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KLEINIAN GROUPS WITH SINGLY CUSPED PARABOLIC FIXED POINTS

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Abstract

We consider geometrically infinite Kleinian groups and, in particular, groups with singly cusped parabolic fixed points. In order to distinguish between different geometric characteristics of such groups, we introduce the notion of horospherical tameness. We give a brief discussion of the fractal nature of their limit sets. Subsequently, we use Jørgensen's analysis of punctured torus groups to give a canonical decomposition into ideal tetrahedra of the geometrically infinite end. This enables us to relate horospherical tameness to Diophantine properties of Thurston's end invariants.

1. Introduction

We consider finitely generated, geometrically infinite Kleinian groups acting on hyperbolic 3-space. Geometrically infinite groups were first shown to exist over 30 years ago by Greenberg in [16]. The first explicit examples were constructed by Jørgensen in [17]. Subsequently, these groups have attracted a great deal of attention from various different points of view.

In this paper we characterise geometrically infinite groups by introducing carefully so called *Jørgensen points*. The term 'Jørgensen end' was introduced loosely by Sullivan in his proof of the Cusp Finiteness Theorem. After making this notion more precise, we consider a fundamental class of Jørgensen points which we term *singly cusped parabolic fixed points*. These are of particular interest as they display both geometrically finite and infinite characteristics. For clarity, we only consider finitely generated Kleinian groups whose only obstruction to being geometrically finite is the existence of such singly cusped parabolic fixed points. To distinguish between the different types of behaviour that such groups can exhibit, we introduce the notion of *horospherical tameness*

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for finitely generated Kleinian groups. This turns out to be distinct from geometrical tameness considered by Thurston, Canary and others.

Subsequently, we exploit the interplay between the geometrically finite and infinite features of finitely generated Kleinian groups with singly cusped parabolic fixed points in order to obtain deeper understanding of Jørgensen points. This enables us to give, in this special case, an elementary account of some deep results about geometrically infinite Kleinian groups. We hope that this will be useful to people trying to understand the general case. In particular, we obtain the result that these groups are of 2-convergence-type. This enables us to draw some conclusions concerning the dynamics on the associated 3-manifold. In particular, these groups are shown to have (uniformly perfect) non-porous limit sets. This makes them interesting objects for fractal geometry.

Finally, we build on Jørgensen's study of quasi-Fuchsian punctured torus groups to provide concrete examples of groups with singly cusped parabolic fixed points. We show that an immediate consequence of Jørgensen's analysis of the Ford domain is that there is a canonical choice of cutting surfaces needed to show that the group is geometrically tame. Moreover, these cutting surfaces lead to a canonical decomposition into ideal tetrahedra of the geometrically infinite end of the associated manifold. This tetrahedral decomposition is related to the well known decomposition of the figure eight knot complement into ideal tetrahedra. By studying these cutting surfaces (or equivalently the tetrahedra) we consider the different types of behaviour that can occur, and relate these to the Diophantine approximation properties of Thurston's end invariants. Specifically, we show that, in this context, horospherical tameness is equivalent to the end invariant being badly approximable. In particular, this clarifies a remark of Sullivan concerning such groups made in [35]. In conclusion, we give two examples of geometrically tame groups one of which is horospherically tame and the other of which is not. The former is the Fibonacci example of Mumford, McMullen and Wright. The latter has end invariant whose continued fraction expansion is given by an arithmetic progression.

We would like to emphasise that the main goal of this paper is not exclusively to produce new theorems which apply in a wide context, but rather to analyse a particular concrete family of examples. Within this context, we study some well known deep theorems which hold in greater generality and we show how particular cases of these theorems may be proved in a straightforward manner. We hope that this will be useful to others working in the field. While doing this, we also develop new structural analysis that makes various intuitive connections much more precise.

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2. Singly cusped parabolic fixed points

2.1. Geometrically finite versus geometrically infinite; Jørgensen limit points

First, we give a brief description of what it means for a hyperbolic 3-manifold to be geometrically finite, respectively geometrically infinite.

Throughout we shall let G denote a finitely generated Kleinian group and $M = \boldsymbol{H}^3/G$ the associated hyperbolic 3-manifold (M is always assumed to be oriented). In order to locate the dynamically and homologically interesting part of M, let $m \in M$ be an arbitrary observation point, and consider $\mathscr{L}_m(M)$, the set of geodesic loops that start and terminate at m (these loops are not necessarily closed geodesics). Further, a geodesic γ in M is called *loop-approximable*, if and only if there exists some observation point $m \in M$ such that each finite segment of γ may be approximated with arbitrary accuracy by segments of elements in $\mathscr{L}_m(M)$[1].

We may now define $\mathscr{G}(M)$, the *geodesic core of M*, by

$$\mathscr{G}(M) := \{\gamma \text{ geodesic in } M : \gamma \text{ is loop-approximable}\}.$$

This enables us to distinguish between the following two different classes of finitely generated Kleinian groups. We remark that our dynamical definition of geometrically finite Kleinian groups is based on an 'observation of Thurston', see Definition 8.4.1 on page 8.15 of [37], who was the first to realise the dynamical significance of the concept geometrical finiteness.

Definition. A finitely generated Kleinian group G and its associated hyperbolic 3-manifold M are called *geometrically finite* if the convex hull of the geodesic core is of finite hyperbolic volume. They are called *geometrically infinite* if this volume is infinite.

Recall that $\Lambda(G)$, the *limit set* of G, is the derived set of some arbitrary point in hyperbolic space, that is the set of accumulation points of the G-orbit of that point. An element of $\Lambda(G)$ is called *radial limit point* if it admits a conical approach by orbit points from inside hyperbolic space [7, 28]. Furthermore, recall that a parabolic fixed point of G is called *bounded* if it is either of rank 2 or else is doubly cusped (see next section for further details).

The following classical result of Beardon and Maskit [8] (see also [12]) characterises geometrically finite Kleinian groups in terms of their limit set.

Beardon-Maskit. *A finitely generated Kleinian group G is geometrically finite if and only if every point of* $\Lambda(G)$ *is either a radial limit point or a bounded parabolic fixed point, that is* $\Lambda(G)$ *'splits'*.

[1] i.e., each finite segment of γ is contained in an arbitrarily small neighbourhood of some closed loop.

In order to define a class of limit points which is generic for geometrically infinite Kleinian groups we require the notion of visibility at infinity. A limit point $\xi \in \Lambda(G)$ is called *visible* if and only if, for some Dirichlet domain F of G based at a point z_0 in hyperbolic space, there exists a group element $g \in G$ such that $g(F)$ contains the hyperbolic geodesic ray from $g(z_0)$ to ξ. The following definition clarifies Sullivan's notion 'Jørgensen end', which he loosely introduced in Figure 1 of [36].

Definition. An element of the limit set of a Kleinian group is called *Jørgensen point* if and only if it is visible and not a bounded parabolic fixed point.

In contrast to the classical result of Beardon and Maskit above, the following result gives a characterisation of geometrically infinite Kleinian groups in terms of the limit set.

Proposition 2.1. *A finitely generated Kleinian group is geometrically infinite if and only if its limit set contains a Jørgensen point.*

Proof. The first assertion, namely that the existence of a Jørgensen point already implies geometrical infiniteness, is an immediate consequence of the result of Beardon and Maskit.

For the second assertion, assume that the finitely generated Kleinian group G is geometrically infinite with no visible points except possibly bounded parabolic fixed points. Since, by Beardon and Maskit, $\Lambda(G)$ does not split, there exists a point $\xi \in \Lambda(G)$ which is neither a radial limit point nor a bounded parabolic fixed point. In the Poincaré ball model let σ denote the ray connecting the origin with ξ. Now, if σ intersects at most finitely many G-images of the Dirichlet domain F based at the origin, then there exist an element $g \in G$ such that σ is eventually contained in $g(F)$. This implies that ξ is visible, and hence a Jørgensen point.

Thus, σ must intersect infinitely many G-translates of F, denoted $g_i(F)$, and in particular their boundaries. Let $\tilde{x}_i$ denote the points at which σ intersects these boundaries, and let $x_i = g_i^{-1}(\tilde{x}_i)$ denote their canonical pull backs onto F. Since ξ is not a radial limit point, it follows that there exists a subsequence (x_{i_k}) which eventually leaves every subset of F with bounded hyperbolic diameter. Thus (x_{i_k}) accumulates at the boundary of hyperbolic space, and hence has a subsequence converging to a point in the intersection of the boundary of hyperbolic space and the boundary of F. By convexity, this accumulation point is visible and not a bounded parabolic fixed point; hence it must be a Jørgensen point. □

2.2. Parabolic fixed points

In this section we classify parabolic fixed points for Kleinian groups. In particular, we introduce singly cusped parabolic fixed points and discuss some of their general properties.

The following facts are well known. A parabolic fixed point p of a Kleinian group G is called rank 1 or rank 2 depending on the type of its stabiliser G_p. Namely,

- p has *rank* 1 when G_p is isomorphic to a finite extension of $\boldsymbol{Z}$, and so is necessarily cyclic or infinite dihedral,
- p has *rank* 2 when G_p is isomorphic to a finite extension of $\boldsymbol{Z}^2$.

A rank 1 parabolic fixed point p is called *doubly cusped* if and only if there exist a pair of disjoint open discs in $\Omega(G)$ tangent at p, where as usual $\Omega(G) = \hat{\boldsymbol{C}} - \Lambda(G)$ denotes the set of ordinary points of G. Furthermore, a parabolic fixed point is called *bounded* if and only if it is either of rank 2 or else is doubly cusped.

In contrast, we are now going to set up the concept of singly cusped rank 1 parabolic fixed points. It will be clear from Proposition 2.1 that groups with such points are necessarily geometrically infinite. In particular, the class of singly cusped parabolic points provides simple examples of Jørgensen points, and hence of geometrically infinite ends for hyperbolic 3-manifolds.

Definition. Let G be a Kleinian group containing parabolic elements. A parabolic fixed point p of G is called *singly cusped* if and only if the following conditions hold:

(i) there exists an open disc h_p contained in $\Omega(G)$ with p on its boundary;
(ii) any open disc in the Riemann sphere disjoint from h_p and with p on its boundary contains a point of the limit set $\Lambda(G)$.

For the rest of the paper we will assume, unless stated otherwise, that G is a group with a singly cusped rank 1 parabolic fixed point p. For simplicity we assume that there are no other geometrically infinite ends of $M = \boldsymbol{H}^3/G$ (i.e., all Jørgensen points in $\Lambda(G)$ are in the orbit of p). In section 3 we shall show that such groups exist.

Observe that the stabiliser of a singly cusped parabolic fixed point is necessarily a cyclic group of parabolic transformations. There are two directions to approach a singly cusped parabolic fixed point along a horosphere in the quotient manifold M. The singly cusped parabolic fixed point has a different appearance depending on our direction of approach. Seen from a certain direction it is cusped and so looks like a rank 1 parabolic fixed point of a geometrically finite Kleinian group. Seen from the opposite direction, it looks like the parabolic fixed point of one of the doubly degenerate groups considered by Jørgensen and Marden in [20]. Thus, heuristically, singly cusped parabolic fixed points combine both geometrically finite and infinite behaviour.

Also, observe that a finitely generated Kleinian can only have finitely many singly cusped parabolic fixed points. This fact is an immediate consequence of the Ahlfors' Finiteness Theorem [1] or of Sullivan's Cusp Finiteness Theorem [35] because singly cusped parabolic fixed points lie on the boundary of some component of $\Omega(G)$, and hence correspond to one of the punctures of one of the finitely many boundary surfaces of finite type.

2.3. Geometry of singly cusped parabolic fixed points

We now consider the geometry associated to a singly cusped parabolic fixed point. The following theorem of Leutbecher [23], generalising work of Shimizu [32], gives a uniform bound on invariant horoballs at parabolic fixed points of a Kleinian group. Recall that if a group G acts on a space X, a subset Y of X is said to be *precisely invariant* under a subgroup G' of G if and only if $g(Y) = Y$ for every $g \in G'$ and $g(Y)$ is disjoint from Y for every element of $G - G'$.

LEUTBECHER. *Let G be a Kleinian group containing the parabolic transformation $z \mapsto z + t$ for some positive t. Then the horoball H_∞ at height t centred at the parabolic fixed point ∞ is precisely invariant under G_∞, the stabiliser of ∞ in G.*

In what follows we always assume, without loss of generality, that $t = 1$. With this normalisation, we refer to H_∞ as the *Leutbecher horoball.* Also, the Leutbecher horoball may clearly be defined for any parabolic fixed point p in terms of its stabiliser, and the Leutbecher horoballs so obtained are pairwise disjoint by construction.

Let p be any singly cusped parabolic fixed point of a Kleinian group G. Let G_p be the subgroup of G fixing p. By definition, there is a horodisc h_p in $\Omega(G)$ with p on its boundary. Without loss of generality, we take this to be the largest precisely invariant horodisc at p in $\Omega(G)$. Let h_p^* be the hyperbolic half space with ideal boundary h_p. The precisely invariant horodisc h_p and Leutbecher horoball H_p will be called the *horopair* (h_p, H_p) at p. This means that the part of a fundamental domain for G_p not in $H_p \cup h_p^*$ is a 'semi-infinite box', see Figure 1.

In what follows we make the following normalisation. We use the Poincaré extension from the Riemann sphere to the upper half space model of hyperbolic

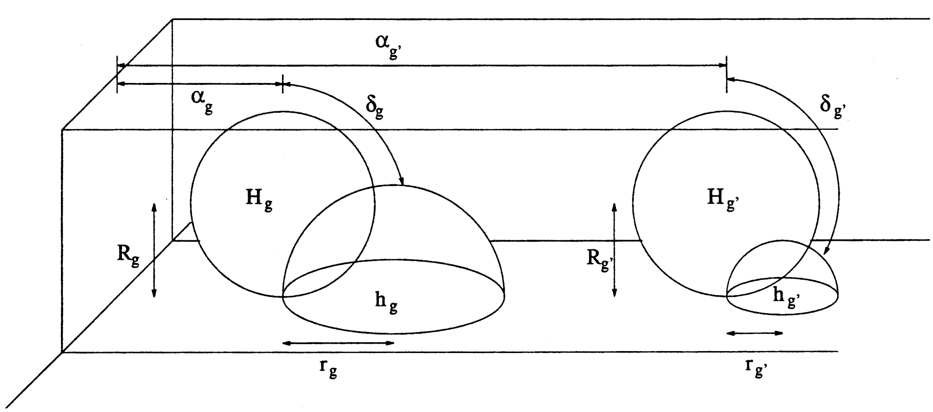

FIGURE 1. Horopairs for a map g with small drift and altitude, and a map g' with large drift and altitude.

3-space, and write points as $(z, t) \in \boldsymbol{C} \times \boldsymbol{R}_+$ or in quaternion notation $z + tj$ [2, 7]. We take p to be the point at infinity, and its stabiliser in G to be $G_p = G_\infty = \langle z \mapsto z + 1 \rangle$. Moreover, we assume that $h_\infty = \{z \in \boldsymbol{C} : \Im(z) > 0\}$. Thus, the half space h_∞^* is $\{(z, t) \in \boldsymbol{H}^3 : \Im(z) > 0\}$. The condition that ∞ is singly cusped implies that there exists a sequence of points $z_j \in \Lambda(G)$ so that $\Im(z_j)$ tends to $-\infty$. Applying elements of G_∞ if necessary, we assume, without loss of generality, that $\Re(z_j) \in [0, 1]$. Also, Leutbecher's theorem implies that $H_\infty = \{(z, t) \in \boldsymbol{H}^3 : t > 1\}$.

We now consider points in the orbit of ∞. Let $g : z \mapsto (az + b)/(cz + d)$ with $ad - bc = 1$ be any element of G not in the stabiliser of ∞ (so $c \neq 0$). We consider the image of the above configuration under g. To be more precise, we actually consider the image of this configuration under the coset gG_∞. Clearly, the point $g(\infty) = a/c$ is a singly cusped rank 1 parabolic fixed point. The horoball $H_g = H_{g(\infty)}$ is a Euclidean ball in $\boldsymbol{H}^3$ with

$$\text{radius } R_g = \frac{1}{2|c|^2} \quad \text{and centre} \quad \left(\frac{a}{c}, \frac{1}{2|c|^2}\right).$$

The horodisc $h_g = h_{g(\infty)}$ is a Euclidean disc in the Riemann sphere with

$$\text{radius } r_g = \frac{1}{|c\bar{d} - d\bar{c}|} = \frac{1}{2|c|^2 \Im(d/c)} \quad \text{and centre} \quad \frac{a\bar{d} - b\bar{c}}{c\bar{d} - d\bar{c}}.$$

The half space h_g^* is the Euclidean hemisphere in $\boldsymbol{H}^3$ with the same centre and radius as h_g. Also I_g', the isometric sphere of g^{-1}, is a Euclidean hemisphere in $\boldsymbol{H}^3$ with radius $\rho_g = 1/|c|$ and centre $g(\infty) = a/c$ on the Riemann sphere. For obvious reasons, we call $g(\infty) = a/c$ the *pole* of g^{-1} and $g^{-1}(\infty) = -d/c$ the pole of g.

We define the altitude and the drift of a horopair which will be crucial in our analysis.

- The *altitude* α_g of the horoball H_g is defined to be the Euclidean distance of $g(\infty)$ from the boundary of h_∞. (It is clear that $\alpha_g = -\Im(a/c)$ and that α_g measures how far H_g is along the cusp at ∞.)
- The *drift* δ_g of the horopair (h_g, H_g) measures the relative sizes of H_g and h_g. It is defined to be

$$\delta_g = \frac{R_g}{r_g} = \Im(d/c).$$

Intuitively, the larger the drift the further one has to 'travel' along ∂H_g from the 'top' of H_g until one reaches the boundary of h_g^*.

Observe that the altitude and the drift are dual to one another in the sense that $\alpha_{g^{-1}} = \delta_g$ and $\delta_{g^{-1}} = \alpha_g$. This is immediate from the definition in terms of a, c and d. Alternatively, this can be deduced geometrically as follows. Consider the hyperbolic geodesic l_g from ∞ to $g(\infty)$. The distance from l_g along ∂H_∞ to h_∞^* is α_g. Applying g^{-1}, the horopairs (h_∞, H_∞) and (h_g, H_g) are mapped to the

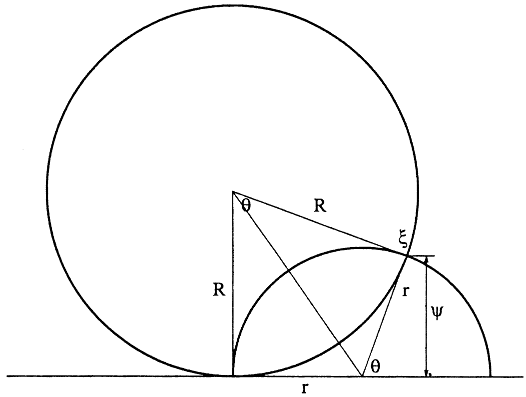

FIGURE 2. The construction of ψ.

horopairs $(h_{g^{-1}}, H_{g^{-1}})$ and (h_∞, H_∞) respectively, and l_g is mapped to $l_{g^{-1}}$. The drift of g has become the altitude of g^{-1}.

Let ψ_g be the maximal height of a point lying in $\partial H_g \cap \partial h_g^*$. The point ξ_g attaining this height is the point of $\partial H_g \cap \partial h_g^*$ closest to $\partial H_g \cap l_g$. We have the following lemma which gives a concrete (Euclidean) estimate of the drift.

LEMMA 2.2. *With the above notation, we have that* $\psi_g = 2R_g/(1+\delta_g^2)$.

Proof. We refer to Figure 2. For simplicity we drop the subscript g from all quantities.

Let θ be the angle subtended at the centre of H between $g(\infty)$ and ξ. An elementary Euclidean argument implies that the angle subtended at the centre of H between $g(\infty)$ and the centre of h is equal to $\theta/2$. Thus,

$$\sin(\theta/2) = r/\sqrt{R^2+r^2}, \quad \cos(\theta/2) = R/\sqrt{R^2+r^2}.$$

In particular, we see that $\sin(\theta) = 2Rr/(R^2+r^2)$.

Now, θ is also the angle subtended at the centre of h between the Riemann sphere and ξ. Thus, using the equality $r = R/\delta$,

$$\begin{aligned}\psi &= r\sin(\theta) \\ &= \frac{2Rr^2}{R^2+r^2} \\ &= \frac{2R(R/\delta)^2}{R^2+(R/\delta)^2} \\ &= \frac{2R}{1+\delta^2}.\end{aligned}$$

□

Proposition 2.3. *Let G be a Kleinian group with a singly cusped parabolic fixed point normalised as above. Let (g_n) be a sequence of elements of G with unbounded altitude. Then as n tends to infinity, either δ_{g_n} tends to infinity or R_{g_n} tends to zero.*

Proof. Suppose the result is false. That is, there exist positive constants δ_0 and R_0 so that $\delta_{g_n} < \delta_0$ and $R_{g_n} > R_0$, for all n.

Consider the horoballs $H_{g_n^{-1}}$. These have altitude less than δ_0, since $\alpha_{g_n^{-1}} = \delta_{g_n}$. By choosing different coset representatives if necessary, we may assume that the centres of these horoballs have imaginary part between $-\delta_0$ and 0 and real part between 0 and 1. That is, their centres lie in a compact set. Moreover, the radii of these horoballs are bounded from below, since $R_{g_n^{-1}} = R_{g_n} > R_0 > 0$. This implies that the horoballs cannot be disjoint, which is a contradiction. □

Corollary 2.4. *Let G be as in the previous proposition. For any $g \in G$ we have $\psi_g \le \Delta(\alpha_g)$, where $\Delta : \boldsymbol{R}^+ \to (0,1]$ is a function with $\lim_{x\to\infty} \Delta(x) = 0$.*

Let $N(G)$ denote the Nielsen region of G, that is the convex hull in $\boldsymbol{H}^3$ of the limit set $\Lambda(G)$. The quotient $\mathscr{C} = N(G)/G$ is called the *convex core* of $M = \boldsymbol{H}^3/G$. It is standard to divide the Nielsen region or the convex core into two parts, called the thick part and the thin part. Given a positive number ε, the *ε-thick part* of $N(G)$ consists of all those points $x \in N(G)$ for which the hyperbolic ball of radius 2ε centred at x contains no points $g(x)$, for all $g \in G - \{I\}$. Equivalently, the hyperbolic balls of radius ε centred at each point in the orbit of x are disjoint. Likewise, the *ε-thick part* of $\mathscr{C}$ consists of those points $x \in \mathscr{C}$ whose ε-neighbourhood in $\mathscr{C}$ is an embedded ball. In both cases, the *ε-thin part* is defined to be the complement of the ε-thick part. By a well known result of Margulis [24], generalising Leutbecher's theorem, there exists a universal constant ε_0, such that the ε_0-thin part of $N(G)$ is a disjoint union of horoballs centred at parabolic fixed points and of tubes around the axes of loxodromic elements with translation length at most $2\varepsilon_0$.

In what follows we choose ε so that the Leutbecher horoballs H_g correspond to the ε-thin part of M. In fact, by a simple calculation we see that $\varepsilon = \log((1+\sqrt{5})/2)$, although we do not use this here. Our analysis above leads to the following result which combines several well known properties of geometrically infinite groups (see [11], [12], [15] for example). In this case, the proof is elementary and illustrates the geometrical construction involved.

Proposition 2.5. *Let G be a Kleinian group with a singly cusped parabolic fixed point, let $M = \boldsymbol{H}^3/G$ be the associated 3-manifold and let $\mathscr{C}$ be the convex core of M. Then there exists a sequence (x_n) of points in the ε-thick part of $\mathscr{C}$ whose hyperbolic distance to any fixed reference point in M tends to infinity with n. Moreover, the hyperbolic volume of $\mathscr{C}$ is infinite.*

Proof. Without loss of generality, we normalise G in the manner described above. That is, we assume that ∞ is a singly cusped parabolic fixed point with stabiliser $G_\infty = \langle z \mapsto z+1 \rangle$. This enables us to define the horopairs, drift and altitude of elements of $G - G_\infty$. Let (g_n) be a sequence of group elements for which α_{g_n} tends to infinity with n. Let (x_n) be the sequence of points at height 1 lying above the centres of H_{g_n}.

The points (x_n) clearly have unbounded hyperbolic distance from any given base point in $\boldsymbol{H}^3$. Moreover, by the previous corollary, they also have unbounded distance from the orbit of this base point.

For $\delta > 0$ sufficiently small, let B_n denote δ-balls about the points x_n. These δ-balls are contained in the convex core. From this set of balls we can extract a sequence of pairwise disjoint balls whose hyperbolic volume then gives rise to the infinite hyperbolic volume of the convex core of M. $\square$

We now introduce the class of horospherically tame Kleinian groups. We first require the notion of the K-neighbourhood of a horoball $H \subset \boldsymbol{H}^3$, which is defined as the collection of points a (hyperbolic) distance at most K from H.

Definition. Let G denote a Kleinian group with a singly cusped parabolic fixed point p. The group G is said to be *horospherically tame* if and only if there exists a positive constant K such that the Nielsen region $N(G)$ is contained in the union over all $g \in G$ of the K-neighbourhoods of the Leutbecher horoballs $H_{g(p)}$. In other words, the whole of $\mathscr{C}$ is contained in the K-neighbourhood of the cusp of $\mathscr{C}$.

Note, in our definition of horospherical tameness we have assumed for simplicity that all parabolic fixed points of G lie in one single orbit. Clearly, one can extend this definition to the case of finitely many orbits of parabolic fixed points. It is also possible to extend this definition in the obvious way to doubly degenerate parabolic fixed points, that is rank 1 fixed point which does not lie on the boundary of any disc in the limit set.

Observe that, for finitely generated groups, horospherical tameness and geometrical tameness are not the same. Recall that a hyperbolic manifold M is said to be *geometrically tame* if every end is either geometrically finite or simply degenerate (see §8.11 of [37] or [15]). That is, the end has a neighbourhood U homeomorphic to $S \times [0, \infty)$ (where S is a finite volume surface), and there exist a sequence of simplicial hyperbolic surfaces $\{f_n : S \to U\}$ such that $\{f_n(S)\}$ leaves every compact set in U and $f_n(S)$ is homotopic to $S \times \{0\}$ within U. One consequence of geometrical tameness in our situation is that there is a positive constant R_0 so that for any positive, arbitrarily large α there is a group element g with altitude $\alpha_g > \alpha$ and horoball H_g of radius $R_g > R_0$. In other words, for geometrically tame groups there is a sequence of group elements g_n with both α_{g_n} and δ_{g_n} tending to infinity with n. In section 3.7 we will give an example of a geometrically tame group which is not horospherically tame.

We remark that in section 3 we shall see that for punctured torus groups horospherical tameness can be detected from the bounded excursion pattern of the tail of the continued fraction expansion of the associated end invariant. Specifically, it will turn out that the end invariant is badly approximable if and only if the group is horospherically tame.

2.4. Fractal geometry and singly cusped parabolic fixed points

In this section we make a few qualitative and quantitative observations on limit sets of finitely generated Kleinian groups with singly cusped parabolic fixed points. Much of the material in this section is true for wider classes of geometrically infinite Kleinian group. We restrict our attention to groups with singly cusped parabolic points, since here many of the proofs are more straightforward. In particular, the results should be of interest to people working in fractal geometry. We show that limit sets of geometrically infinite Kleinian groups provide a wide class of examples that are uniformly perfect but not porous. Hence, they are of particular interest to fractal geometers.

It is known that the Hausdorff dimension of the limit set of a geometrically infinite, geometrically tame Kleinian group is equal to 2 [14]. Recently, in [10] this result was generalised to the case of an arbitrary geometrically infinite Kleinian group. In fact, the most delicate part of the proof in [10] shows that a geometrically infinite Kleinian group G whose exponent of convergence[2] $\delta(G)$ is strictly less than 2 has a limit set of positive 2-dimensional Lebesgue measure (contradicting the *Ahlfors' conjecture*). However, this means that its Hausdorff dimension is equal to 2. (We remark that in this statement the assumption that $\delta(G)$ is strictly less than 2 is purely hypothetical.)

Recall that a Kleinian group G is of 2-*convergence type* if and only if the series $\sum_{g\in G}(1-|g(v)|)^2$ converges, for some point v in the Poincaré ball model of hyperbolic space. The following proposition is proved using ideas of Sullivan.

PROPOSITION 2.6. *A finitely generated Kleinian group with singly cusped parabolic fixed points is of* 2*-convergence type.*

Proof. We refer to Figure 3. We assume that the reader is familiar with passage between the two equivalent models of hyperbolic space, the upper half space model and the Poincaré ball model. Where it is clear which of these two models we are in, we shall make no notational difference. Let G denote the finitely generated Kleinian group with singly cusped parabolic fixed points under consideration. We use the notation and normalisation introduced earlier, and assume in particular that the origin in the Poincaré ball model is contained in $N(G)$, the Nielsen region of G.

[2] The exponent of convergence of the Poincaré series $\sum_{g\in G}(1-|g(v)|)^s$, for some point v in the Poincaré ball model of hyperbolic space.

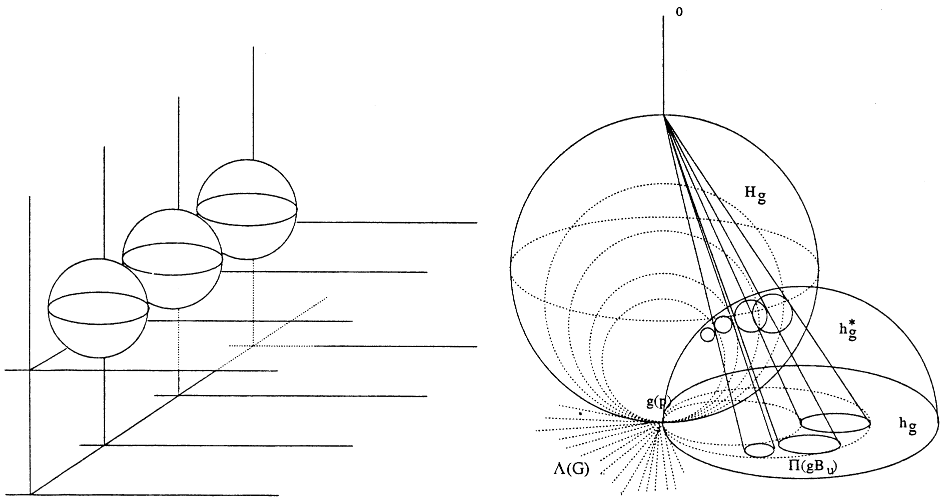

FIGURE 3. The balls $g(B_u)$ and their shadows.

Fix a hyperbolic ball contained in $H_\infty \cap h_\infty^*$, tangent to the boundary of H_∞ as well as the boundary of h_∞^*, and which is contained in a fundamental domain for the stabiliser of ∞ in G. Let B_u denote the corresponding hyperbolic ball, centred at u, in the Poincaré ball model. Let Π denote the *shadow map*, which projects subsets of the Poincaré ball radially from the origin to the boundary of hyperbolic space.

A straight forward Euclidean argument then shows that the set of shadows $\{\Pi(g(B_u)) : g \in G - E\}$ represents a packing of finite multiplicity of $\Omega(G)$; where E denotes some suitably chosen, finite set of exceptional elements of G. The finite multiplicity of the packing arises when dealing with horoballs with small drift. Together with the trivial fact that, in the Poincaré ball model, $\Omega(G)$ is of finite 2-dimensional Lebesgue measure, this gives

$$\sum_{g \in G-E} (\operatorname{diam}(\Pi(g(B_u))))^2 < \infty.$$

Combining this with the finiteness of E and the well known fact that, for $g \in G - E$, the quotient of $\operatorname{diam}(\Pi(g(B_u)))$ and $1 - |g(u)|$ is universally bounded from above and below, the assertion of the proposition follows. $\square$

We remark that the above proof in fact only uses the existence of a rank 1 cusped parabolic fixed point, and hence works for general Kleinian groups which have parabolic elements of that type.

It is well known that this result closely relates to some interesting properties of the dynamics on the associated 3-manifold [3, 28]. The following corollary gives a few of these.

Corollary 2.7. *Let G be a finitely generated Kleinian group with singly cusped parabolic fixed points, and let M denote the associated hyperbolic* 3-*manifold. Then*

- *the geodesic flow on M is not ergodic*;
- *M supports a Green's function*;
- *the harmonic measure of the ideal boundary of M is strictly positive.*

We end this section by giving a further immediate implication of the above results, which clearly exhibits the rich fractal structure of geometrically infinite limit sets. Recall the following notion, where $B(z,r)$ denotes the ball of radius r centred at z.

Definition. A set $X \subset \boldsymbol{R}^n$ is called *porous* if and only if there exist positive constants K and R such that for all $\xi \in X$ and $0 < r < R$ there exists $\eta = \eta(\xi, r) \in \boldsymbol{R}^n$ with the properties

- $B(\eta, Kr) \subset B(\xi, r)$,
- $B(\eta, Kr) \cap X = \emptyset$.

As Tukia has shown implicitly in [38], the limit set of a geometrically finite Kleinian group without rank 2 parabolic fixed points is porous. For geometrically infinite Kleinian groups we have the following statement.

Corollary 2.8. *The limit set of a geometrically infinite Kleinian group is not porous.*

Proof. It is an easy exercise in the theory of fractals to see that the porosity of a compact subset in $\boldsymbol{R}^n$ implies that its box-counting dimension is strictly less than n, and hence in particular the same is true for its Hausdorff dimension. Since, as mentioned above, the Hausdorff dimension of the limit set is 2, the result follows. □

The limit set of a finitely generated Kleinian group is uniformly perfect, see [30] (see also the survey articles [33], [34]). It hence follows that limit sets of finitely generated Kleinian groups with singly cusped parabolic fixed points are examples of uniformly perfect fractal sets which are not porous.

3. The punctured torus

3.1. Punctured torus groups

In this section we look at a family of concrete examples in some detail. These examples arise from so called *punctured torus groups*. Parts of the material we present may be found in the unpublished papers of Jørgensen [19], Wright [39] and McMullen-Mumford-Wright [27] as well as the recent paper of Minsky [26]. We now recall some standard terminology.

Definition. A Kleinian group $G < \mathrm{PSL}(2, \boldsymbol{C})$ is called a *punctured torus group* if and only if it is freely generated by two maps whose commutator is parabolic.

Clearly, there is a certain amount of freedom in choosing these generators. With this in mind, we define a *generator* g to be any element of the punctured torus group G with the property that there exists an element h of G so that the commutator $gh^{-1}g^{-1}h$ is parabolic. It is clear that h is also a generator, and (g, h) will be called a pair of *neighbours*. Indeed, (g, hg^n) is a pair of neighbours for any integer n. From an algebraic viewpoint, a pair of neighbours is a generating set for G, viewed as the free group on two generators. From a geometric point of view, conjugacy classes of generators correspond to homotopy classes of simple closed curves on the punctured torus, and the commutator corresponds to a loop around the puncture. In this setting, a pair of neighbours corresponds to a marking, i.e. a pair of simple closed curves intersecting exactly once.

If g and h are neighbours, then gh is also a generator and, moreover, (g, gh) and (h, gh) are two pairs of neighbours. We call the unordered set (g, h, gh) a *generator triple*. To each pair of neighbours (g, h) there are two generator triples, namely (g, h, gh) and (g, h, gh^{-1}). A classical result of Nielsen [29] states that, given a pair of neighbours, we may obtain any other pair of neighbours by passing through a sequence of successive generator triples. These changes in the generators are called *Nielsen moves*. A convenient geometric model describing this set up is the *Farey tessellation*. The sides of this tessellation form an infinite graph whose vertices correspond to generators and where two vertices are connected by an edge if and only if the associated generators are neighbours. In addition, each generator is associated to the same vertex as its inverse. In particular, Nielsen's result referred to above states that this graph is connected.

There is a standard way of embedding this graph into the closed upper half plane such that the generators (up to conjugacy and taking inverses) are in one to one correspondence to the rational numbers. This gives an infinite triangulation where every triangle corresponds to a generator triple. Identifying the upper half plane with the Teichmüller space of the punctured torus in the usual way, one obtains that groups with singly cusped parabolic fixed points correspond to irrational numbers. One of our goals will be to see how the geometry of such Kleinian groups relates to Diophantine approximation patterns of these irrational numbers.

We remark that most of the results in this section do not use singly cusped parabolic points in an essential way. Properties such as having a badly approximable end invariant could apply to doubly degenerate groups but would involve keeping track of the end invariants for both ends. Another context where similar ideas are used is the construction of explicit Ford domains for fibre bundles over the circle with fibre the punctured torus by Alestalo and Helling [6]. This is related to the Farey tessellation by Bowditch [13]. These have cyclic covers whose end invariants have a periodic continued fraction expansion (related to the combinatorics of the Ford domain) and so are well approximable. Thus these groups are horospherically tame (with an appropriate extension of the definition to cover doubly degenerate groups). However, it is clear that as the automorphism used becomes more complicated, the canonical horoballs need to be expanded more and more. This phenomenon is encapsulated in Proposition 3.16. Furthermore, cutting surfaces may be defined as in section 3.5 below and decompose such manifolds into ideal tetrahedra. The combinatorics of this decomposition come from the arithmetic of the end invariants (see page 329 of [13]).

3.2. Normalising punctured torus groups

We begin this section by deriving some general consequences about pairs of elements in $\mathrm{SL}(2,\boldsymbol{C})$ with parabolic commutator. This leads to Jørgensen's normalisation for punctured torus groups [19]. Even though these results are well known, the proofs are short and we include them. One consequence of the first result is that, if (g,h) is a pair of neighbours, then the trace of their commutator must be -2.

LEMMA 3.1. *Let g and h be elements of $\mathrm{SL}(2,\boldsymbol{C})$ whose commutator has trace $+2$. Then either g and h commute or they have a common fixed point.*

Proof. Without loss of generality we may assume that $k = gh^{-1}g^{-1}h$ fixes infinity. Thus $k : z \mapsto z + t$ for some $t \in \boldsymbol{C}$. If we have

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad h^{-1}gh = g' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

then $g' = k^{-1}g$, $a' = a - tc$ and $d' = d$. As g and g' are conjugate, their traces are equal, which implies $tc = 0$. Thus we have that either $t = 0$ or $c = 0$. That is, k is the identity or g fixes infinity. Letting $h' = ghg^{-1}$ and using $h'k = h$, a similar argument gives that either k is the identity or h fixes infinity. This proves the result. □

LEMMA 3.2. *Let g and h be elements of $\mathrm{SL}(2,\boldsymbol{C})$ whose commutator has the form*

$$gh^{-1}g^{-1}h = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}.$$

Then g and h have the form

$$\begin{pmatrix} a & b \\ a+d & d \end{pmatrix}.$$

Proof. Let g and g' have the same form as in the proof of the previous lemma. Since g and g' are conjugate, their traces are equal, namely $\mathrm{tr}(g') = a' + d' = a + d = \mathrm{tr}(g)$. Also $g' = (gh^{-1}g^{-1}h)^{-1}g$. That is

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a+2c & -b+2d \\ -c & -d \end{pmatrix}.$$

Thus $a + d = \mathrm{tr}(g) = \mathrm{tr}(g') = -a - d + 2c$, and so $c = a + d$ as claimed. It is clear that a similar argument works for h and $h' = ghg^{-1}$. □

We shall now return to the situation of a punctured torus group G. For the rest of the paper we shall normalise the commutator of the generators, which is parabolic by hypothesis, to be the translation of length 2, as already done in Lemma 3.2. This enables us to express the radii of the isometric sphere and the Leutbecher horoball of a generator in terms of its trace. Using the notation established in section 2.3, we obtain the following result.

Lemma 3.3. *Let G be a punctured torus group normalised as above. For a generator g of G we have that*

$$\rho_g = \frac{1}{|\mathrm{tr}(g)|}, \quad R_g = \frac{1}{2|\mathrm{tr}(g)|^2}, \quad \alpha_g = \delta_g.$$

Proof. The first two facts follow by definition. For the third, as $c = a + d$, we have $a/c = 1 - d/c$, and so

$$\alpha_g = \Im(-a/c) = \Im(-1 + d/c) = \Im(d/c) = \delta_g. \qquad \square$$

Proposition 3.4. *Let G denotes a punctured torus group normalised as above. For each g in G we have $|\mathrm{tr}(g)| \geq 1$.*

Proof. Let g and h be as in Lemma 3.2 and let f be the translation by 1. We will show that $G = \langle g, h \rangle$ has index 2 in $G' = \langle f, g, h \rangle$. This then implies that G' is discrete. By Leutbecher's lemma, the isometric sphere of g has radius at most 1. But this radius is equal to $1/|\mathrm{tr}(g)|$ by the previous lemma.

It is easy to see that $f^2 = gh^{-1}g^{-1}h$. An argument similar to that in the proof of Lemma 3.2 gives $\mathrm{tr}(gf^{-1}) = 0$ and $\mathrm{tr}(hf^{-1}) = 0$. Thus gf^{-1} and hf^{-1} have order 2 (they correspond to the hyperelliptic involution of the punctured torus). Hence $gf = fh^{-1}g^{-1}h$ and $hf = fh^{-1}gh^{-1}g^{-1}h$. This implies that $G' = G \cup fG$. □

3.3. The structure of the Ford domain—Jørgensen's theorem

In what follows we shall assume that the reader is familiar with the definition of the Ford domain of a Kleinian group acting on hyperbolic 3-space for which ∞ is an ordinary point. If ∞ is a parabolic fixed point of a group, then the Ford domain is defined to be the set of points lying above the isometric spheres of all group elements not fixing ∞. In order to obtain a fundamental domain, we have to intersect the Ford domain with some fundamental domain for the stabiliser of ∞. The details for Fuchsian groups are given in [22] (pages 57–58) and [7] (page 239), and may be extended to Kleinian groups in the obvious way. We also assume that the reader is familiar with the concepts faces, edges, edge cycles, vertices and vertex cycles for Ford domains. In [19] Jørgensen developed his so called 'method of geometric continuity' to show that the Ford domain of punctured torus groups has the following structure.

JØRGENSEN. *The Ford domain D of a quasi-Fuchsian punctured torus group G, normalised as above, has the following structure*:

- *every face of D is contained in the isometric sphere of a generator*;
- *every edge of D is contained in the intersection of the isometric spheres of a pair of neighbours*;
- *every edge of D with endpoints in the interior of hyperbolic* 3*-space lies in a cycle of length* 3 *and this cycle corresponds to the pairwise intersection of the isometric spheres of a generator triple*;
- *every vertex of D in hyperbolic* 3*-space is the common intersection of the isometric spheres of a generator triple*;
- *if the isometric spheres of a generator triple intersect pairwise, then their common intersection is non-empty.*

This result appears in a paper that has not been published or widely circulated (Theorem 1 of [19]). Therefore, we now give a brief description of Jørgensen's method of proof. Recently, another account of this result has been announced by Akiyoshi, Sakuma, Wada and Yamashita [5]. We will be interested in an extension of Jørgensen's result to groups with singly cusped parabolic fixed points. This extension may be proved using Jørgensen's methods and an outline was given by Akiyoshi in [4], who calls the theorem Condition (J). In what follows we will continuously vary the punctured torus group. Technically we should define a punctured torus group to be a type preserving representation of the fundamental group of a punctured torus (which we refer to as the abstract punctured torus group) into $\mathrm{SL}(2, \boldsymbol{C})$. A deformation is a continuous path of such representations.

The main technique is Jørgensen's so called method of geometric continuity which he had earlier used to classify the Ford domains for cyclic groups of loxodromic transformations, [18]. As the two methods are broadly similar, we now describe the methods in [18] as an aid to understanding [19]. A cyclic loxodromic group is completely classified, up to conjugacy, by the trace of the generator. Therefore, for each point in the trace plane one can find a suitably normalised group. The Ford domain is then specified by this information and one can determine its combinatorics. A small change of parameter in the trace plane effects a small change in the normalised group. This small change will generally not effect the combinatorics of the Ford domain. However there are places where a small change causes a face to disappear or a new face to be created. Therefore, by continuously varying the trace parameter (that is continuously varying the group) one may watch these continuous changes in the

Ford domain and the associated changes in the combinatorics. The details of this process are clearly laid out in [18].

For the punctured torus the process is similar. If the group is Fuchsian, then it is completely determined up to conjugation by a point in the upper half plane. Moreover, one may check that the faces of the Ford domain correspond to a particular generator triple and that there are two edge cycles, both with length three. (There is also a degenerate case where the sides of the Ford domain are contained in the isometric spheres of a pair of neighbours and the edge cycle has length 4). Thus we can subdivide the upper half plane into regions according to which faces arise. This decomposition is just the Farey decomposition. One can easily check that the Ford domain of a Fuchsian punctured torus group satisfies the conditions of the theorem.

Now we extend this to quasi-Fuchsian punctured torus groups. Roughly speaking, if a quasi-Fuchsian group is sufficiently close to being Fuchsian the faces in both components of the ideal boundary of the Ford domain correspond to the same triples of generators. (If we are in the degenerate case mentioned above then the group must be Fuchsian.) If we deform further, the combinatorics of the Ford domain will change. This process is illustrated by the figures on page 36 of [5]. The main idea of the theorem is to show that when this happens the resulting Ford domains also satisfy the conditions of the theorem. This is the method of geometric continuity. Jørgensen's proof of this assertion involves careful study of the structure of the Ford domain and the properties of isometric spheres.

The ideal boundary of the Ford domain consists of two polygons in the Riemann sphere. These polygons each glue up to form a punctured torus. Part of the ingredients of Jørgensen's proof is that, just as we had for Fuchsian groups, each of these polyhedra has edges in a generator triple (or a generator pair). Each of these triples (pairs) corresponds to a triangle (edge) in the Farey tessellation. He then goes on to show that the faces of the Ford domain correspond to generators which lie on a path between these two triangles (edges). Moreover, this path never crosses an edge of the Farey tessellation more than once. Thus, this path lies in a finite number of triangles which (apart from the end ones) abut exactly two more triangles in the path. Each of these triangles corresponds to a generator triple and Jørgensen proves that the generator triples we see along this path precisely correspond to the faces of the Ford domain, Theorem 3 of [19].

Another way of thinking of this is that the upper half plane is the Teichmüller space of the punctured torus. Bers' simultaneous uniformisation theorem [9] states that a quasi-Fuchsian group corresponds to two points in Teichmüller space each associated to one of the ends (with the restriction that points in a certain diagonal are not allowed). Thus a quasi-Fuchsian punctured torus group corresponds to two points in the upper half plane. If the group is Fuchsian, the ends are related by complex conjugation (reflection across the limit set) and the two points are the same. What Jørgensen is doing is putting the additional structure of the Farey tessellation onto the upper half plane and showing that this records information about the combinatorics of the Ford domain. It is clear that for a Fuchsian group the Ford domains in the upper and lower half planes are related by complex conjugation and so Jørgensen's point of view is consistent with Bers' theorem. This connection was exploited by Minsky in his pivot theorem, Theorem 4.1 of [26], where the details are worked out precisely.

Now suppose we deform our group towards the boundary of quasi-Fuchsian space. Using Minsky's theorem [26], which says that all punctured torus groups are limits of quasi-Fuchsian ones, we can extend Jørgensen's theorem to cover all punctured torus groups [4]. We give a sketch here. For simplicity, we suppose that, throughout the deformation, one of the two polygons in the ideal boundary corresponds to the same generator triple or pair. In other words, one of the end points of the path through the Farey tessellation remains in a given triangle or edge while the other end moves towards the boundary. There are exactly three things that can happen (see page 117 of [4]). First, two faces may degenerate at the same time (pages 17–18 of [19]). In this case a generator has become an accidental parabolic (see §3 of [27]). Secondly, there may be an element g of the abstract punctured torus group so that g corresponds to a face of the Ford domain whose number of sides grows unboundedly as we go towards the boundary of quasi-Fuchsian space (g is an "infinite pivot"). This corresponds to passing through infinitely many triangles of the Farey tessellation with the same (rational) vertex. This group element becomes parabolic in the limit (see Lemma 4 of [19] or compare Proposition 3.15 below). Therefore we again have an accidental parabolic element. In both of these cases, this accidental parabolic element corresponds to the vertex of one (and hence infinitely many) of the triangles in the Farey tessellation. This vertex is the end-point of our deformed path through the Farey tessellation from which we read off the faces of the Ford domain and corresponds to the end-invariant. Hence the end-invariant is rational, [26] or page 119 of [4], and these rational end-invariants occur with their natural ordering. We will not be concerned with the details of this case. Indeed one needs to be careful because, as Jørgensen points out on page 36 of [19], these two constructions give different paths towards a cusp the Ford domains which look very different. In particular, for some paths the Ford domains do not converge to the Ford domain of the limit group.

The third possibility, which is the one which interests us here, is the following. We continually add more and more faces to the Ford domain in such a way that, for each g in the abstract punctured torus group, the number of sides of the face corresponding to g (if any) remains bounded. In the limit we have faces corresponding to infinitely many generator triples and each generator lies in only finitely many of these triples. In this case one of the polygons in the ideal boundary degenerates, and we obtain a group with a singly cusped parabolic fixed point (see §6 of [27]). The ordinary set now has one component and the limit set looks like a tree. Such groups are called *singly degenerate*. The corresponding path through the Farey tessellation passes through infinitely many triangles and ends at an irrational number which we define to be the *end invariant* (see Figure 11 of [4]). We will relate this to the usual definition below. Minsky shows that the group is determined by the end-invariants [26]. Moreover, a consequence of Jørgensen's methods is that the generator triples giving faces of the Ford domain correspond to the triangles in the path through the Farey tessellation (this uses Theorem 3 of [19] and geometric continuity and is outlined below). The arrangement of these triangles is, in turn, related to the continued fraction expansion of the end-invariant.

For completeness, we mention that if we allow both ends of the path through the Farey tessellation to tend to the boundary then (again using Minsky's theorem) there are three possibilities (see [27]). First, both end invariants may be rational and the group is called doubly cusped. The group is geometrically finite [21]. In this case the ordinary set has infinitely many components, each of which is a round disc. There are two orbits of such discs corresponding to the two ends of the original group. The limit set is a circle packing (such as those illustrated in [25]). Secondly, one end invariant may be rational and the other irrational. In this case we again obtain a group with a singly cusped parabolic fixed point. Now the ordinary set again has infinitely many components, each of which is a round disc but now there is a single orbit of these. The limit set has become a circle packing considerably more complicated than the circle packings mentioned above (see Figure 7). Finally, both end invariants may be irrational. In this case the limit set is the whole Riemann sphere and the group is called doubly degenerate (see [20]).

In order to make this account as self contained as possible, we now indicate how the above discussion relates to Thurston's construction of end-invariants for a geometrically infinite end (compare the discussion in [26]). There exists a sequence $\{\gamma_j\}$ of geodesics in the associated manifold M which is eventually contained in any neighbourhood of the end. To each of these geodesics there is a unique rational number, the "slope" of the corresponding curve on the square torus. This rational number is the corresponding vertex in the Farey tessellation. Thurston then shows that these rational numbers tend to a unique limit [37]. He defines this limit to be the end invariant. From our construction above, it is clear that the simple closed geodesic corresponding to any generator giving rise to sides of the Ford domain is the axis of this generator. Its end points are contained in the isometric spheres. Thus, for any sequence of faces of the Ford domain that move out to the end (that is the altitude α_{g_n} of the corresponding horoballs tends to infinity) the axes of the associated group elements form a sequence of geodesics that are eventually contained in any neighbourhood of the end. The slope of these geodesics are the vertices of the corresponding triangles in the Farey tessellation. It is clear that as we move deeper into the tessellation then these triangles get smaller and smaller. Eventually they tend to a limit, which is the end invariant.

Fix a starting point y in the upper half plane and consider the geodesic path from y to an irrational number x in the boundary of the upper half plane. (Note that, if we allow finitely many changes as the beginning, the following construction is independent of the starting point, and also the path does not have to be geodesic [26].) We observe the pattern of edges in the Farey tessellation that one crosses while travelling along this path from y to x. It is well known that this pattern is reflected by the continued fraction expansion of x, see for example [31]. Moreover, the points y and x determine a punctured torus group G with a singly cusped parabolic fixed point [26], the number x being the end invariant. By Minsky's theorem [26] we can express G as a limit of quasi-Fuchsian punctured torus groups G_n. These correspond to finite geodesic segments (y, x_n) of the path from y to $x = \lim_{x\to\infty} x_n$. In Theorem 3 of [19], Jørgensen has shown that the patterns mentioned above on the arc (y, x_n) are also seen in the arrangements of the faces of the Ford domain of G_n. Now using geometric continuity we see that as n tends to infinity more faces are added to the Ford domain but the patterns already established do not change (see section 4.2.1 of [4]). Hence the the pattern of triangles containing the arc (y, x) corresponds to the combinatorial arrangement of the faces of the Ford domain of G (essentially they are dual to one another).

We now outline our method for producing cutting surfaces and decomposing the geometrically infinite end into ideal tetrahedra. The details will be given in the next sections. By Jørgensen's theorem, every edge cycle is contained in the pairwise intersection of the isometric spheres of a generator triple. Consider a generator triple (g, h, gh) giving rise to such an edge cycle of the Ford domain of G. We shall show that associated to this generator triple there are two edge cycles, each consisting of three edges. Moreover, transverse to these six edges there is a canonical cutting surface whose quotient under the group is a pair of ideal triangles. These triangles may be glued together with constant bending angle along their edges to form a punctured torus. This surface separates the geometrically infinite end from the rest of the manifold. Moving along the boundary of the Ford domain towards the Jørgensen point, we next encounter a vertex cycle of the Ford domain. This vertex cycle has length four. Also, by Jørgensen's theorem, each vertex is contained in the common intersection of the isometric spheres of a generator triple. This triple is either the triple we started with, or else a triple corresponding to an adjacent triangle in the Farey tessellation (both possibilities occur, each for two of the vertices in the cycle). In the latter case, the new generator triple may be obtained from the initial triple by applying a Nielsen move. For instance, the vertex formed by the intersection of the isometric spheres of g, gh^{-1} and h is the endpoint of two edges associated with the triple (g, h, gh^{-1}) and one edge associated to the triple (g, h, gh). Similarly the vertex formed by the intersection of the isometric spheres of g, h and gh is the endpoint of one edge associated to (g, h, gh^{-1}) and two edges associated to (g, h, gh). The latter vertex is the image of the former under the map h^{-1}. We can repeat this process to obtain a cutting surface associated to this new generator triple. We shall see that the region between these two surfaces is an ideal tetrahedron.

Moving from the generator triple (g, h, gh) to the triple (g, h, gh^{-1}) may be interpreted as crossing the edge in the Farey tessellation joining g and h. Following [26], such an edge is called a *spanning edge*. If a generator is the endpoint of at least two spanning edges, then, again following [26], it is called a *pivot*. It is easy to see that a generator is a pivot if and only if the corresponding face in the Ford domain has at least 6 edges.

3.4. The structure of the Ford domain—finer details

We now give a quantitative analysis of the faces, edges and vertices of the Ford domain of our punctured torus group G.

LEMMA 3.5. *If g is a generator, then the difference between the centres of the isometric spheres of g and g^{-1} is equal to -1.*

If g and h are neighbours, then the difference between the centres of their isometric spheres is equal to

$$\frac{\mathrm{tr}(gh^{-1})}{\mathrm{tr}(g)\,\mathrm{tr}(h)}.$$

Proof. The centres of the isometric spheres of g, g^{-1} and h respectively are the points $-d_g/c_g = -d_g/(a_g+d_g)$, $a_g/c_g = a_g/(a_g+d_g)$ and $-d_h/c_h = -d_g/(a_h+d_h)$. The difference of the first two is clearly equal to -1. The difference between the first and last is equal to

$$\frac{-d_g}{c_g}+\frac{d_h}{c_h}=\frac{-d_gc_h+c_gd_h}{c_gc_h}=\frac{c_{gh^{-1}}}{c_gc_h}=\frac{\mathrm{tr}(gh^{-1})}{\mathrm{tr}(g)\,\mathrm{tr}(h)}. \qquad \square$$

We now compare the relative sizes of traces of the members of a generator triple giving rise to faces of the Ford domain.

PROPOSITION 3.6. *Suppose that (g,h,gh) is a generator triple giving rise to faces of the Ford domain. Then*

$$|\mathrm{tr}(gh)| < |\mathrm{tr}(g)| + |\mathrm{tr}(h)|,$$
$$|\mathrm{tr}(g)| < |\mathrm{tr}(h)| + |\mathrm{tr}(gh)|,$$
$$|\mathrm{tr}(h)| < |\mathrm{tr}(g)| + |\mathrm{tr}(gh)|.$$

Proof. Since gh gives rise to a face of the Ford domain, the isometric spheres of g and h^{-1} intersect. Thus, the distance between their centres is less than the sum of their radii. That is

$$\frac{1}{|\mathrm{tr}(g)|}+\frac{1}{|\mathrm{tr}(h)|}>\frac{|\mathrm{tr}(gh)|}{|\mathrm{tr}(g)|\,|\mathrm{tr}(h)|}.$$

This gives the first stated inequality. The others follow similarly. $\square$

The following proposition gives an explicit expression for the height of each vertex of the Ford domain.

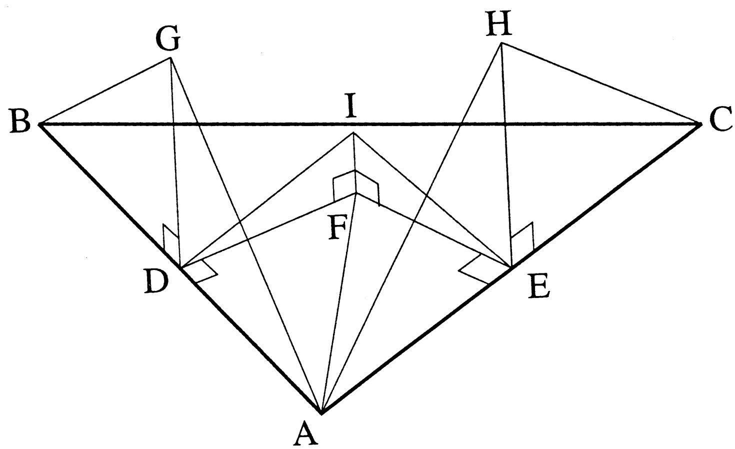

FIGURE 4. Finding the height of a vertex of the Ford domain.

PROPOSITION 3.7. *Suppose that* (g, h) *is a pair of neighbours, and that there are faces of the Ford domain in the isometric spheres of* g, h, gh, gh^{-1} *and their inverses (so necessarily either* g *or* h *is a pivot). Then the faces of the Ford domain contained in the isometric spheres of* g, h *and* gh *intersect in a point* (z_0, t_0), *where*

$$t_0^2 = \frac{2(|\mathrm{tr}(g)|^2 + |\mathrm{tr}(h)|^2) - (|\mathrm{tr}(gh)|^2 + |\mathrm{tr}(gh^{-1})|^2)}{(|\mathrm{tr}(g)|^2 + |\mathrm{tr}(h)|^2)^2 - |\mathrm{tr}(gh)|^2|\mathrm{tr}(gh^{-1})|^2}.$$

COROLLARY 3.8. *If* g *and* h *are as in Proposition* 3.7, *then*

$$2|\mathrm{tr}(g)|^2 + 2|\mathrm{tr}(h)|^2 > |\mathrm{tr}(gh)|^2 + |\mathrm{tr}(gh^{-1})|^2.$$

Proof. The proof is a simple exercise in Euclidean geometry. We refer to Figure 4. Let A, B and C be the centres of the isometric spheres of gh, g and h respectively. The heavy lines in Figure 4 represent the triangle in the Riemann sphere with vertices A, B and C. We know that the lengths of the sides of this triangle are given by

$$|AB| = \frac{|\mathrm{tr}(h)|}{|\mathrm{tr}(g)|\,|\mathrm{tr}(gh)|}, \quad |AC| = \frac{|\mathrm{tr}(g)|}{|\mathrm{tr}(h)|\,|\mathrm{tr}(gh)|}, \quad |BC| = \frac{|\mathrm{tr}(gh^{-1})|}{|\mathrm{tr}(g)|\,|\mathrm{tr}(h)|}.$$

For simplicity, for the rest of the proof we write τ_g instead of $|\mathrm{tr}(g)|$, etc. Let ϕ be the internal angle at A of the triangle ABC. Using the cosine rule, we see that

$$\cos(\phi) = \frac{\tau_g^4 + \tau_h^4 - \tau_{gh}^2\tau_{gh^{-1}}^2}{2\tau_g^2\tau_h^2}.$$

Let G be the point of intersection of the isometric spheres of g and gh lying above AB, and let θ be the internal angle at A of the triangle ABG. The lengths of AG and BG are equal to the radii of the isometric spheres of gh and g respectively, that is they are equal to $1/\tau_{gh}$ and $1/\tau_g$. Using the cosine rule on this triangle, we have that

$$\cos(\theta) = \frac{\tau_g^2 + \tau_h^2 - \tau_{gh}^2}{2\tau_g\tau_h}.$$

As this expression is symmetric in τ_g and τ_h, it follows that θ is also the internal angle at A of the triangle ACH, where H is the point of intersection of the isometric spheres of h and gh lying above AC.

Let D and E be the feet of perpendiculars from G and H to AB and AC respectively. Let I be the common intersection point of the isometric spheres of g, h and gh, and let F be the foot of the perpendicular from I to the Riemann sphere. As the intersection of the isometric spheres of g and gh is a semicircle centred at D, we have that $|DG| = |DI|$ and similarly that $|EH| = |EI|$. Now, the triangles ADG and AEH are congruent, and so are also the triangles ADF and AEF. This means that the internal angle at A of the triangle ADF is equal to $\phi/2$.

The distance $t_0 = |IF|$ can now be computed as follows.

$$\begin{aligned}
t_0^2 &= |DI|^2 - |DF|^2 \\
&= |DG|^2 - |AD|^2 \tan^2(\phi/2) \\
&= |AG|^2 \sin^2(\theta) - |AG|^2 \cos^2(\theta) \tan^2(\phi/2) \\
&= \frac{1}{\tau_{gh}^2} \frac{1 + \cos(\phi) - 2\cos^2(\theta)}{1 + \cos(\phi)} \\
&= \frac{1}{\tau_{gh}^2} \frac{(\tau_g^2 + \tau_h^2)^2 - \tau_{gh}^2\tau_{gh^{-1}}^2 - (\tau_g^2 + \tau_h^2 - \tau_{gh}^2)^2}{(\tau_g^2 + \tau_h^2)^2 - \tau_{gh}^2\tau_{gh^{-1}}^2} \\
&= \frac{2(\tau_g^2 + \tau_h^2) - (\tau_{gh}^2 + \tau_{gh^{-1}}^2)}{(\tau_g^2 + \tau_h^2)^2 - \tau_{gh}^2\tau_{gh^{-1}}^2}.
\end{aligned}$$

□

In order to show that a group is horospherically tame, it is sufficient to give a lower bound on the height of points which are both in the Ford domain and the Nielsen region of our punctured torus group G. By (Euclidean) concavity of the faces and edges of the Ford domain, it is clear that the height is locally minimal at (certain) vertices of the Ford domain. Thus, we say that a vertex (z_0, t_0) of the Ford domain leads to a *local minimum of the height function*, if there exists a neighbourhood U of z_0 such that all points (z, t) in the closure of the Ford domain which lie above U (that is $z \in U$) have the property that $t \geq t_0$.

We need to characterise vertices of the Ford domain which lead to a local minimum of the height function. We also express this minimal height in terms of the radii of the isometric spheres. This is done in the next two propositions.

PROPOSITION 3.9. *Suppose that the generator triple (g, h, gh) gives rise to a vertex (z_0, t_0) of the Ford domain leading to a local minimum of the height function. Then we have that*

$$|\mathrm{tr}(g)|^2 + |\mathrm{tr}(h)|^2 > |\mathrm{tr}(gh)|^2,$$
$$|\mathrm{tr}(g)|^2 + |\mathrm{tr}(gh)|^2 > |\mathrm{tr}(h)|^2,$$
$$|\mathrm{tr}(h)|^2 + |\mathrm{tr}(gh)|^2 > |\mathrm{tr}(g)|^2.$$

Proof. Since the vertex corresponding to (g, h, gh) leads to a local minimum of the height function, it follows, by concavity, that its projection to the Riemann sphere is contained in the interior of the triangle whose vertices are the centres of the isometric spheres of g, h and gh. Using the notation in the proof of Proposition 3.7, we deduce that F is in the interior of the triangle ABC. A consequence of this is that the triangle ABG has an acute internal angle θ at A. Applying the cosine rule to this triangle, we obtain that $|\mathrm{tr}(g)|^2 + |\mathrm{tr}(h)|^2 > |\mathrm{tr}(gh)|^2$. Similarly, the triangles ABG and ACH have acute internal angles at B and C respectively. This gives the other two inequalities. □

PROPOSITION 3.10. *Suppose that g and h are as in Proposition 3.7. Suppose also that $|\mathrm{tr}(g)|^2 + |\mathrm{tr}(h)|^2 > |\mathrm{tr}(gh)|^2$ and $|\mathrm{tr}(g)|^2 + |\mathrm{tr}(h)|^2 > |\mathrm{tr}(gh^{-1})|^2$. Then the intersection of the isometric spheres of g, h and gh has height t_0, where*

$$t_0^2 > \frac{1}{|\mathrm{tr}(g)|^2 + |\mathrm{tr}(h)|^2 + |\mathrm{tr}(gh)|\,|\mathrm{tr}(gh^{-1})|}.$$

Proof. Without loss of generality we assume that $|\mathrm{tr}(gh)| \le |\mathrm{tr}(gh^{-1})|$. Therefore, we have that

$$|\mathrm{tr}(gh)|^2 \le |\mathrm{tr}(gh)|\,|\mathrm{tr}(gh^{-1})| \le |\mathrm{tr}(gh^{-1})|^2 < |\mathrm{tr}(g)|^2 + |\mathrm{tr}(h)|^2.$$

As in the proof of Proposition 3.7, let $\tau_g = |\mathrm{tr}(g)|$. We see that

$$\begin{aligned} t_0^2 &= \frac{2(\tau_g^2 + \tau_h^2) - (\tau_{gh}^2 + \tau_{gh^{-1}}^2)}{(\tau_g^2 + \tau_h^2)^2 - \tau_{gh}^2\tau_{gh^{-1}}^2} \\ &> \frac{\tau_g^2 + \tau_h^2 - \tau_{gh}^2}{(\tau_g^2 + \tau_h^2 + \tau_{gh}\tau_{gh^{-1}})(\tau_g^2 + \tau_h^2 - \tau_{gh}\tau_{gh^{-1}})} \\ &\ge \frac{1}{(\tau_g^2 + \tau_h^2 + \tau_{gh}\tau_{gh^{-1}})}. \end{aligned}$$ □

3.5. Cutting surfaces and the tetrahedral decomposition

In this section we give an explicit construction of the geometrically infinite end of our singly cusped manifold associated to the punctured torus group G. We construct a sequence of simplicial hyperbolic surfaces which we call cutting surfaces. This sequence is then used to build up the tiling with ideal tetrahedra of the geometrically infinite end. Moreover, an immediate consequence of our construction will be that all punctured torus groups with a singly cusped parabolic fixed point are geometrically tame.

The cutting surfaces will be associated to pairs of edge cycles in the boundary of the Ford domain. Suppose that the isometric spheres of g^{-1} and h^{-1} intersect to give such an edge. Let $T(g,h)$ denote the ideal triangle in $\boldsymbol{H}^3$ with vertices at $\infty, g(\infty)$ and $h(\infty)$ (these last two points are the centres of the isometric spheres that give rise to the edge). By Jørgensen's theorem, this edge lies in a cycle of length 3 and the three group elements form a generator triple. It is clear that the other members of this cycle are the intersections of the isometric spheres of the pairs $(g, h^{-1}g)$ and $(h, g^{-1}h)$. We analogously define the triangles $T(g^{-1}h, g^{-1})$ and $T(h^{-1}, h^{-1}g)$. The edge cycle extends to these triangles, namely:

$$g^{-1} : T(g,h) \mapsto T(g^{-1}h, g^{-1}),$$
$$h^{-1}g : T(g^{-1}h, g^{-1}) \mapsto T(h^{-1}, h^{-1}g),$$
$$h : T(h^{-1}, h^{-1}g) \mapsto T(g,h).$$

The intersection of $T(g,h)$ with the Ford domain is a quadrilateral (denoted by E in Figure 5). Similarly, the intersection of $T(g^{-1}h, g^{-1})$ and $T(h^{-1}, h^{-1}g)$ with the Ford domain of G are quadrilaterals (denoted by C and A respectively). The pull backs of these three quadrilaterals then tile $T(g,h)$. In Figure 5

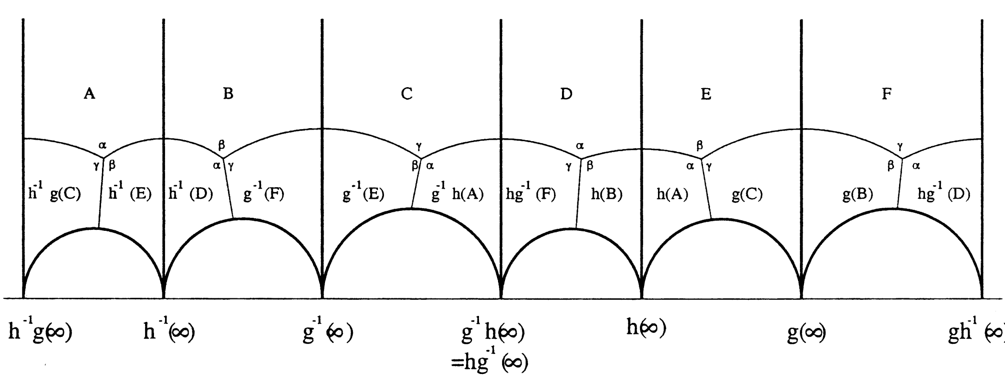

FIGURE 5. A cutting surface corresponding to the generator triple (g, h, gh^{-1}). In order to embed this into the manifold, bend along the bold lines.

these quadrilaterals are E, $h(A)$ and $g(C)$. Also, consider the ideal triangle $T(g^{-1}, h^{-1})$. Analogously, this lies in a cycle of length 3, namely:

$$g : T(g^{-1}, h^{-1}) \mapsto T(gh^{-1}, g),$$
$$hg^{-1} : T(gh^{-1}, g) \mapsto T(h, hg^{-1}),$$
$$h^{-1} : T(h, hg^{-1}) \mapsto T(g^{-1}, h^{-1}).$$

Figure 5 illustrates the tessellation of $T(g^{-1}, h^{-1})$ by the quadrilaterals B, $h^{-1}(D)$ and $g^{-1}(F)$. The six quadrilaterals A, B, C, D, E and F may be glued together to form a polygon inside the Ford domain of G, as indicated in Figure 5. If we now build up $M = \boldsymbol{H}^3/G$ by identifying the sides of the Ford domain as usual, then this particular polygon becomes a punctured torus embedded in M. This punctured torus consists of two flat triangles (corresponding to $T(g, h)$ and $T(g^{-1}, h^{-1})$) which are glued together at constant 'bending angle' along three disjoint geodesic arcs which begin and end at the puncture. We call such a surface a *cutting surface*. The way in which a cutting surface is embedded in the Ford domain makes it clear that the cutting surface disconnects the manifold.

We remark that Figure 5 illustrates the generic situation, where the intersection of the cutting surface and the isometric spheres is contained in the triangles $T(g, h)$ and $T(g^{-1}, h^{-1})$. Our construction also works for more general configurations; the crucial point is that $T(g, h)$ and $T(g^{-1}, h^{-1})$ may be glued to form an embedded simplicial surface.

We now repeat this construction for every generator triple that gives rise to an edge cycle of the Ford domain. To each of these triples there corresponds a cutting surface. The set of these cutting surfaces is naturally ordered by the path through the Farey tessellation. In particular this gives a sequence of surfaces which allow us to deduce that the group is geometrically tame. We remark that it may not be immediately clear that the end of the manifold is a topological product (which is also necessary for geometrical tameness). This follows immediately because our group is in the closure of quasi-Fuchsian space (by Minsky's theorem [26]), and hence its quotient manifold is a topological product. Alternatively, one may interpolate between adjacent cutting surfaces using surfaces with the following properties. They are the union of pieces of vertical planes and they intersect the boundary of the Ford domain in points which are identified pairwise by elements of the group. Hence, we have proved the following special case of a result of Thurston (Theorem 9.2 of [37]).

THEOREM 3.11. *A punctured torus group with a singly cusped parabolic fixed point is geometrically tame.*

We now turn our attention from the cutting surfaces and investigate the regions between adjacent cutting surfaces. It will turn out that these regions are in fact ideal tetrahedra and the boundary of each of these tetrahedra consists of two pairs of ideal triangles arising from the cutting surfaces.

Let us begin with a cutting surface associated to a generator triple (g, h, gh^{-1}) (compare Figure 5). There are two edge cycles of the Ford domain associated to this cutting surface. One of these is contained in the pairwise intersection of the isometric spheres of $(g^{-1}h, h)$, (g, gh^{-1}), (h^{-1}, g^{-1}), the other in the intersections of the isometric spheres of (h, g), (gh^{-1}, h^{-1}), (g^{-1}, hg^{-1}). We need to find the generator triple which is associated to an adjacent cutting surface. In order to do this, choose one of the six edges mentioned above. Move along this edge towards the geometrically infinite end. If we do not meet a vertex, then we have come to the end of the manifold, and the end was not geometrically infinite after all. Therefore we must reach a vertex. By Jørgensen's theorem, this vertex must have valence 3. Hence, the edge we are considering bifurcates into two new edges, a new face appearing between them. Again using Jørgensen's theorem, we see that this new face must be associated to a neighbour of the two faces giving our first edge. For example, let us suppose that we are considering the edge which is the intersection of the isometric spheres of (h, g). The new face must be a common neighbour, so is in the isometric sphere of one of gh, hg, gh^{-1}, $g^{-1}h$ or their inverses.

There are two possibilities. First, the new face may not be one we have already seen in our cutting surface. In Figure 5 we illustrate this in the case where the new face is in the isometric sphere of gh. Thus the new cutting surface is associated to the triple (g, h, gh). Secondly, the new face may already give a face from our cutting surface. In this case two of the edges associated to our cutting surface meet. We again illustrate this from Figure 5 but need to use a different edge. Let us suppose that these are the edges arising from the intersection of the isometric spheres of $(g, h^{-1}g)$ and $(h^{-1}g, h^{-1})$. By Jørgensen's theorem, this vertex has valence 3 and the third edge is in the intersection of the isometric spheres of (g, h^{-1}). Therefore the triangle $T(g^{-1}, h)$ is part of the next cutting surface. This determines the new triple as (g, h, gh).

These processes are the reverse of each other. It is not hard to see that for each cutting surface, both possibilities must occur. Two of the edges each bifurcate to give a pair of new faces and four new edges while the other four edges meet in pairs and give two new edges. These six new edges are the ones associated to the new cutting surface. That they form two edge cycles of length three may be seen using the gluing patterns derived from our first cutting surface.

We now characterise the region between adjacent cutting surfaces.

PROPOSITION 3.12. *Suppose the generator triples* (g, h, gh) *and* (g, h, gh^{-1}) *give rise to edge cycles of the Ford domain* (*and hence cutting surfaces*). *Consider the ideal tetrahedron with vertices* ∞, $g(\infty)$, $h(\infty)$ *and* $gh(\infty)$. *Then the projection of this tetrahedron to* M *has as its boundary the cutting surfaces for the generator triples* (g, h, gh) *and* (g, h, gh^{-1}).

Proof. The boundary of the tetrahedron considered in the statement of the proposition consists of four ideal triangles. Clearly, two of these are the triangles $T(gh, g) = g(T(g^{-1}, h))$ and $T(h, gh) = h(T(g, h^{-1}))$. By definition,

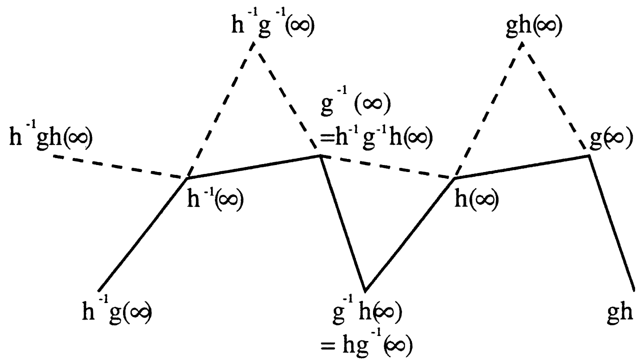

FIGURE 6. Two adjacent cutting surfaces seen from above. Here either g or h (or both) is a pivot.

$T(g,h^{-1}) \cup T(g^{-1},h)$ is the cutting surface associated to the generator triple (g,h,gh). The other two faces of the tetrahedron are $T(g,h)$ and $gh(T(g^{-1},h^{-1}))$. The union of $T(g,h)$ and $T(g^{-1},h^{-1})$ gives the cutting surface associated to (g,h,gh^{-1}). Thus, the boundary of the tetrahedron is in the G-orbit of the union of these two cutting surfaces. $\square$

Figure 6 shows the vertical projection onto the Riemann sphere of the cutting surfaces associated to (g,h,gh^{-1}) (solid line) and (g,h,gh) (dashed line). Each of the ideal triangles shown in Figure 5 has projected to a line. The union of these lines over all cutting surfaces is a graph dual to the projection of the Ford domain. The four triangles between the two cutting surfaces are the vertical projections of four hyperbolic polyhedra. Each of these polyhedra has three infinite faces (corresponding to quadrilaterals in the cutting surfaces) and three finite faces (corresponding to pieces of the isometric spheres associated to the vertices of the triangle). When we use the side identifications to glue these polyhedra together we obtain the ideal tetrahedron of Proposition 3.12 (in just the same way that the triangle $T(g,h)$ is formed by identifying sides of the three quadrilaterals A, C and E).

One could perform the analogous construction in the case of fibre bundles over the circle with fibre the punctured torus. This would verify the assertion of Bowditch on page 329 of [13].

3.6. Pivots

In this section we shall give an estimate on the size of the isometric sphere of a pivot. For this we first require the following lemma which improves Lemma 2 of [19].

LEMMA 3.13. *If g and h are neighbours, and either g or h is a pivot, then*

$$\frac{1}{|\mathrm{tr}(g)|^2} + \frac{1}{|\mathrm{tr}(h)|^2} > \frac{1}{4}.$$

Proof. Without loss of generality, we assume that there exist faces of the Ford domain which are contained in the isometric spheres of g, h, gh, gh^{-1} and their inverses. Using the estimate of the height of a vertex in Proposition 3.7, it follows that

$$\begin{aligned} 4|\mathrm{tr}(g)|^2 + 4|\mathrm{tr}(h)|^2 &> 2|\mathrm{tr}(gh)|^2 + 2|\mathrm{tr}(gh^{-1})|^2 \\ &\geq |\mathrm{tr}(gh) + \mathrm{tr}(gh^{-1})|^2 \\ &= |\mathrm{tr}(g)\ \mathrm{tr}(h)|^2; \end{aligned}$$

where we have used the well known relation

$$\mathrm{tr}(g)\ \mathrm{tr}(h) = \mathrm{tr}(gh) + \mathrm{tr}(gh^{-1}). \qquad \square$$

We now give the estimate on the size of isometric spheres of pivots, showing that the absolute value of the trace of a pivot is bounded. The estimate is an adaptation of Jørgensen's method in [19], where the weaker bound $4 + 2\sqrt{5}$ is obtained.

Proposition 3.14. *If g is a pivot, then $|\mathrm{tr}(g)| < 2\sqrt{3}$.*

Proof. Using Lemma 3.13 and the fact that a pivot is by definition the end of two spanning edges, we have for the neighbours h and gh of the pivot g so that

$$\frac{1}{|\mathrm{tr}(g)|^2} + \frac{1}{|\mathrm{tr}(h)|^2} > \frac{1}{4}, \quad \frac{1}{|\mathrm{tr}(g)|^2} + \frac{1}{|\mathrm{tr}(gh)|^2} > \frac{1}{4}.$$

If $|\mathrm{tr}(g)| \leq 2$ we already may deduce the estimate in the proposition. Hence we assume without loss of generality that $|\mathrm{tr}(g)| > 2$. Rearranging the above estimates gives

$$|\mathrm{tr}(h)|^2 < \frac{4|\mathrm{tr}(g)|^2}{|\mathrm{tr}(g)|^2 - 4}, \quad |\mathrm{tr}(gh)|^2 < \frac{4|\mathrm{tr}(g)|^2}{|\mathrm{tr}(g)|^2 - 4}.$$

We may also assume that among all vertices of the Ford domain on the isometric sphere of g, the vertex corresponding to the triple (g, h, gh) has the smallest height. This means that, with the notation of Figure 4, the internal angle of the triangle ABG at the vertex B is acute. Using the cosine rule, this gives

$$|\mathrm{tr}(g)|^2 < |\mathrm{tr}(h)|^2 + |\mathrm{tr}(gh)|^2.$$

Combining these three inequalities, we obtain

$$|\mathrm{tr}(g)|^2 < |\mathrm{tr}(h)|^2 + |\mathrm{tr}(gh)|^2$$
$$< \frac{8|\mathrm{tr}(g)|^2}{|\mathrm{tr}(g)|^2 - 4}.$$

It follows that $|\mathrm{tr}(g)|^2 - 4 < 8$, which gives the result. □

3.7. Neighbours and horospherical tameness

In this section we show that a punctured torus group G is horospherically tame if and only if each face of the Ford domain of G, which corresponds to a pivot, has a bounded number of sides. This is equivalent to saying that the associated end invariant is badly approximable. In particular, this will clarify a remark of Sullivan in [35], which we quote verbatim at the end of this section.

As we have already seen, the patterns one sees in the Farey tessellation when going towards an irrational number are (by Jørgensen's theorem) related to the combinatorics of the Ford domain. The goal of this section is to make this connection more explicit and to interpret this in the context of the geometry of the manifold. As one goes towards an irrational point on the boundary of the Farey tessellation (by the above remarks this number will be the end invariant) then one crosses an infinite sequence of edges, each edge corresponding to a pair of neighbours (that is a pair of curves on the punctured torus intersecting exactly once). Adjacent edges that one crosses are boundary arcs of the same triangle in the Farey tessellation. There are two possibilities: the third edge of this triangle is either on the left or on the right (see [31]). Topologically, this corresponds to obtaining the new set curves by doing a Dehn twist about one or the other of them. Algebraically, this corresponds to doing different Nielsen moves on the generators. If we cross a sequence of $m+1$ edges in the Farey tessellation so that the third edge in the m triangles we pass through is always on the left, then these triangles all have a common vertex. This group element is a pivot (see [26]). Arithmetically, this corresponds to seeing the number m in the continued fraction expansion of our irrational number. As the pattern in the Farey tessellation is dual to the pattern we see in the Ford domain, we have a face in the Ford domain, in the isometric sphere of the pivot, with at least $m+1$ edges (in fact $2m+4$).

We can now ask what this means in terms of the cutting surfaces and the tetrahedral decomposition we discussed earlier. When viewing the Ford domain and cutting surfaces from above (as in Figure 6) we see that the cutting surface is a pattern of 6 line segments joining a point z to the point $z+2$. If the point z is the centre of the isometric sphere of our pivot, we see that we must have $m+1$ cutting surfaces from z to $z+2$ and the regions between these are all triangles. Moreover, the cutting surfaces never contain points whose imaginary parts are more than a uniformly bounded distance from the imaginary part of z. In order to see this, recall that the cutting surface is contained in the union of six isometric spheres which pairwise intersect and whose radii are at most 1. Putting all of this together, we see that, if m is large, then some of the triangles one sees between the cutting surfaces become long and thin. This in turn means that when we consider the tetrahedral decomposition of the geometrically infinite end of the manifold, then at least some of these tetrahedra become long and thin. In particular, when we cut off the vertices using the Leutbecher horoballs, then the remaining compact pieces have diameter which increases with m. If m is allowed to increase unboundedly over a sequence of pivots, this means that the manifold is not horospherically tame. Making this statement precise is the object of this section. Also, the tori given by the cutting surfaces become very long and thin: that is, they have a very short closed geodesic and this short geodesic corresponds to the pivot. This in turn means that in the 3-manifold the geodesic associated to the pivot is also short. Thus, when m is large enough, the ε-thin part of the manifold contains Margulis tubes as well as cusp neighbourhoods.

We now want to quantify the above discussion. If g is a pivot with $m+1$ neighbours, each giving rise to sides of the Ford domain as discussed above, then the number m is called the *width* of g. Observe that if g is a pivot of width m, then the neighbours of g giving rise to faces in the Ford domain can be taken to have the form $h, gh, \dots, g^m h$, for some h in G. In this situation, h and $g^m h$ are both pivots, and we shall refer to them as *extreme neighbours*. The following result should be compared to Lemma 5 of [19] or Theorem 4.1(i) of [26].

Proposition 3.15. *Let (g_n) be a sequence of pivots of width m_n. If m_n tends to infinity with n, then, as n tends to infinity, the translation length of g_n tends to zero.*

Proof. Suppose first that $m_n = 2j$ is even. Let h_n and $g_n^{2j} h_n$ be the extreme neighbours of g_n. Since these are both pivots, it follows from Proposition 3.14 that $|\mathrm{tr}(h_n)| \le 2\sqrt{3}$ and $|\mathrm{tr}(g_n^{2j} h_n)| \le 2\sqrt{3}$.

Now suppose that $m_n = 2j+1$ is odd. Let h_n and $g_n^{2j+1} h_n$ be the extreme neighbours of g_n. Since these are both pivots, it follows that $|\mathrm{tr}(h_n)| \le 2\sqrt{3}$ and $|\mathrm{tr}(g_n^{2j+1} h_n)| \le 2\sqrt{3}$. Clearly, $g_n^{2j} h_n$ has two pivots as neighbours (namely, g_n and $g_n^{2j+1} h_n$). It follows that

$$|\mathrm{tr}(g_n^{2j} h_n)| \le |\mathrm{tr}(g_n)| + |\mathrm{tr}(g_n^{2j+1} h_n)| \le 4\sqrt{3}.$$

Hence, in either case we have that

$$|\mathrm{tr}(g_n^{j})| \le |\mathrm{tr}(g_n^{j})\ \mathrm{tr}(g_n^{j} h_n)| = |\mathrm{tr}(h_n) + \mathrm{tr}(g_n^{2j} h_n)| \le 6\sqrt{3}.$$

Note, in this estimate we have implicitly used the fact that $|\mathrm{tr}(g_n^{j} h_n)| \ge 1$. Let $[m_n/2]$ denote the integer part of $m_n/2$. The previous inequality gives a bound on $|\mathrm{tr}(g_n^{[m_n/2]})|$, for all n. Since g_n cannot tend to an elliptic transformation as n tends to infinity, it follows that the translation length of g_n tends to zero. □

The following proposition can be seen in practice from the example constructed by Alestalo and Helling [6].

Proposition 3.16. *Let (g_n) be a sequence of pivots of width m_n. Among all neighbours of g_n which give rise to faces of the Ford domain, let h_n denote the one for which $|\mathrm{tr}(h_n)|$ is maximal. Then, the sequence $(|\mathrm{tr}(h_n)|)$ is unbounded if and only if (m_n) is unbounded.*

Proof. In the following we consider the intersection of an isometric sphere with the complex plane. As usual, this intersection will be called a *isometric circle*.

Without loss of generality, we first assume that m_n tends to infinity. The distance between the poles of g_n and h_n is at most equal to 2, since the radii of their isometric circles (which intersect) are at most equal to 1. Thus, every neighbour of g_n giving rise to a face in the Ford domain has its pole within a circle of radius 2 centred at the pole of g_n. Let l_n denote the shortest distance between the centres of two of these isometric circles. By construction, there exist $m_n + 1$ pairwise disjoint discs of radius $l_n/2$ which are contained in the disc of radius $2 + l_n/2$ centred at the pole of g_n. Comparing the areas of these discs, we obtain the estimate $(m_n + 1)(l_n/2)^2\pi \leq (2 + l_n/2)^2\pi$, and thus $l_n \leq 4/(\sqrt{m_n + 1} - 1) \leq 4(1 + \sqrt{2})/\sqrt{m_n}$. Hence, l_n tends to zero as m_n tends to infinity. Observing that $l_n \geq 1/|\mathrm{tr}(h_n)|^2$, it follows that $|\mathrm{tr}(h_n)|$ tends to infinity.

Now, suppose that m_n is bounded, say g_n has width at most M, for all n. Then there exists $j \leq m_n \leq M - 1$ such that $g_n^j h_n$ is a pivot and $g_n^i h_n$ gives rise to a face of the Ford domain, for all $0 \leq i \leq j$. This implies that

$$
\begin{aligned}
|\mathrm{tr}(h_n)| &\leq |\mathrm{tr}(g_n)| + |\mathrm{tr}(g_n h_n)| \\
&\leq j|\mathrm{tr}(g_n)| + |\mathrm{tr}(g_n^j h_n)| \\
&< 2(j+1)\sqrt{3} \leq 2M\sqrt{3}.
\end{aligned}
$$

□

The following theorem gives the main result of this section. Recall that an irrational number is badly approximable if the entries in its continued fraction expansion are bounded from above by some positive fixed number.

Theorem 3.17. *Let G be a punctured torus group with a singly cusped parabolic fixed point. The end invariant of $M = \boldsymbol{H}^3/G$ is badly approximable if and only if G is horospherically tame.*

Proof. By construction, the width of a pivot of G can be interpreted as an entry in the continued fraction expansion of the associated end invariant x (see the remark on page 10 of [26]).

Suppose first that the entries in the continued fraction expansion of the end invariant x are unbounded. Proposition 3.16 implies that there exists a sequence (h_n) of generators of G giving rise to sides in the Ford domain, such that the radii of the isometric spheres of the h_n become arbitrarily small. It follows that there exists a sequence of points in the intersection of the Nielsen region and the Ford domain with unbounded distance from the Leutbecher horoball H_∞. Since these points are contained in the Ford domain, they are at least as far from all other Leutbecher horoballs. It follows that G is not horospherically tame.

Suppose now that the entries in the continued fraction expansion of x are bounded. Therefore, the radii of all isometric spheres which give rise to faces in the boundary of the Ford domain are bounded away from zero. We shall see that this means that the heights of the vertices of the Ford domain are bounded away from zero. Clearly, this then implies that the group is horospherically tame.

It is sufficient to show that if all isometric spheres containing faces of the Ford domain have radius at least ρ, for some positive ρ, then there exists a positive lower bound T for the heights of the vertices of the Ford domain. Let (g, h) be a pair of neighbours which are ends of a spanning edge. Suppose first that $|\mathrm{tr}(g)|^2 + |\mathrm{tr}(h)|^2 > |\mathrm{tr}(gh)|^2$ and $|\mathrm{tr}(g)|^2 + |\mathrm{tr}(h)|^2 > |\mathrm{tr}(gh^{-1})|^2$. An immediate consequence of Proposition 3.10 is that the heights of the vertices corresponding to (g, h, gh) and (g, h, gh^{-1}) are at least equal to $\rho/\sqrt{3}$. Now suppose that (g_0, h_0, g_0h_0) is a generator triple giving rise to the vertex (z_0, t_0) of the Ford domain and, furthermore, that $|\mathrm{tr}(g_0)|^2 + |\mathrm{tr}(h_0)|^2 \le |\mathrm{tr}(g_0h_0)|^2$. Proposition 3.9 implies that t_0 does not lead to a local minimum of the height function. By construction, the points near (z_0, t_0) on the boundary of the Ford domain with height less than t_0 are contained in the isometric sphere of g_0h_0. In particular, there is a vertex (z_1, t_1) of the Ford domain with $t_1 \le t_0$, such that (z_1, t_1) corresponds to a generator triple (g_1, h_1, g_1h_1), where $g_1 = g_0h_0$. If these generators satisfy the conditions of Proposition 3.10, then t_1, and hence also t_0, is at least equal to $\rho/\sqrt{3}$. When these conditions are not satisfied, we repeat this process to obtain a sequence of vertices of the Ford domain with decreasing height. We claim that there is a universal bound on the number of times we need to do this until we obtain a vertex corresponding to a generator triple satisfying the conditions of Proposition 3.10.

Specifically, suppose that for each $0 \le i \le j$ we have a vertex (z_i, t_i) of the Ford domain corresponding to a generator triple (g_i, h_i, g_ih_i) such that $g_{i+1} = g_ih_i$, $t_{i+1} \le t_i$ and $|\mathrm{tr}(g_i)|^2 + |\mathrm{tr}(h_i)|^2 \le |\mathrm{tr}(g_{i+1})|^2$, for each $0 \le i \le j-1$. Then we have in particular that

$$\begin{aligned} |\mathrm{tr}(g_j)|^2 &\ge |\mathrm{tr}(g_{j-1})|^2 + |\mathrm{tr}(h_{j-1})|^2 \\ &\ge |\mathrm{tr}(g_{j-1})|^2 + 1 \\ &\ge |\mathrm{tr}(g_0)|^2 + j. \end{aligned}$$

Since by assumption $|\mathrm{tr}(g_j)| \le 1/\rho$, it follows that there exists $j \le 1/\rho^2$ for which $|\mathrm{tr}(g_j)|^2 + |\mathrm{tr}(h_j)|^2 > |\mathrm{tr}(g_jh_j)|^2$ and $|\mathrm{tr}(g_j)|^2 + |\mathrm{tr}(h_j)|^2 > |\mathrm{tr}(g_jh_j^{-1})|^2$. Now, since we have that $\rho/\sqrt{3} < t_j \le t_i \le t_0$, the result follows. □

Remark. In [35] Sullivan defines a *hyperbolic half cylinder* to be a manifold M for which

• the fundamental group G of M is isomorphic to $\pi_1(S)$, where S is a compact surface;

- the ordinary set $\Omega(G)$ is non-empty;
- there are embeddings $\{f_n : S \to M\}$ of S with bounded diameter so that $f_n(S)$ has distance between n and $n+1$ from some initial embedding $f_0(S)$.

Sullivan then makes the following assertion ([35], page 142), which we now quote verbatim and which should be compared with Theorem 3.17.

Sullivan. *In Jorgensen's description* [19] *of the punctured torus case, Teichmuller space is the Poincaré disk and the geometry of the hyperbolic* 3-*manifold corresponding to the limit group is controlled by the tail of the continued fraction expansion of a limiting point on the boundary of the disk. A hyperbolic half cylinder results* (*we ignore the cusp*) *iff the partial convergents are bounded.*

3.8. Two examples

We conclude with two examples of punctured torus groups with singly cusped parabolic fixed points. The group in the first example is horospherically tame, whereas the group in the second is not. Since both groups are geometrically tame, this illustrates that, for punctured torus groups, horospherical tameness is in fact a stronger notion than geometrical tameness.

Example 1 (a punctured torus group which is horospherically tame).

This example is considered by McMullen, Mumford, Wright in [27], who call it the *Fibonacci example*. Here, G is the punctured torus group whose end invariant x is the golden ratio $x = (1+\sqrt{5})/2 = [1,1,1,\ldots]$ (where, in particular, the edges in the Farey tessellation, crossed by the path towards x, alternate between left and right). A part of the limit set for this group in the case where the geometrically finite end has been pinched to a thrice punctured sphere is shown in Figure 7. Every generator g of G is a pivot (of width 1), and for its trace we have $|\mathrm{tr}(g)| < 2\sqrt{3}$. Also, every face in the Ford domain is a hexagon (compare this to the first example in [20]). Using similar arguments to those in the proof of Theorem 3.17, we see that every vertex of the Ford domain has height at least equal to $1/6$. Therefore, if in the definition of horospherical tameness we choose $K = \log(6)$, then it follows that G is horospherically tame.

Example 2 (a punctured torus group which is not horospherically tame).

Let G denote a punctured torus group whose associated end invariant is given by the irrational number with continued fraction expansion $[1,2,3,4,5,\ldots]$. Now, if g is the mth pivot of G, then g is of width m. Furthermore, using the estimates obtained in the proof of Proposition 3.16, g has a neighbour h_m whose isometric sphere has radius

$$\frac{1}{|\mathrm{tr}(h_m)|} \le \sqrt{\frac{4}{\sqrt{m+1}-1}} < \frac{4}{m^{1/4}}.$$

Obviously, isometric spheres of this type contribute to the boundary of the Ford domain of G. This implies that there exists no finite hyperbolic enlargement

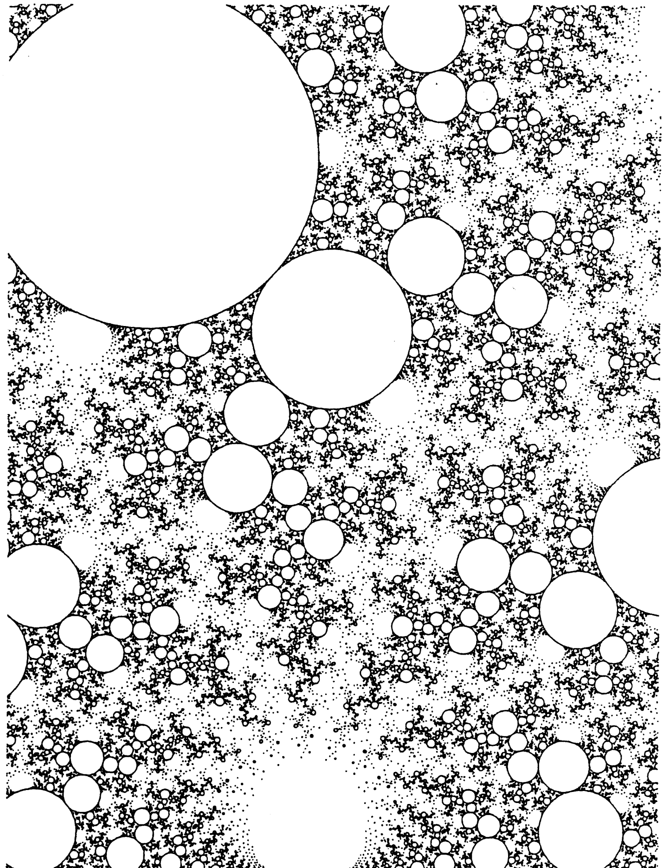

FIGURE 7. The Fibonacci example conjugated so that the singly cusped parabolic point is at -1. This figure was drawn using a computer programme written by Ian Redfern and described in [25]. The white area in the centre of this picture immediately above -1.0 is filled with circles that are smaller than a pixel. If these circles were drawn, then the this area would appear totally black and much of the detail would be lost.

of the Leutbecher horoball which 'swallows up' the convex core of $M = \boldsymbol{H}^3/G$. Hence, the group G is not horospherically tame.

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