r, a_2, f) &\leq \frac{1}{k} T(r, f) + S(r, f), \\ \bar{N}_{(k+1)}(r, a_3, f) + \bar{N}_{(k+1)}(r, a_4, f) &\leq \frac{1}{k} T(r, f) + S(r, f).\end{aligned}$$

From the above we get (2.12).

3. Proof of Lemma 2.2

If $\bar{N}_0(r, a_4, f, g) = S(r, f) + S(r, g)$, (2.5) holds obviously. In the following we suppose

$$(3.1) \quad \bar{N}_0(r, a_4, f, g) \neq S(r, f) + S(r, g).$$

Set

$$(3.2) \quad L(w) := \frac{w - a_1}{a_2 - a_1}.$$

Let $F(z) := L(f(z))$, $G(z) := L(g(z))$, $b_j := L(a_j)$ ($j = 1, 2, 3, 4$). From (3.2) b_1 , b_2 , b_3 , b_4 are distinct and $b_1 = 0$, $b_2 = 1$,

$$(3.3) \quad T(r, F) = T(r, f) + S(r, f), \quad T(r, G) = T(r, g) + S(r, g),$$

and $b_3, b_4 \in S(F) \cap S(G)$. Noting $f \not\equiv g$, we have

$$(3.4) \quad F \not\equiv G.$$

From (3.1) and (3.3), we get

$$(3.5) \quad \bar{N}_0(r, b_4, F, G) \neq S(r, F) + S(r, G).$$

Set

$$(3.6) \quad H := \frac{F'(a'G - aG')(F - G)}{F(F - 1)G(G - a)} - \frac{G'(a'F - aF')(F - G)}{G(G - 1)F(F - a)},$$

where $a = b_3$. Then we have from (3.6)

$$(3.7) \quad H = \frac{(F - G)Q}{F(F - 1)(F - a)G(G - 1)(G - a)},$$

where

$$\begin{aligned}(3.8) \quad Q &= F'(a'G - aG')(G - 1)(F - a) - G'(a'F - aF')(F - 1)(G - a) \\ &= a'FF'G^2 - a'FF'G - a(a - 1)FF'G' - aa'F'G^2 + aa'F'G \\ &\quad - a'F^2GG' + a'FGG' + a(a - 1)F'GG' + aa'F^2G' - aa'FG'.\end{aligned}$$

Suppose that $H \equiv 0$. From (3.4) and (3.6) we obtain

$$(3.9) \quad \frac{F'(a'G - aG')}{(F - 1)(G - a)} \equiv \frac{G'(a'F - aF')}{(G - 1)(F - a)}.$$

If a is a constant, from (3.4) and (3.9) we get $a = 1$, which is a contradiction. Thus, a is not a constant. By Lemma 2.1,

$$a'F - aF' \neq 0.$$

From (3.9) we have

$$\frac{F'(a'G - aG')}{G'(a'F - aF')} - 1 \equiv \frac{(F-1)(G-a)}{(G-1)(F-a)} - 1.$$

Thus,

$$\frac{a'[(F'-G')G - (F-G)G']}{G'(a'F - aF')} \equiv \frac{(1-a)(F-G)}{(G-1)(F-a)}.$$

From this we get

$$(3.10) \quad \frac{F' - G'}{F - G} \equiv \frac{(1-a)G'(a'F - aF')}{a'G(G-1)(F-a)} + \frac{G'}{G}.$$

By (3.5), we know that there is a point z_0 such that z_0 is a common zero of $F - b_4$ and $G - b_4$ that is not a zero or a pole of $a, a', b_4, b_4 - 1, b_4 - a$. It is obvious that z_0 is a pole of the left-hand side of (3.10), and not a pole of the right-hand side of (3.10), which is a contradiction. Thus,

$$(3.11) \quad H \neq 0.$$

From (3.7) and (3.11), we have

$$(3.12) \quad \begin{aligned} \bar{N}_0(r, b_4, F, G) &\leq N(r, 0, H) + S(r, F) + S(r, G) \\ &\leq m(r, H) + N(r, H) + S(r, F) + S(r, G). \end{aligned}$$

From (3.6) we have

$$\begin{aligned} H &= \frac{F'}{F-1} \cdot \frac{a'G - aG'}{G(G-a)} - \left(\frac{F'}{F-1} - \frac{F'}{F} \right) \cdot \frac{a'G - aG'}{G-a} \\ &\quad - \left(\frac{G'}{G-1} - \frac{G'}{G} \right) \cdot \frac{a'F - aF'}{F-a} + \frac{G'}{G-1} \cdot \frac{a'F - aF'}{F(F-a)}. \end{aligned}$$

Again by Lemma 2.1 we obtain

$$(3.13) \quad m(r, H) = S(r, F) + S(r, G).$$

Next we make an estimate of $N(r, H)$.

By (3.6) we know that the poles of H occur possibly only from the zeros of $F, G, F-1, G-1, F-a$ and $G-a$, and the poles of a . Let S_0 be the set of all zeros, 1-points and poles of a and let for $j = 1, 2, 3$

$$A_j := \{z \mid F(z) - b_j(z) = 0\} \setminus S_0, \quad B_j := \{z \mid G(z) - b_j(z) = 0\} \setminus S_0,$$

where $b_1 = 0$, $b_2 = 1$ and $b_3 = a$. Thus, the poles of H occur possibly only from the set

$$\left(\bigcup_{1 \leq p \leq 3} A_p \right) \cup \left(\bigcup_{1 \leq q \leq 3} B_q \right) \cup S_0.$$

Let

$$\begin{aligned} S_1 &:= \bigcup_{1 \leq p \leq 3} \{A_p \cap B_p\}, \\ S_2 &:= \left\{ \bigcup_{1 \leq p \leq 3} A_p \right\} \setminus \left\{ \bigcup_{1 \leq q \leq 3} B_q \right\}, \\ S_3 &:= \left\{ \bigcup_{1 \leq q \leq 3} B_q \right\} \setminus \left\{ \bigcup_{1 \leq p \leq 3} A_p \right\}, \\ S_4 &:= \bigcup_{\substack{1 \leq p \leq 3 \\ 1 \leq q \leq 3 \\ p \neq q}} \{A_p \cap B_q\}. \end{aligned}$$

It is clear that S_1 is the set of common zeros of $F - b_j$ and $G - b_j$ ($j = 1, 2, 3$); S_2 is the set of zeros of $F - b_j$ ($j = 1, 2, 3$) which are not zeros of $G - b_k$ ($k = 1, 2, 3$); S_3 is the set of zeros of $G - b_j$ ($j = 1, 2, 3$) which are not zeros of $F - b_k$ ($k = 1, 2, 3$); S_4 is the set of zeros of $F - b_j$ which are zeros of $G - b_k$, where $1 \leq j, k \leq 3$ and $j \neq k$. From this we have

$$\bigcup_{1 \leq j \leq 4} S_j = \left(\bigcup_{1 \leq p \leq 3} A_p \right) \cup \left(\bigcup_{1 \leq q \leq 3} B_q \right).$$

Thus, the poles of H occur possibly only from the set

$$\left(\bigcup_{1 \leq j \leq 4} S_j \right) \cup S_0.$$

Since $a \in S(F) \cap S(G)$, the contribution of S_0 to $N(r, H)$ is at most $S(r, F) + S(r, G)$. We next proceed to estimate the contribution of $\bigcup_{1 \leq j \leq 4} S_j$ to $N(r, H)$. We discuss the following four cases:

CASE 1. The contribution of S_1 to $N(r, H)$.

We distinguish the following three subcases:

SUBCASE 1.1. Suppose that $z_{11} \in A_1 \cap B_1$, and assume that z_{11} is a zero of F of order p_1 and G of order q_1 , from (3.8), we know that z_{11} is a zero of Q of order at least $p_1 + q_1 - 1$. Noting that z_{11} is a zero of $F - G$, from (3.7) we know that z_{11} is not a pole of H .

SUBCASE 1.2. Suppose that $z_{12} \in A_2 \cap B_2$. Noting that z_{12} is a zero of $F - G$,

a simple pole of $F'/(F-1)$ and $G'/(G-1)$, from (3.6) we know that z_{12} is not a pole of H .

SUBCASE 1.3. Suppose that $z_{13} \in A_3 \cap B_3$, from (2.4), we know that z_{13} is a simple pole of $(a'F - aF')/(F(F-a))$ and $(a'G - aG')/(G(G-a))$. Noting that z_{13} is a zero of $F - G$, from (3.6) we know that z_{13} is not a pole of H .

From the above, we know that the points in S_1 are not poles of H . Thus, the contribution of S_1 to $N(r, H)$ is 0.

CASE 2. The contribution of S_2 to $N(r, H)$.

We distinguish the following three subcases:

SUBCASE 2.1. Suppose that $z_{21} \in A_1$ and $z_{21} \notin \bigcup_{1 \leq q \leq 3} B_q$. Then z_{21} is a zero of F , not a zero of G , $G-1$ and $G-a$. From (3.6), we know that z_{21} is a pole of H of order at most 1.

SUBCASE 2.2. Suppose that $z_{22} \in A_2$ and $z_{22} \notin \bigcup_{1 \leq q \leq 3} B_q$. Then z_{22} is a zero of $F-1$, not a zero of G , $G-1$ and $G-a$. From (3.6), we know that z_{22} is a pole of H of order at most 1.

SUBCASE 2.3. Suppose that $z_{23} \in A_3$ and $z_{23} \notin \bigcup_{1 \leq q \leq 3} B_q$. Then z_{23} is a zero of $F-a$, not a zero of G , $G-1$ and $G-a$. From (2.4) and (3.6), we know that z_{23} is a pole of H of order at most 1.

From the above, we know that the points in S_2 are poles of H of order at most 1.

CASE 3. The contribution of S_3 to $N(r, H)$.

Similar to Case 2, we can obtain that the points in S_3 are poles of H of order at most 1.

CASE 4. The contribution of S_4 to $N(r, H)$.

Suppose that $z_4 \in S_4$. Then $z_4 \in A_p$ and $z_4 \in B_q$, where $1 \leq p, q \leq 3$ and $p \neq q$. Without loss of generality we can assume that $z_4 \in A_1$ and $z_4 \in B_2$. Then, z_4 is a zero of F and $G-1$. From (3.6), we know that z_4 is a pole of H of order at most 2. Thus, the points in S_4 are poles of H of order at most 2.

Noting that each point of S_2 and S_3 is counted once, each point of S_4 is counted twice in

$$\sum_{j=1}^3 \bar{N}_{12}(r, b_j, F, G),$$

from the above we obtain that the contribution of $\bigcup_{1 \leq j \leq 4} S_j$ to $N(r, H)$ is at most

$$\sum_{j=1}^3 \bar{N}_{12}(r, b_j, F, G).$$

Thus,

$$(3.14) \quad N(r, H) \leq \sum_{j=1}^3 \bar{N}_{12}(r, b_j, F, G) + S(r, F) + S(r, G).$$

Substituting (3.13) and (3.14) into (3.12) we derive

$$\bar{N}_0(r, b_4, F, G) \leq \sum_{j=1}^3 \bar{N}_{12}(r, b_j, F, G) + S(r, F) + S(r, G).$$

From this we get (2.5). Noting that

$$\bar{N}(r, a_4, f) + \bar{N}(r, a_4, g) = 2\bar{N}_0(r, a_4, f, g) + \bar{N}_{12}(r, a_4, f, g),$$

from this and (2.5) we obtain (2.6), and Lemma 2.2 is thus proved.

4. Preparatory theorems

Using Lemma 2.2, we can prove the following theorems:

THEOREM 4.1. *Let f and g be nonconstant entire functions and let a_j ($j = 1, 2, 3, 4$) be four distinct elements in $S(f) \cap S(g)$. If $f \not\equiv g$, then*

$$(4.1) \quad 2T(r, f) + 2T(r, g) \leq 7 \sum_{j=1}^4 \bar{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g).$$

Proof. By Lemma 2.2 we have

$$(4.2) \quad \begin{aligned} \bar{N}(r, a_4, f) + \bar{N}(r, a_4, g) &\leq 2 \sum_{j=1}^4 \bar{N}_{12}(r, a_j, f, g) - \bar{N}_{12}(r, a_4, f, g) \\ &\quad + S(r, f) + S(r, g). \end{aligned}$$

In the same manner as above, we have

$$(4.3) \quad \begin{aligned} \bar{N}(r, a_3, f) + \bar{N}(r, a_3, g) &\leq 2 \sum_{j=1}^4 \bar{N}_{12}(r, a_j, f, g) - \bar{N}_{12}(r, a_3, f, g) \\ &\quad + S(r, f) + S(r, g), \end{aligned}$$

$$(4.4) \quad \begin{aligned} \bar{N}(r, a_2, f) + \bar{N}(r, a_2, g) &\leq 2 \sum_{j=1}^4 \bar{N}_{12}(r, a_j, f, g) - \bar{N}_{12}(r, a_2, f, g) \\ &\quad + S(r, f) + S(r, g), \end{aligned}$$

$$(4.5) \quad \begin{aligned} \bar{N}(r, a_1, f) + \bar{N}(r, a_1, g) &\leq 2 \sum_{j=1}^4 \bar{N}_{12}(r, a_j, f, g) - \bar{N}_{12}(r, a_1, f, g) \\ &\quad + S(r, f) + S(r, g). \end{aligned}$$

By Lemma 2.6, from (4.2), (4.3), (4.4) and (4.5) we obtain (4.1), and Theorem 4.1 is thus proved.

THEOREM 4.2. *Let f and g be nonconstant entire functions and let a_j ($j = 1, 2, 3, 4$) be four distinct elements in $S(f) \cap S(g)$. If f and g share a_1 “IM”, and $f \not\equiv g$, then*

$$(4.6) \quad \frac{3}{2}T(r, f) + \frac{3}{2}T(r, g) \leq 5 \sum_{j=2}^4 \bar{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g).$$

Proof. By Lemma 2.2 we may obtain (4.2), (4.3) and (4.4). Noting that f and g share a_1 “IM”, from (4.2), (4.3) and (4.4) we have for $i = 2, 3, 4$

$$(4.7) \quad \bar{N}(r, a_i, f) + \bar{N}(r, a_i, g) \leq 2 \sum_{j=2}^4 \bar{N}_{12}(r, a_j, f, g) - \bar{N}_{12}(r, a_i, f, g) + S(r, f) + S(r, g).$$

By Lemma 2.5, from (4.7) we obtain (4.6).

THEOREM 4.3. *Let f and g be nonconstant entire functions and let a_j ($j = 1, 2, 3, 4$) be four distinct elements in $S(f) \cap S(g)$. If f and g share a_1, a_2 “IM”, and $f \not\equiv g$, then*

$$(4.8) \quad T(r, f) + T(r, g) \leq 3 \sum_{j=3}^4 \bar{N}_{12}(r, a_j, f, g) + S(r, f) + S(r, g).$$

Proof. By Lemma 2.2 we obtain (4.2) and (4.3). Noting that f and g share a_1, a_2 “IM”, from (4.2) and (4.3) we have for $i = 3, 4$

$$(4.9) \quad \bar{N}(r, a_i, f) + \bar{N}(r, a_i, g) \leq 2 \sum_{j=3}^4 \bar{N}_{12}(r, a_j, f, g) - \bar{N}_{12}(r, a_i, f, g) + S(r, f) + S(r, g).$$

By Lemma 2.4, from (4.9) we obtain (4.8).

THEOREM 4.4. *Let f and g be nonconstant entire functions and let a_j ($j = 1, 2, 3, 4$) be four distinct elements in $S(f) \cap S(g)$. If f and g share a_1, a_2, a_3 “IM”, and $f \not\equiv g$, then*

$$(4.10) \quad T(r, f) + T(r, g) \leq 3\bar{N}_{12}(r, a_4, f, g) + S(r, f) + S(r, g).$$

Proof. Noting that f and g share a_3 “IM”, from Theorem 4.3 we obtain (4.10).

5. Proof of main results

5.1. Proof of Theorem 1.1

Suppose that $f \not\equiv g$. By Theorem 4.1, we can obtain (4.1).

If $k = \infty$, from (1.3) we have

$$(5.1) \quad \bar{N}_{12}(r, a_j, f, g) = 0 \quad (j = 1, 2, 3, 4).$$

Substituting (5.1) into (4.1) we obtain

$$2T(r, f) + 2T(r, g) = S(r, f) + S(r, g),$$

which is a contradiction. Next, we assume that k is a positive integer.

Noting (1.3), we have

$$(5.2) \quad \bar{N}_{12}(r, a_j, f, g) \leq \bar{N}_{(k+1)}(r, a_j, f) + \bar{N}_{(k+1)}(r, a_j, g) \quad (j = 1, 2, 3, 4).$$

Substituting (5.2) into (4.1) we obtain

$$(5.3) \quad 2T(r, f) + 2T(r, g) \leq 7 \sum_{j=1}^4 \bar{N}_{(k+1)}(r, a_j, f) + 7 \sum_{j=1}^4 \bar{N}_{(k+1)}(r, a_j, g) \\ + S(r, f) + S(r, g).$$

By Lemma 2.6, we have

$$(5.4) \quad \sum_{j=1}^4 \bar{N}_{(k+1)}(r, a_j, f) \leq \frac{2}{k} T(r, f) + S(r, f),$$

$$(5.5) \quad \sum_{j=1}^4 \bar{N}_{(k+1)}(r, a_j, g) \leq \frac{2}{k} T(r, g) + S(r, g).$$

Substituting (5.4) and (5.5) into (5.3) we obtain

$$2T(r, f) + 2T(r, g) \leq \frac{14}{k} T(r, f) + \frac{14}{k} T(r, g) + S(r, f) + S(r, g),$$

which contradicts the assumption $k \geq 8$. Thus, $f \equiv g$.

5.2. Proof of Theorem 1.2

Suppose that $f \not\equiv g$. By Theorem 4.4, we can obtain (4.10). Noting (1.4), we have

$$(5.6) \quad \bar{N}_{12}(r, a_4, f, g) \leq \bar{N}_{(k+1)}(r, a_4, f) + \bar{N}_{(k+1)}(r, a_4, g) \\ \leq \frac{1}{k+1} T(r, f) + \frac{1}{k+1} T(r, g) + S(r, f) + S(r, g).$$

Substituting (5.6) into (4.10) we obtain

$$T(r, f) + T(r, g) \leq \frac{3}{k+1} T(r, f) + \frac{3}{k+1} T(r, g) + S(r, f) + S(r, g),$$

which contradicts the assumption $k \geq 3$. Thus, $f \equiv g$.

5.3. Proof of Theorem 1.3

Suppose that $f \not\equiv g$. By Theorem 4.4, we can obtain (4.10). From (4.10), we get

$$\lambda(a_4, f, g) \leq \frac{2}{3},$$

which contradicts (1.5). Thus, $f \equiv g$.

6. Supplement to main results

Proceeding as in the proof of Theorem 1.1, by Theorem 4.2, 4.3 respectively, we have the following theorems, which are improvements of Theorem A.

THEOREM 6.1. *Let f and g be nonconstant entire functions and let a_j ($j = 1, 2, 3, 4$) be four distinct elements in $S(f) \cap S(g)$. If f and g share a_1 “IM”, and*

$$\bar{E}(a_j, k, f) = \bar{E}(a_j, k, g) \quad (j = 2, 3, 4),$$

where k is a positive integer or ∞ with $k \geq 6$, then $f \equiv g$.

THEOREM 6.2. *Let f and g be nonconstant entire functions and let a_j ($j = 1, 2, 3, 4$) be four distinct elements in $S(f) \cap S(g)$. If f and g share a_1 and a_2 “IM”, and*

$$\bar{E}(a_j, k, f) = \bar{E}(a_j, k, g) \quad (j = 3, 4),$$

where k is a positive integer or ∞ with $k \geq 4$, then $f \equiv g$.

Proceeding as in the proof of Theorem 1.3, by Theorem 4.1, 4.2, 4.3 respectively, we have the following theorems, which are improvements of Theorem A.

THEOREM 6.3. *Let f and g be nonconstant entire functions and let a_j ($j = 1, 2, 3, 4$) be four distinct elements in $S(f) \cap S(g)$. If*

$$\sum_{j=1}^4 \lambda(a_j, f, g) > \frac{26}{7},$$

then $f \equiv g$.

THEOREM 6.4. *Let f and g be nonconstant entire functions and let a_j ($j = 1, 2, 3, 4$) be four distinct elements in $S(f) \cap S(g)$. If f and g share a_1 “IM”, and*

$$\sum_{j=2}^4 \lambda(a_j, f, g) > \frac{27}{10},$$

then $f \equiv g$.

THEOREM 6.5. *Let f and g be nonconstant entire functions and let a_j ($j = 1, 2, 3, 4$) be four distinct elements in $S(f) \cap S(g)$. If f and g share a_1 and a_2 “IM”, and*

$$\lambda(a_3, f, g) + \lambda(a_4, f, g) > \frac{5}{3},$$

then $f \equiv g$.

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