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ON THE MAXIMAL POLAR QUOTIENT OF AN ANALYTIC PLANE CURVE

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Abstract

We give an explicit formula for the maximal polar quotient of a plane curve singularity with some applications to the Łojasiewicz exponent and the C^0-degree of sufficiency.

Introduction

Let $\boldsymbol{C}\{X, Y\}$ be the ring of convergent power series in two variables X, Y. If $f, \phi \in \boldsymbol{C}\{X, Y\}$, we denote by $(f, \phi)_0$ the intersection multiplicity of f and ϕ, equal to the $\boldsymbol{C}$-codimension of the ideal (f, ϕ) generated by f and ϕ in $\boldsymbol{C}\{X, Y\}$. We use ord f to denote the order of the series f. If f, ϕ are without constant term, then $(f, \phi)_0 \geq (\text{ord}\, f)(\text{ord}\, \phi)$, the equality holding if and only if f, ϕ are transverse. Let $t = t(X, Y) \in \boldsymbol{C}\{X, Y\}$ be a regular parameter i.e. a series of order 1. Assume that $f \in \boldsymbol{C}\{X, Y\}$ is a reduced (i.e. without multiple factors) power series and t does not divide f. The rational numbers $(f, \phi)_0/(t, \phi)_0$, where ϕ runs over irreducible factors of the Jacobian $J = \partial(t, f)/\partial(X, Y)$, are called polar quotients of f with respect to t. If $t = bX - aY$ then we speak about polar quotients with respect to the direction $(a : b) \in \boldsymbol{P}^1(\boldsymbol{C})$. Clearly the set of polar quotients is finite. It is empty if and only if $(\partial(t, f)/\partial(X, Y))(0, 0) \neq 0$. If t and f are transverse, then the polar quotients are of the form $(f, \phi)_0/\text{ord}\, \phi$ and are also called polar invariants. They are topological invariants of the singularity $f = 0$ (see [T], [LMW]). If t and f are not transverse the notion of polar quotient is also interesting, especially in the case of the singularities at infinity of plane algebraic curves (see [Eph] and [Lê]). Casas-Alvero ([C-A], Chapter 6, Theorem 6.11.5) calculated polar quotients in the general case of reduced power series using the infinitely near points. However, the formulas he got are not so explicit as that given by Merle ([M], Théorème 3.1) in the case of one branch. In this note we study the maximal polar quotient $q_0(f, t)$ of f with respect to t:

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$$q_0(f,t) = \sup\{q \in \boldsymbol{R} : q \text{ is a polar quotient of } f \text{ with respect to } t\}.$$

If $(\partial(t,f)/\partial(X,Y))(0,0) \neq 0$ then $q_0(f,t) = -\infty$ by convention.

Our aim is to give an explicit formula for $q_0(f,t)$ (Theorem 1.3 of this paper) by means of the maximal polar quotients of the branches and some intersection multiplicities. Our result is inspired by Kuo and Lu's paper [KL], to prove it we use the Kuo-Lu lemma that enables us to locate the Puiseux roots of $\partial f/\partial Y = 0$ relatively to the roots of $f = 0$ ([KL], Lemma 3.3). As an application of our formula we show that for a given f the $\sup\{q_0(f,t) : t$ does not divide $f\}$ is attained if f and t are transverse (Corollary 1.4). The maximal polar quotient are of particular interest. Teissier in his fundamental paper [T] proved (in the case of isolated hypersurface singularities) that the Łojasiewicz exponent $\mathscr{L}_0(f)$ and the C^0-degree of sufficiency are determined by the maximal polar quotient with respect to the generic direction. Using Teissier's result we give formulas for $\mathscr{L}_0(f)$ and $\mathrm{Suff}_0(f)$ (Corollary 1.5) and a correct version (Corollary 1.8) of a formula for $\mathrm{Suff}_0(f)$ proposed by Lichtin in [Li]. In the appendix we reprove the main result of [KL]. We give also an example showing that the geometric interpretation of this result given in [KL] is not exact.

1. Main result

Let $f \in \boldsymbol{C}\{X,Y\}$ be an irreducible power series. Recall that the semigroup $\Gamma_0(f)$ of f is the set of all intersection numbers $(f,\phi)_0$ where ϕ runs over all power series $\phi \in \boldsymbol{C}\{X,Y\}$ such that $\phi \notin (f)\boldsymbol{C}\{X,Y\}$. Let $\bar{\beta}_0,\dots,\bar{\beta}_g$ be the minimal system of generators of $\Gamma_0(f) : \bar{\beta}_0 = \min(\Gamma_0(f)\backslash\{0\}) = \mathrm{ord}\, f$, $\bar{\beta}_i = \min(\Gamma_0(f)\backslash(\boldsymbol{N}\bar{\beta}_0 + \cdots + \boldsymbol{N}\bar{\beta}_{i-1}))$ for $i = 1,\dots,g$, $\boldsymbol{N}\bar{\beta}_0 + \cdots + \boldsymbol{N}\bar{\beta}_g = \Gamma_0(f)$. Let $\mathrm{GCD}(\bar{\beta}_0,\dots,\bar{\beta}_i)$ stand for the greatest common divisor of $\bar{\beta}_0,\dots,\bar{\beta}_i$.

PROPOSITION 1.1. *Suppose that f is an irreducible power series with* $\mathrm{ord}\, f > 1$. *Then for every regular parameter t we have*

$$q_0(f,t) = \frac{\mathrm{GCD}(\bar{\beta}_0,\dots,\bar{\beta}_{g-1})\bar{\beta}_g}{(t,f)_0}.$$

The proposition above can be easily deduced from the generalization of Merle's result ([M], Théorème 3.1) given by Ephraim ([Eph], Lemma 1.6). We give a direct proof of (1.1) in Section 4 of this paper.

Remark 1.2. If $\mathrm{ord}\, f = \mathrm{ord}\, t = 1$ and $(\partial(t,f)/\partial(X,Y))(0,0) \neq 0$ then $q_0(f,t) = -\infty$, because the set of polar quotients is empty (by convention $\sup\emptyset = -\infty$). If $\mathrm{ord}\, f = \mathrm{ord}\, t = 1$ and $(\partial(t,f)/\partial(X,Y))(0,0) = 0$ then there is only one polar quotient and it is equal to 1. Hence $q_0(f,t) = 1$.

In the sequel we put $\max\{-\infty, a\} = a$ for every $a \in \boldsymbol{R}$. The main result of this note is

Theorem 1.3. *Let* $f = f_1 \cdots f_r$ $(r \geq 2)$, *with* $f_i \in \boldsymbol{C}\{X, Y\}$ *irreducible, be a reduced power series. Let* t *be a regular parameter that does not divide* f. *Then*

$$q_0(f,t) = \max_{i=1}^{r}\left\{\max\left\{q_0(f_i,t), \max_{j\neq i}\frac{(f_i,f_j)_0}{(t,f_j)_0}\right\} + \frac{1}{(t,f_i)_0}\sum_{j\neq i}(f_i,f_j)_0\right\}.$$

We give the proof of (1.3) in Section 4. If f and t are transverse then our theorem is an intersection theoretical counterpart of the main theorem of Kuo and Lu (see [KL], Theorem A). Eggers in [E] developed Kuo and Lu's ideas to calculate the polar quotients with respect to the generic direction in terms of characteristics of branches and their intersection multiplicities coded by means of tree-models (see [Ga] for further developments). In this case our result could be also proved by using Eggers' theorem ([E], Satz 2.1). Let us define for any reduced power series f the invariant $q_0(f)$ by putting

(i) if $\operatorname{ord} f = 1$, then $q_0(f) = -\infty$,

(ii) if f is irreducible, $\operatorname{ord} f > 1$ and $\bar{\beta}_0, \bar{\beta}_1, \ldots, \bar{\beta}_g$ is the minimal system of generators of the semigroup $\Gamma_0(f)$ then

$$q_0(f) = \frac{\mathrm{GCD}(\bar{\beta}_0, \ldots, \bar{\beta}_{g-1})\bar{\beta}_g}{\bar{\beta}_0},$$

(iii) if $f = f_1 \cdots f_r$, $r \geq 2$ with irreducible f_i $(i = 1, \ldots, r)$ then

$$q_0(f) = \max_{i=1}^{r}\left\{\max\left\{q_0(f_i), \max_{j\neq i}\frac{(f_i,f_j)_0}{\operatorname{ord} f_j}\right\} + \frac{1}{\operatorname{ord} f_i}\sum_{j\neq i}(f_i,f_j)_0\right\}.$$

Using (1.1), (1.2) and (1.3) we get

Corollary 1.4. *Suppose* $\operatorname{ord} f > 1$. *Then for every regular parameter* t *that does not divide* f *we have* $q_0(f,t) \leq q_0(f)$. *If* t *and* f *are transverse then* $q_0(f,t) = q_0(f)$.

Let S be an analytic set near $0 \in \boldsymbol{C}^2$. We put $\mathscr{L}_0(f,S) = \inf\{\theta > 0 : |\operatorname{grad} f(z)| \geq C|z|^{\theta}$ for $z \in S$ near $0\}$ and call $\mathscr{L}_0(f) = \mathscr{L}_0(f, \boldsymbol{C}^2)$ the Łojasiewicz exponent of f. We say that $\mathscr{L}_0(f)$ is attained along S if $\mathscr{L}_0(f,S) = \mathscr{L}_0(f)$. Let $\mathrm{Suff}_0(f)$ be the C^0-degree of sufficiency i.e. the smallest integer r such that f is topologically equivalent to $f + g$ for all g with $\operatorname{ord} g \geq r+1$. A historical note about these two notions is given in [LW].

Corollary 1.5. *If* f *is reduced and* $\operatorname{ord} f > 1$ *then* $\mathscr{L}_0(f) = q_0(f) - 1$ *and* $\mathrm{Suff}_0(f) = [q_0(f)]$.

Proof. According to [T] (Sec. 1, Corollaire 2 and Sec. 3, Théorème 8) there is a Zariski-open subset $U \subset \boldsymbol{P}^1(\boldsymbol{C})$ such that $\mathscr{L}_0(f) = q_0(f, bX - aY) - 1$, $\mathrm{Suff}_0(f) = [q_0(f, bX - aY)]$ for $(a:b) \in U$. On the other hand if the line $bX - aY = 0$ is not tangent to the curve $f(X,Y) = 0$ then $q_0(f, bX - aY) = q_0(f)$ by Corollary 1.4.

The following result was proved independently by Bogusławska ([B], Theorem 2) and Kuo and Parusiński ([KP], Theorem 3.1).

Corollary 1.6 ([B], [KP]). *If the line $bX - aY = 0$ is not tangent to the curve $f(X, Y) = 0$ then the Łojasiewicz exponent $\mathscr{L}_0(f)$ is attained on the polar curve $a(\partial f/\partial X) + b(\partial f/\partial Y) = 0$.*

Proof. Assume that the line $bX - aY = 0$ is not tangent to the curve $f(X, Y) = 0$. Using parametrizations of the branches of the polar curve $a(\partial f/\partial X) + b(\partial f/\partial Y) = 0$ we check that $\mathscr{L}_0(f, \{a(\partial f/\partial X) + b(\partial f/\partial Y) = 0\}) = q_0(f, bX - aY) - 1$. Then we use (1.4) and (1.5).

Remark 1.7. Let $f = f_1 \cdots f_r$ $(r \geq 2)$ be a reduced power series such that the irreducible factors f_i are pairwise transverse that is $(f_i, f_j)_0 = (\operatorname{ord} f_i)(\operatorname{ord} f_j)$ for $i \neq j$. Let $I = \{i \in \{1, \dots, r\} : \operatorname{ord} f_i > 1\}$. Then

$$q_0(f) = \max_{i \in I}\left\{q_0(f_i) + \sum_{j \neq i} \operatorname{ord} f_j\right\} = \max_{i \in I}\{q_0(f_i) - \operatorname{ord} f_i\} + \operatorname{ord} f$$

if $I \neq \emptyset$ and $q_0(f) = \operatorname{ord} f$ if $I = \emptyset$.

Corollary 1.8 ([Li]). *Suppose that $f = f_1 \cdots f_r$ $(r \geq 2)$ where f_i are irreducible, pairwise transverse and* $\operatorname{ord} f_i > 1$ *for some $i \in \{1, \dots, r\}$. Then*

$$\mathrm{Suff}_0(f) = \max_{i \in I}\left\{[q_0(f_i)] + \sum_{j \neq i} \operatorname{ord} f_j\right\}.$$

Proof. We apply the second part of Corollary 1.5 and Remark 1.7. In [Li] (p. 160) the above formula is given without assumptions imposed on f_i.

2. Polar quotients and Puiseux series

Let $\boldsymbol{C}\{X\}^* = \bigcup_{n \geq 1} \boldsymbol{C}\{X^{1/n}\}$ be the ring of Puiseux series. If $f(X, Y) \in \boldsymbol{C}\{X, Y\}$ is a power series Y-regular of order $p > 0$ i.e. such that $\operatorname{ord} f(0, Y) = p$, then $f(X, Y) = \prod_{i=1}^{p}(Y - y_i(X))U(X, Y)$ where $y_i(X) \in \boldsymbol{C}\{X\}^*$ are without constant term and $U(X, Y) \in \boldsymbol{C}\{X, Y\}$ is a unit i.e. $U(0, 0) \neq 0$. We denote by $\operatorname{Zer} f = \langle y_1(X), \dots, y_p(X)\rangle$ the sequence $y_1(X), \dots, y_p(X)$ regarded as unordered. To simplify the notation we write y_i for $y_i(X)$. Let $f(X, Y) \in \boldsymbol{C}\{X, Y\}$ be a power series Y-regular of order $p > 1$, then $(\partial f/\partial Y)(X, Y) \in \boldsymbol{C}\{X, Y\}$ is Y-regular of order $p - 1 > 0$ and we can consider the roots of both sereis: $\operatorname{Zer} f = \langle y_1, \dots, y_p\rangle$ and $\operatorname{Zer}(\partial f/\partial Y) = \langle z_1, \dots, z_{p-1}\rangle$. The following is basic for us

Lemma 2.1 (The Kuo-Lu lemma, [KL] Lemma 3.3). *Suppose that $f \in \boldsymbol{C}\{X, Y\}$ has no multiple factors. Then for every $i, j \in \{1, \dots, p\}$, $i \neq j$, there exists a $k \in \{1, \dots, p - 1\}$ such that*

$$(*) \quad \operatorname{ord}(y_i - y_j) = \operatorname{ord}(y_i - z_k).$$

Moreover, for every $i \in \{1, \dots, p\}$ *and* $k \in \{1, \dots, p-1\}$ *there exists a* $j \in \{1, \dots, p\}$ *such that* (*) *holds.*

A simple proof of the Kuo-Lu lemma without using perturbations of power series is given in [GP1], Lemma 3.2.

Proposition 2.2. *With the notations and assumptions introduced above*

$$q_0(f, X) = \max_{i=1}^{p} \left\{ \sum_{j \neq i} \operatorname{ord}(y_j - y_i) + \max_{j \neq i} \{\operatorname{ord}(y_j - y_i)\} \right\}.$$

Proof. By definition

$$q_0(f, X) = \sup\left\{ \frac{(f, g)_0}{(X, g)_0} : g \text{ is an irreducible factor of } \frac{\partial f}{\partial Y} \right\}.$$

Using Zeuthen's rule for intersection multiplicity we get

$$q_0(f, X) = \max_{k=1}^{p-1} \left\{ \sum_{i=1}^{p} \operatorname{ord}(z_k - y_i) \right\}. \tag{1}$$

Put $l_i = \sum_{j \neq i} \operatorname{ord}(y_j - y_i) + \max_{j \neq i}\{\operatorname{ord}(y_j - y_i)\}$ and choose $i_0 \in \{1, \dots, p\}$ such that $\max_{i=1}^{p}\{l_i\} = l_{i_0}$. Let $j_0 \in \{1, \dots, p\}$ be such that

$$\max_{j \neq i_0}\{\operatorname{ord}(y_j - y_{i_0})\} = \operatorname{ord}(y_{j_0} - y_{i_0}). \tag{2}$$

By the first part of the Kuo-Lu lemma there is a root $z = z_k$ of $\partial f / \partial Y = 0$ such that

$$\operatorname{ord}(z - y_{i_0}) = \operatorname{ord}(y_{j_0} - y_{i_0}). \tag{3}$$

We will check that

$$\operatorname{ord}(z - y_i) \geq \operatorname{ord}(y_{i_0} - y_i) \quad \text{for all } i \neq i_0. \tag{4}$$

Indeed, we have

$$\operatorname{ord}(z - y_i) \geq \min\{\operatorname{ord}(z - y_{i_0}), \operatorname{ord}(y_{i_0} - y_i)\} = \operatorname{ord}(y_{i_0} - y_i)$$

for

$$\operatorname{ord}(z - y_{i_0}) = \operatorname{ord}(y_{j_0} - y_{i_0}) = \max_{j \neq i}\{\operatorname{ord}(y_j - y_{i_0})\} \geq \operatorname{ord}(y_i - y_{i_0})$$

for a given $i \neq i_0$. We get

$$\begin{aligned}\max_{i=1}^{p}\{l_i\} = l_{i_0} &= \sum_{j\neq i_0}\operatorname{ord}(y_j - y_{i_0}) + \operatorname{ord}(y_{j_0} - y_{i_0})\\ &\leq \sum_{j\neq i_0}\operatorname{ord}(z - y_j) + \operatorname{ord}(z - y_{i_0})\\ &= \sum_{j=1}^{p}\operatorname{ord}(z - y_j)\end{aligned}$$

by (4) and (3). Consequently $\max_{i=1}^{p}\{l_i\} \leq q_0(f, X)$ by (1). To check the inequality $q_0(f, X) \leq \max_{i=1}^{p}\{l_i\}$ let $z = z_k$ be a root of $\partial f/\partial Y = 0$ such that

$$q_0(f, X) = \sum_{i=1}^{p}\operatorname{ord}(z - y_i). \tag{5}$$

Let $i_1 \in \{1, \dots, p\}$ be such that

$$\max_{i=1}^{p}\{\operatorname{ord}(z - y_i)\} = \operatorname{ord}(z - y_{i_1}). \tag{6}$$

We will check that

$$\operatorname{ord}(y_i - y_{i_1}) \geq \operatorname{ord}(y_i - z) \quad \text{for } i \neq i_1. \tag{7}$$

We have

$$\begin{aligned}\operatorname{ord}(y_i - y_{i_1}) &\geq \min\{\operatorname{ord}(y_i - z), \operatorname{ord}(y_{i_1} - z)\}\\ &= \min\left\{\operatorname{ord}(y_i - z), \max_{j=1}^{p}\{\operatorname{ord}(z - y_j)\}\right\}\\ &= \operatorname{ord}(y_i - z).\end{aligned}$$

By the second part of the Kuo-Lu lemma there is a $j_1 \in \{1, \dots, p\}$ such that

$$\operatorname{ord}(z - y_{i_1}) = \operatorname{ord}(y_{j_1} - y_{i_1}). \tag{8}$$

Therefore we get

$$\begin{aligned}q(f, X) = \sum_{i=1}^{p}\operatorname{ord}(z - y_i) &= \sum_{i\neq i_1}\operatorname{ord}(z - y_i) + \operatorname{ord}(z - y_{i_1})\\ &\leq \sum_{i\neq i_1}\operatorname{ord}(y_i - y_{i_1}) + \operatorname{ord}(y_{j_1} - y_{i_1})\\ &\leq \sum_{i\neq i_1}\operatorname{ord}(y_i - y_{i_1}) + \max_{i\neq i_1}\{\operatorname{ord}(y_i - y_{i_1})\}\\ &= l_{i_1} \leq \max_{i=1}^{p}\{l_i\}.\end{aligned}$$

This ends the proof of (2.2).

3. Characteristic, order of contact and intersection multiplicity

In this section we recall some notions of the theory of branches. Our main reference is [Z] (see also [D], [GP1], [GP2] for nontransverse case). Let $f = f(X, Y) \in \boldsymbol{C}\{X, Y\}$ be an irreducible power series Y-regular of order $p > 1$. Clearly $(f, X)_0 = p$. The Propositions 3.1, 3.2 and 3.3 are well-known.

Proposition 3.1. *Let* $\operatorname{Zer} f = \langle y_1, \dots, y_p\rangle$. *Put* $b_0 = p$. *Then there exists a sequence of strictly positive integers* $b_1 < \cdots < b_h$ *such that for every* $i \in \{1, \dots, p\}$:

(i) $\{b \in \boldsymbol{R} : b = \operatorname{ord}(y_i - y_j)$ *for some* $j \neq i\} = \{b_1/b_0, \dots, b_h/b_0\}$

(ii) $\sharp\{j \in \{1, \dots, p\} : \operatorname{ord}(y_i - y_j) = b_k/b_0\} = B_{k-1} - B_k$ *where* $B_k = \mathrm{GCD}\cdot(b_0, \dots, b_k)$.

Proof. Let $i \in \{1, \dots, p\}$. Then $y_i(X) = \eta(X^{1/p})$ where $\eta(T) \in \boldsymbol{C}\{T\}$. The roots of $f(X, Y) = 0$ form a cycle $\eta(wX^{1/p})$ where w runs over the set $U(p)$ of p-th roots of unity. There exists strictly increasing sequence $b_1, \dots, b_h$ such that $\operatorname{ord}(\eta(wX^{1/p}) - \eta(X^{1/p})) = b_k/p$ if $w \in U(B_{k-1})\backslash U(B_k)$ and $k = 1, \dots, p$. Clearly $\sharp(U(B_{k-1})\backslash U(B_k)) = B_{k-1} - B_k$ and the proposition follows. For more details see [Z] (Chapter II) and [GP2] (Section 3).

We call $(b_0, b_1, \dots, b_h)$ the characteristic sequence of f (with respect to coordinates X, Y). We put $\bar{b}_k = b_k + (1/B_{k-1})\sum_{i=1}^{k-1}(B_{i-1} - B_i)b_i$ for $k = 1, \dots, h$. The sum of an empty family is equal to zero. Thus we have $\bar{b}_1 = b_1$. We put $\bar{b}_0 = b_0$.

Proposition 3.2. *The sequence* $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_h$ *is a system of generetors of the semigroup* $\Gamma_0(f)$ *with respect to* $\bar{b}_0 = (f, X)_0$ *i.e.* $\bar{b}_k = \min\{\Gamma_0(f)\backslash(\boldsymbol{N}\bar{b}_0 + \cdots + \boldsymbol{N}\bar{b}_{k-1})\}$ *for* $k = 1, \dots, h$ *and* $\boldsymbol{N}\bar{b}_0 + \cdots + \boldsymbol{N}\bar{b}_h = \Gamma_0(f)$.

Proof. See [Z] (Théorème 3.9) and [GP2] (Proposition 3.2).

Let us consider two Y-regular power series $f, g \in \boldsymbol{C}\{X, Y\}$. Let $\operatorname{Zer} f = \langle y_1(X), \dots, y_p(X)\rangle$ and $\operatorname{Zer} g = \langle z_1(X), \dots, z_q(X)\rangle$. We define the order of contact of f and g (in coordinates X, Y) by putting

$$\operatorname{cont}(f, g) = \max\{\operatorname{ord}(y_i(X) - z_j(X)) : 1 \le i \le p, 1 \le j \le q\}.$$

Clearly $\operatorname{cont}(f, g) = \operatorname{cont}(g, f)$. Moreover it is easy to check that $\operatorname{cont}(f, g) = \max_{i=1}^{p}\{\operatorname{ord}(y_i(X) - z_j(X))\}$ for every $j \in \{1, \dots, q\}$.

Proposition 3.3. *Suppose that* $\operatorname{ord} f(0, Y) > 1$ *and let* $(b_0, \dots, b_h)$ *be the characteristic sequence of* f. *Let* $k > 0$ *be the smallest integer such that* $\operatorname{cont}(f, g) \le b_k/b_0$ (*we put* $b_{h+1}/b_0 = +\infty$). *Then*

$$\frac{(f, g)_0}{(X, g)_0} = \sum_{i=1}^{k-1}(B_{i-1} - B_i)\frac{b_i}{b_0} + B_{k-1}\operatorname{cont}(f, g).$$

Proof (see also [M], Proposition 2.4 where the case of generic coordinates is considered). We may assume that $\operatorname{cont}(f,g) = \operatorname{ord}(y_1 - z_1)$. By Zeuthen's rule we get

$$\frac{(f,g)_0}{(X,g)_0} = \sum_{i=1}^{p} \operatorname{ord}(y_i - z_1).$$

Using Proposition 3.1 we check that

(a) the set $\{\operatorname{ord}(y_1 - z_1), \ldots, \operatorname{ord}(y_p - z_1)\}$ is equal to the set $\{b_1/b_0, \ldots, b_{k-1}/b_0, \operatorname{cont}(f,g)\}$,

(b) $\sharp\{j \in \{1,\ldots,p\} : \operatorname{ord}(y_j - z_1) = b_i/b_0\} = B_{i-1} - B_i$,

(c) $\sharp\{j \in \{1,\ldots,p\} : \operatorname{ord}(y_j - z_1) = \operatorname{cont}(f,g)\} = B_{k-1}$

and Proposition 3.3 follows.

Using (3.1) and (3.3) we prove

PROPOSITION 3.4. *Let $f = f(X,Y) \in \boldsymbol{C}\{X,Y\}$ be irreducible, Y-regular,* $\operatorname{ord} f(0,Y) > 1$ *of characteristic* $(b_0, b_1, \ldots, b_h)$. *Then*

(i) $q_0(f,X) = \sum_{i=1}^{h}(B_{i-1} - B_i)b_i/b_0 + b_h/b_0 = B_{h-1}\bar{b}_h/\bar{b}_0$,

(ii) *for every irreducible, Y-regular power series* $g \in \boldsymbol{C}\{X,Y\}$

$$\frac{(f,g)_0}{(X,g)_0} \le q_0(f,X) \text{ if and only if } \operatorname{cont}(f,g) \le \frac{b_h}{b_0}.$$

Proof. To prove (i) let us observe that by Proposition 3.1:

$$\sum_{j\neq i} \operatorname{ord}(y_i - y_j) + \max_{j\neq i}\{\operatorname{ord}(y_j - y_i)\} = \sum_{i=1}^{h}(B_{i-1} - B_i)\frac{b_i}{b_0} + \frac{b_h}{b_0} = B_{h-1}\frac{\bar{b}_h}{\bar{b}_0}$$

for every $i \in \{1,\ldots,p\}$. Then we use Proposition 2.2. To check (ii) let us suppose first that $\operatorname{cont}(f,g) \le b_h/b_0$. Then the smallest $k > 0$ such that $\operatorname{cont}(f,g) \le b_k/b_0$ is less than or equal to h. By Proposition 3.3 we get

$$\begin{aligned}\frac{(f,g)_0}{(X,g)_0} &\le \sum_{i=1}^{k-1}(B_{i-1} - B_i)\frac{b_i}{b_0} + B_{k-1}\frac{b_k}{b_0} = B_{k-1}\frac{\bar{b}_k}{\bar{b}_0}\\ &\le \frac{B_{h-1}\bar{b}_h}{\bar{b}_0} = q_0(f,X)\end{aligned}$$

for the sequence $B_{k-1}\bar{b}_k$ is increasing for $k > 0$ $(\bar{b}_{k+1} - (B_{k-1}/B_k)\bar{b}_k = b_{k+1} - b_k > 0$ for $k = 1,\ldots,h-1)$.

Now, suppose that $\operatorname{cont}(f,g) > b_h/b_0$. Again by Proposition 3.3 we get

$$\frac{(f,g)_0}{(X,g)_0} = \sum_{i=1}^{h}(B_{i-1}-B_i)\frac{b_i}{b_0} + \mathrm{cont}(f,g) > \sum_{i=1}^{h}(B_{i-1}-B_i)\frac{b_i}{b_0} + \frac{b_h}{b_0}$$
$$= \frac{B_{h-1}\bar{b}_h}{\bar{b}_0} = q_0(f,X).$$

This ends the proof.

4. Proof

It is easy to check

Lemma 4.1. *If Φ is a local isomorphism i.e. a pair of power series without constant term such that* $\mathrm{Jac}\,\Phi(0,0) \neq 0$ *then*

$$q_0(f,t) = q_0(f \circ \Phi, t \circ \Phi).$$

Therefore to prove (1.1) and (1.3) it suffices to consider the case $t = X$.

Proof of Proposition 1.1. Let $(\bar{\beta}_0, \dots, \bar{\beta}_g)$ be the minimal system of generators of $\Gamma_0(f)$. It is known ([D], pp. 332–333) that three cases are possible

(a) $\bar{b}_0 = \bar{\beta}_0$ then $h = g$ and $(\bar{b}_0, \dots, \bar{b}_g) = (\bar{\beta}_0, \dots, \bar{\beta}_g)$,

(b) $\bar{b}_0 > \bar{\beta}_0$ is a multiple of $\bar{\beta}_0$, then $h = g+1$ and $(\bar{b}_0, \dots, \bar{b}_{g+1}) = (\bar{b}_0, \bar{\beta}_0, \dots, \bar{\beta}_g)$

(c) $\bar{b}_0 > \bar{\beta}_0$ and $\bar{b}_0$ is not a multiple of $\bar{\beta}_0$, then $h = g$ and $(\bar{b}_0, \bar{b}_1, \dots, \bar{b}_g) = (\bar{\beta}_1, \bar{\beta}_0, \bar{\beta}_2, \dots, \bar{\beta}_g)$.

Therefore $B_{h-1}\bar{b}_h = \mathrm{GCD}(\bar{b}_0, \dots, \bar{b}_{h-1})\bar{b}_h = \mathrm{GCD}(\bar{\beta}_0, \dots, \bar{\beta}_{g-1})\bar{\beta}_g$ and Proposition 1.1 follows from (3.4)(i).

Proof of Theorem 1.3. It suffices to prove the following: if $f = f_1 \cdots f_r$ $(r \geq 2)$ with f_i irreducible, is Y-regular, then

$$q_0(f,X) = \max_{i=1}^{r}\left\{\max\left\{q_0(f_i,X), \max_{j\neq i}\frac{(f_i,f_j)_0}{(X,f_j)_0}\right\} + \frac{1}{(X,f_i)_0}\sum_{j\neq i}(f_i,f_j)_0\right\}.$$

According to Proposition 2.2 we have to calculate the quantities

$$l_i = \sum_{j\neq i}\mathrm{ord}(y_j - y_i) + \max_{j\neq i}\{\mathrm{ord}(y_j - y_i)\}.$$

Without restriction of the generality we may assume that $i = 1$ and $y_1 = y_1(X)$ is a root of the equation $f_1(X,Y) = 0$. Let us suppose that $\mathrm{ord}\, f_1(0,Y) > 1$ and let $(b_0, \dots, b_h)$ be the characteristic of f_1. Set

$$I_1 = \{i \neq 1 : y_i = y_i(X) \text{ is a root of } f_1(X,Y) = 0\}$$

and

$$I_2 = \{i : y_i = y_i(X) \text{ is a root of } f_2(X,Y)\cdots f_r(X,Y) = 0\}.$$

Therefore we can write

$$l_1 = \sum_{i\in I_1} \operatorname{ord}(y_i(X) - y_1(X)) + \sum_{i\in I_2} \operatorname{ord}(y_i(X) - y_1(X))$$
$$+ \max\left\{\max_{i\in I_1}\{\operatorname{ord}(y_1(X) - y_i(X))\}, \max_{i\in I_2}\{\operatorname{ord}(y_i(X) - y_1(X))\}\right\}.$$

We get

$$\sum_{i\in I_1} \operatorname{ord}(y_i(X) - y_1(X)) = \sum_{i=1}^{h} (B_{i-1} - B_i)\frac{b_i}{b_0} \text{ by Proposition 3.1,}$$

$$\sum_{i\in I_2} \operatorname{ord}(y_i(X) - y_1(X)) = \frac{1}{(X,f_1)_0}\sum_{i\neq 1}(f_1,f_i)_0 \text{ by Zeuthen's rule,}$$

$$\max_{i\in I_1}\{\operatorname{ord}(y_i(X) - y_1(X))\} = \frac{b_h}{b_0} \text{ by Proposition 3.1,}$$

$$\max_{i\in I_2}\{\operatorname{ord}(y_i(X) - y_1(X))\} = \max_{i\neq 1}\{\operatorname{cont}(f_1,f_i)\} \text{ by definition.}$$

Consequently we get

$$(*)\quad l_1 = \frac{1}{(X,f_1)_0}\sum_{i\neq 1}(f_1,f_i)_0 + \sum_{i=1}^{h}(B_{i-1} - B_i)\frac{b_i}{b_0}$$
$$+ \max\left\{\frac{b_h}{b_0}, \max_{i\neq 1}\ \operatorname{cont}(f_1,f_i)\right\}.$$

Let us consider two cases.

Case 1. $\max_{i\neq 1}\{\operatorname{cont}(f_1,f_i)\} \le b_h/b_0$. By Proposition 3.4 we have $q_0(f_1,X) = \sum_{i=1}^{h}(B_{i-1} - B_i)b_i/b_0 + b_h/b_0$ and $(f_1,f_i)_0/(X,f_i)_0 \le q_0(f_1,X)$ for all $i \neq 1$. Therefore we get by (*):

$$l_1 = \frac{1}{(X,f_1)_0}\sum_{i\neq 1}(f_1,f_i)_0 + \sum_{i=1}^{h}(B_{i-1} - B_i)\frac{b_i}{b_0} + \frac{b_h}{b_0}$$
$$= \frac{1}{(X,f_1)_0}\sum_{i\neq 1}(f_1,f_i) + q_0(f_1,X)$$
$$= \frac{1}{(X,f_1)_0}\sum_{i\neq 1}(f_1,f_i)_0 + \max\left\{q_0(f_1,X), \max_{i\neq 1}\frac{(f_1,f_i)_0}{(X,f_i)_0}\right\}.$$

Case 2. $\max_{i\neq 1}\{\operatorname{cont}(f_1,f_i)\} > b_h/b_0$. Let us fix $i \neq 1$. If $\operatorname{cont}(f_1,f_i) > b_h/b_0$ then

$$\frac{(f_1, f_i)_0}{(X, f_i)_0} = \sum_{i=1}^{h} (B_{i-1} - B_i) \frac{b_i}{b_0} + \mathrm{cont}(f_1, f_i)$$

by (3.3), if $\mathrm{cont}(f_1, f_i) \le b_h/b_0$ then

$$\frac{(f_1, f_i)_0}{(X, f_i)_0} \le \sum_{i=1}^{h} (B_{i-1} - B_i) \frac{b_i}{b_0}$$

again by (3.3). Therefore

$$\max_{i \neq 1} \frac{(f_1, f_i)_0}{(X, f_i)_0} = \sum_{i=1}^{h} (B_{i-1} - B_i) \frac{b_i}{b_0} + \max_{i \neq 1} \{\mathrm{cont}(f_1, f_i)\}.$$

Using (*) we get

$$\begin{aligned} l_1 &= \frac{1}{(X, f_1)_0} \sum_{i \neq 1} (f_1, f_i)_0 + \sum_{i=1}^{h} (B_{i-1} - B_i) \frac{b_i}{b_0} + \max_{i \neq 1} \{\mathrm{cont}(f_1, f_i)\} \\ &= \frac{1}{(X, f_1)_0} \sum_{i \neq 1} (f_1, f_i)_0 + \max_{i \neq 1} \frac{(f_1, f_i)_0}{(X, f_i)_0} \\ &= \frac{1}{(X, f_1)_0} \sum_{i \neq 1} (f_1, f_i)_0 + \max\left\{ q_0(f_1, X), \max_{i \neq 1} \frac{(f_1, f_i)_0}{(X, f_i)_0} \right\} \\ &\quad \text{for } q_0(f_1, X) = \sum_{i=1}^{h} (B_{i-1} - B_i) \frac{b_i}{b_0} + \frac{b_h}{b_0} < \max_{i \neq 1} \frac{(f_1, f_i)_0}{(X, f_i)_0}. \end{aligned}$$

It remains to consider the case $\mathrm{ord}\, f_1(0, Y) = 1$. Then $(X, f_1)_0 = 1$, $q_0(f_1, X) = -\infty$ and we get easily

$$l_1 = \sum_{j \neq 1} (f_1, f_j)_0 + \max_{j \neq 1} \frac{(f_1, f_j)_0}{(X, f_j)_0}.$$

This ends the proof.

Appendix

We reprove here the main result of [KL].

Let $f = f(X, Y) \in \boldsymbol{C}\{X, Y\}$ be a power series Y-regular of order $p = \mathrm{ord}\, f$. Let $y_i = y_i(X) \in \boldsymbol{C}\{X\}^*$ $(i = 1, \ldots, p)$ be the sequence of all solutions (without constant term) of the equation $f(X, Y) = 0$. We put

$$l_i = \sum_{j \neq i} \mathrm{ord}(y_i - y_j) + \max_{j \neq i} \{\mathrm{ord}(y_i - y_j)\} \quad \text{for } i = 1, \ldots, p.$$

Then we have

Kuo and Lu's formula for the Łojasiewicz exponent ([KL], Theorem A). *With the above notation assume that f is a reduced power series with* ord $f > 1$. *Then, we have*

$$\mathscr{L}_0(f) = \max_{i=1}^{p}\{l_i - 1\}.$$

Proof. By Proposition 2.2 we have $q_0(f, X) = \max_{i=1}^{p}\{l_i\}$. On the other hand $q_0(f, X) = q_0(f)$ by the second part of Corollary 1.4. Then we use the first part of Corollary 1.5.

Note here that the original result was proved for distinguished polynomials. The quantities l_i was defined in Kuo and Lu's work by means of some perturbations of the roots $y_i = y_i(X)$. The simple expressions for l_i we use are due to Bogusławska ([B], Lemma 3).

Let $c_i = \max_{j \neq i}\{\mathrm{ord}(y_i - y_j)\}$ for $i = 1, \dots, p$. According to [KL] a root y_i is minimal if for any root y_j with $\mathrm{ord}(y_i - y_j) = c_i$ we get $c_j \le c_i$. If is easy to check that if $c_i < c_k$ then $l_i < l_k$. Hence we get

Corollary to Kuo and Lu's formula ([KL], Theorem A′). *We have*

$$\mathscr{L}_0(f) = \max_{i \in I}\{l_i - 1\}.$$

where I is the set of all $i \in \{1, \dots, p\}$ such that the root y_i is minimal.

For every branch f we denote by $K_0(f)$ the knot corresponding to f. Let f be a branch of characteristic $(\beta_0, \dots, \beta_g)$. We define the self-link $\sigma_0(f)$ of f to be $\sigma_0(f) = \mathrm{link}(K_0(f), K_0(\tilde{f}))$ where $\tilde{f}$ is a branch of characteristic $(\beta_0, \dots, \beta_g)$ such that $\mathrm{cont}(f, \tilde{f}) = \beta_g/\beta_0$ (see [KL], pp. 300–301). Then $\sigma_0(f) = (f, \tilde{f})_0 = \mathrm{GCD}(\bar{\beta}_0, \dots, \bar{\beta}_{g-1})\bar{\beta}_g = q_0(f)(\mathrm{ord}\, f)$ by Proposition 3.3.

In [KL] the authors assert ([KL], Corollary to Theorem B) the following: "suppose that $y_1 = y_1(X)$ is a minimal root of $f = f_1 \cdots f_r$ and let $f_i = f_i(X, Y)$ be such that $f_i(X, y_1(X)) = 0$. Assume that knots $K_0(f_i)$ and $K_0(f_j)$ are not isotopic for $i \neq j$. Then

$$(\mathrm{ord}\, f_i)l_1 = \sum_{j \neq i}(f_i, f_j)_0 + \sigma_0(f_i).\text{''}$$

The example given below shows that the above statement (and consequently Theorem B of [KL]) are not true.

Example. Let $1 < \beta_0 < \cdots < \beta_g$, $g > 1$ be a sequence of coprime integers such that the divisors $\mathrm{GCD}(\beta_0, \dots, \beta_i)$ form a strictly decreasing sequence. Let $f_1(X, Y) = 0$ (resp. $f_2(X, Y) = 0$) be the minimal equation of the Puiseux series $y_1 = X^{\beta_1/\beta_0} + \cdots + X^{\beta_{g-1}/\beta_0}$ (resp. $y_2 = X^{\beta_1/\beta_0} + \cdots + X^{\beta_g/\beta_0}$). Let $f = f_1 f_2$. The series f_1 and f_2 have different characteristics and hence the knots $K_0(f_1)$ and $K_0(f_2)$ are not isotopic. For every root y_i of f we get $c_i = \mathrm{cont}(f_1, f_2) = \beta_g/\beta_0$, therefore all roots of f are minimal. The calculation from the proof of Theorem 1.3 (Section 4 of this paper, Case 2) gives

$$l_1 = \frac{(f_1, f_2)_0}{\text{ord } f_1} + \frac{(f_1, f_2)_0}{\text{ord } f_2}$$

and

$$q_0(f_1) < \frac{(f_1, f_2)_0}{\text{ord } f_2}.$$

Therefore

$$\begin{aligned}(\text{ord } f_1)l_1 &= (f_1, f_2)_0 + \text{ord } f_1 \frac{(f_1, f_2)_0}{\text{ord } f_2} \\ &> (f_1, f_2)_0 + (\text{ord } f_1)q_0(f_1) = (f_1, f_2)_0 + \sigma_0(f_1).\end{aligned}$$

This contradicts the statement quoted above.

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