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SOLUTION OF A DIFFERENTIAL EQUATION AND ITS APPLICATIONS*†

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Abstract

In this paper, we proved that the order of entire solutions of the differential equation $f^{(n)} - fe^{Q(z)} = 1$ are infinite, where $Q(z)$ is a nonconstant polynomial, and gave some of its applications.

1. Introduction

In this paper a meromorphic function will mean meromorphic in the whole complex plane. We say that two meromorphic functions f and g share a finite value a IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities). It is assumed that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [3], [7].

L. Rubel and C. C. Yang proved the following result.

THEOREM A ([6]). *Let f be a nonconstant entire function. If f and f' share two finite, distinct values CM, then $f \equiv f'$.*

Regarding Theorem A, a natural question is:

QUESTION 1. What can be said when a nonconstant entire function f shares one finite value CM with one of its derivatives $f^{(k)}$ $(k \geq 1)$?

We mention several results that concern with the Question 1.

THEOREM B ([4]). *Let f be a nonconstant meromorphic function, and let $a \neq 0$ be a finite constant. If f, f', and f'' share the value a CM, then $f \equiv f'$.*

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THEOREM C ([4]). *Let f be a nonconstant entire function, and let $a \neq 0$ be a finite constant. If f and f' share the value a IM, and if $f''(z) = a$ whenever $f(z) = a$, then $f \equiv f'$.*

THEOREM D ([9]). *Let f be a nonconstant entire function, let $a \neq 0$ be a finite constant, and let n be a positive integer. If f and f' share the value a CM, and if $f^{(n)}(z) = f^{(n+1)}(z) = a$ whenever $f(z) = a$, then $f \equiv f^{(n)}$.*

THEOREM E ([1]). *Let f be an entire function which is not constant. If f and f' share the value 1 CM, and if $N(r, 0, f') = S(r, f)$, then*

$$\frac{f' - 1}{f - 1} = c,$$

where c is a non-zero constant.

THEOREM F ([8]). *Let f be a non-constant meromorphic function. If f and f' share the value 1 CM, and if*

$$\bar{N}(r, f) + N\left(r, \frac{1}{f'}\right) < (\lambda + o(1))T(r, f')$$

for some real constant $\lambda \in (0, 1/2)$, then f and f' satisfy

$$\frac{f' - 1}{f - 1} = c.$$

THEOREM G ([8, Corollary 3]). *Let f be a non-constant entire function, k be a positive integer. If f and $f^{(k)}$ share the value 1 CM, and if*

$$\bar{N}\left(r, \frac{1}{f'}\right) < (\lambda + o(1))T(r, f)$$

for some real constant $\lambda \in (0, 1/4)$, then

$$\frac{f^{(k)} - 1}{f - 1} = c$$

for some non-zero constant c.

By considering solutions of the following differential equation

$$\frac{f^{(k)} - 1}{f - 1} = e^z,$$

we know, in general, that one can not get

$$\frac{f^{(k)} - 1}{f - 1} = \text{const.}$$

under the condition of Question 1, for entire functions of infinite order.

In this paper, we will answer Question 1 and give some improvements of the above Theorems A–G for entire functions of finite order.

2. Lemmas

LEMMA 1 ([2]). *Let F be a nonconstant meromorphic function of finite order ρ, and let $\varepsilon > 0$ be a given constant, k be a positive integer. Then there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E$, then there is a constant $R_0 = R_0(\psi_0) > 0$ such that for all z satisfying* $\arg z = \psi_0$ *and* $|z| \geq R_0$, *we have*

$$\left|\frac{F^{(k)}(z)}{F(z)}\right| \leq |z|^{k(\rho-1+\varepsilon)}.$$

LEMMA 2. *Let F be an entire function, and suppose that $|F^{(k)}(z)|$ is unbounded on some ray* $\arg z = \phi$. *Then there exists an infinite sequence of points $z_n = r_n e^{i\phi}$ where $r_n \to +\infty$, such that $F^{(k)}(z_n) \to \infty$ and*

$$\left|\frac{F(z_n)}{F^{(k)}(z_n)}\right| \leq (1 + o(1))|z_n|^k$$

as $z_n \to \infty$.

Proof. Let

$$M(r, F^{(k)}, \phi) = \max\{|F^{(k)}(z)| : 0 \leq |z| \leq r, \arg z = \phi\}.$$

Since $|F^{(k)}(z)|$ is unbounded on the ray $\arg z = \phi$, then there exists an infinite sequence of points $z_n = r_n e^{i\phi}$ where $r_n \to +\infty$, such that $F^{(k)}(z_n) \to \infty$ as $z_n \to \infty$ and

$$M(r_n, F^{(k)}, \phi) = |F^{(k)}(z_n)|.$$

From

$$F^{(k-1)}(z_n) = F^{(k-1)}(0) + \int_0^{z_n} F^{(k)}(u)\, du,$$

we have

$$|F^{(k-1)}(z_n)| \leq |F^{(k-1)}(0)| + |z_n|\, |F^{(k)}(z_n)|.$$

Again, by

$$\begin{aligned} F^{(k-2)}(z_n) &= F^{(k-2)}(0) + \int_0^{z_n} F^{(k-1)}(u)\, du \\ &= F^{(k-2)}(0) + \int_0^{z_n} \left\{ F^{(k-1)}(0) + \int_0^{u} F^{(k)}(s)\, ds \right\} du, \end{aligned}$$

we obtain

$$|F^{(k-2)}(z_n)| \le |F^{(k-2)}(0)| + |F^{(k-1)}(0)|\,|z_n| + |F^{(k)}(z_n)|\,|z_n|^2.$$

By using the same methods for n time, we deduce that

$$\begin{aligned}|F(z_n)| &\le |F(0)| + |F'(0)|\,|z_n| + \cdots + |F^{(k)}(z_n)|\,|z_n|^k \\ &= (1+o(1))|F^{(k)}(z_n)|\,|z_n|^k\end{aligned}$$

and we obtain

$$\left|\frac{F(z_n)}{F^{(k)}(z_n)}\right| \le (1+o(1))|z_n|^k$$

as $z_n \to \infty$.

3. Solutions of a differential equation

THEOREM 1. *Let $Q(z)$ be a nonconstant polynomial and k be a positive integer. Then every solution F of the differential equation*

$$F^{(k)} - e^{Q(z)}F = 1 \tag{1}$$

is an entire function of infinite order.

Proof. It is well known that every solution of equation (1) is entire. We prove Theorem 1 by contradiction. Assume that Theorem 1 is not true, i.e., suppose that F is a solution of equation (1) that has finite order ρ.

From (1),

$$\frac{F^{(k)}}{F} - e^{Q(z)} = \frac{1}{F}. \tag{2}$$

Let $\varepsilon > 0$ be any given constant. Then from Lemma 1, there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E$, then there is a constant $R_0 = R_0(\psi_0) > 0$ such that for all z satisfying $\arg z = \psi_0$ and $|z| \ge R_0$, we have

$$\left|\frac{F^{(k)}(z)}{F(z)}\right| \le |z|^{k(\rho-1+\varepsilon)}. \tag{3}$$

Now suppose that θ is any real number that satisfies $\theta \in [0, 2\pi) - E$, and for every $\alpha > 0$,

$$\frac{|e^{Q(re^{i\theta})}|}{r^\alpha} \to +\infty \tag{4}$$

as $r \to +\infty$. Then from (4), (3), and (2), it follows that

$$F(re^{i\theta}) \to 0 \tag{5}$$

as $r \to +\infty$.

Now suppose that ϕ is any real number that satisfies $\phi \in [0, 2\pi)$, and for every $\beta > 0$,

$$r^{\beta} e^{Q(re^{i\phi})} \to 0 \tag{6}$$

as $r \to +\infty$. We now show that $|F^{(k)}(z)|$ is bounded on the ray $\arg z = \phi$. Assume the contrary, i.e., suppose that $|F^{(k)}(z)|$ is not bounded on the ray $\arg z = \phi$. Then from Lemma 2, there exists an infinite sequence of points $z_n = r_n e^{i\phi}$ where $r_n \to +\infty$, such that $F^{(k)}(z_n) \to \infty$ and

$$\left| \frac{F(z_n)}{F^{(k)}(z_n)} \right| \le (1 + o(1))|z_n|^k \tag{7}$$

as $z_n \to \infty$. Since $F^{(k)}(z_n) \to \infty$, it follows from (6) and (1) that $F(z_n) \to \infty$. Then from (7), (6), and (2), we obtain that $F^{(k)}(z_n) \to 1$, which contradicts $F^{(k)}(z_n) \to \infty$. This contradiction proves that $|F^{(k)}(z)|$ must be bounded on the ray $\arg z = \phi$. By considering the formula

$$F^{(k-1)}(z) = F^{(k-1)}(0) + \int_0^z F^{(k)}(w)\, dw,$$

we obtain that

$$|F^{(k-1)}(z)| \le |F^{(k-1)}(0)| + M|z| \tag{8}$$

for all z satisfying $\arg z = \phi$, where $M = M(\phi) > 0$ is some constant.

Similarly, by using (8) and the formula

$$F^{(j-1)}(z) = F^{(j-1)}(0) + \int_0^z F^{(j)}(w)\, dw, \quad (1 \le j \le k),$$

for k times, we can easily obtain

$$|F(z)| \le p_k(|z|) = (M + o(1))|z|^k, \quad \text{as } r \to \infty \tag{9}$$

for all z satisfying $\arg z = \phi$, where $p_k(x)$ is a polynomial of degree k.

We have shown that (9) holds for any $\phi \in [0, 2\pi)$ with the property (6), and that (5) holds for any $\theta \in [0, 2\pi) - E$ with the property (4). Since $Q(z)$ is a nonconstant polynomial, there exist only finitely many real numbers in $[0, 2\pi)$ that do not satisfy either (6) or (4). We also note that the set E has linear measure zero. Therefore, since F has finite order, it can be deduced from (9), (5), the Phragmén-Lindelöf theorem [5, pp. 270–271], and Liouville's theorem, that F must be a polynomial with deg $F \le k$. But this is impossible because $Q(z)$ is nonconstant in (1). This contradiction proves Theorem 1.

4. Applications of Theorem 1

THEOREM 2. *Let f be a nonconstant entire function of finite order, and let $a \neq 0$ be a finite constant. If f, $f^{(k)}$ share the value a CM, then*

$$\frac{f^{(k)} - a}{f - a} = c \tag{10}$$

for some non-zero constant c.

Proof. Since f has finite order, and since f and $f^{(k)}$ share the value a CM, it follows from the Hadamard factorization theorem that

$$\frac{f^{(k)} - a}{f - a} = e^{Q(z)}, \tag{11}$$

where $Q(z)$ is a polynomial. Set $F = (f/a) - 1$. Then from (11),

$$F^{(k)} - e^{Q(z)}F = 1. \tag{12}$$

If $Q(z)$ is nonconstant, then from (12) and Theorem 1, we obtain that F has infinite order. Since f has finite order, this is impossible. Hence $Q(z)$ is a constant. Then from (11), we obtain (10) for a non-zero constant c.

Obviously, Theorem 2 improves Theorem E, Theorem F and Theorem G for entire functions of finite order. The following theorems follow from Theorem 2, which are the improvements of Theorems A, B, C and D.

THEOREM 3. *Let f be a nonconstant entire function of finite order, and let $a \neq 0$ be a finite constant, k be a positive integer. If f and $f^{(k)}$ share the value a CM, and if there exists one point z_0 such that $f^{(k)}(z_0) = f(z_0) \neq a$, then $f \equiv f^{(k)}$.*

COROLLARY. *Let f be a nonconstant entire function of finite order, k be a positive integer, and let $a \neq 0$, b be two distinct finite values. If f and $f^{(k)}$ share the value a CM, f and $f^{(k)}$ share the value b IM, then $f \equiv f^{(k)}$.*

THEOREM 4. *Let f be a nonconstant entire function of finite order, let $a \neq 0$ be a finite constant, and let n be a positive integer. If f and f' share the value a CM, and if there exists at least one point z_0 such that $f^{(n)}(z_0) = f^{(n+1)}(z_0) \neq 0$, then $f \equiv f'$.*

THEOREM 5. *Let f be a nonconstant entire function of finite order, let $a \neq 0$ be a finite constant, and let n be a positive integer. If f and $f^{(n)}$ share the value a CM, and if there exists at least one point z_0 such that $f'(z_0) = f^{(n+1)}(z_0) \neq 0$, then $f \equiv f^{(n)}$.*

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