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SOME FURTHER RESULTS ON THE ZEROS AND GROWTHS OF ENTIRE SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, we define the hyper-exponent of convergence of zeros of an entire solution $f(z)$ of a second order linear differential equation, and use it to obtain some further estimates on the zeros, growth, and fixed points of $f(z)$.

1. Introduction and results

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (e.g. see [8, or 10]). In addition, we will use the same notations as in [1], such as $\lambda(f)$ and $\bar{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and the sequence of distinct zeros of meromorphic function $f(z)$, $\sigma(f)$ and $\mu(f)$ to denote respectively the order and the lower order of growth of $f(z)$.

We recall the following definition.

DEFINITION 1 ([16]). Let f be a meromorphic function. Then the hyper-order $\sigma_2(f)$ of $f(z)$ is defined by

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log T(r,f)}{\log r}. \tag{1.1}$$

NOTE. Clearly, if $f(z)$ is entire, then

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log\log M(r,f)}{\log r} = \overline{\lim_{t\to\infty}} \frac{\log\log T(r,f)}{\log r}. \tag{1.2}$$

We define:

Key words: differential equation, hyper-order, hyper-exponont of convergence of zeros, Wiman-Valiron theory.

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DEFINITION 2. Let f be a entire function. Then $\lambda_2(f)$, the hyper-exponent of convergence of zeros of $f(z)$, is defined by

$$\lambda_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log N(r, 1/f)}{\log r}, \tag{1.3}$$

and $\bar{\lambda}_2(f)$, hyper-exponent of convergence of distinct zeros of $f(z)$, is defined by

$$\bar{\lambda}_2(f) = \overline{\lim_{r\to\infty}} \frac{\log\log \overline{N}(r, 1/f)}{\log r}. \tag{1.4}$$

For almost two decades, the Nevanlinna's value distribution theory has been a useful tool in investigating the complex oscillation of differential equations. Recently, in [4, 12, 13], the concepts of hyper-order [4, 16] and iterated order [13] were used to further investigate the growth of infinite order solutions of complex differential equations. The following results have been obtaind.

THEOREM A ([12]). *Let A, B be two entire functions such that $\sigma(A) < \sigma(B)$ or $\sigma(B) < \sigma(A) < 1/2$. Then every entire solution $f \not\equiv 0$ of*

$$f'' + Af' + Bf = 0, \tag{1.5}$$

satisfies

$$\sigma_2(f) \geq \max\{\sigma(A), \sigma(B)\}.$$

THEOREM B ([12]). *Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$, and let $A(z)$ and $B(z)$ be entire functions such that for real constants $\alpha(>0), \beta(>0)$,*

$$|A(z)| \leq \exp\{o(1)|z|^{\beta}\} \tag{1.6}$$

and

$$|B(z)| \geq \exp\{(1 + o(1))\alpha|z|^{\beta}\} \tag{1.7}$$

as $z \to \infty$ for $z \in H$. Then every entire solution f of the equation (1.5) *satisfies $\sigma_2(f) \geq \beta$.*

Hence, the upper and the lower densities of H are defined by

$$\overline{dens}\, H = \overline{\lim_{r\to\infty}} \frac{m(H \cap [0, r])}{r},$$

and

$$\underline{dens}\, H = \underline{\lim_{r\to\infty}} \frac{m(H \cap [0, r])}{r},$$

where $m(H)$ is the linear measure of a set H.

THEOREM C ([15]). *Let $a_0, \ldots, a_{k-1}$ be polynomials. If $f(z)$ is an entire solution of the equation*

$$f^{(k)} + a_{k-1}f^{k-1} + \cdots + a_0 f = 0,$$

then

$$\sigma(f) \le 1 + \max_{1 \le j \le k-1} \frac{\deg a_j}{k-j}.$$

The main purposes of this paper are to improve results of Theorems A, B, and C (when $k = 2$) and to investigate the hyper-exponent of convergence of zeros and hyper-order of solutions of non-homogeneous linear differential equations. As an application, we give the estimate of fixed points of solutions of some class of differential equations.

THEOREM 1. *Let a_0, a_1 be nonconstant polynomials with degrees $\deg a_j = n_j$ $(j = 1,2)$. Let $f(z)$ be an entire solution of the differential equation*

$$f'' + a_1(z)f' + a_0(z)f = 0. \tag{1.8}$$

Then

(i) *If $n_0 \ge 2n_1$, then any entire solution $f \not\equiv 0$ of the equation (1.8) satisfies $\sigma(f) = (n_0 + 2)/2$; if $n_0 < n_1 - 1$, then any entire solution $f \not\equiv 0$ of (1.8) satisfies $\sigma(f) = n_1 + 1$; if $n_1 - 1 \le n_0 < 2n_1$, then any entire solution $f \not\equiv 0$ of (1.8) satisfies either $\sigma(f) = n_1 + 1$, or $\sigma(f) = n_0 - n_1 + 1$.*

(ii) *In (i), if $n_0 = n_1 - 1$, then the equation (1.8) possibly has polynomial solutions, and any two polynomial solutions f of (1.8) are linearly dependent, all the polynomial solutions have the form $f_c(z) = cp(z)$, where p is some polynomial, c is an arbitrary constant.*

Example of polynomial solutions in case (ii). The equation

$$f'' - (z^3 + z^2 + z + 1)f' + (z^2 + 1)f = 0$$

has polynomial solutions $f_c = c\,(z + 1)$.

THEOREM 2. *Let a_0, a_1 and b be nonconstant polynomials with degrees: $\deg a_j = n_j$ $(j = 1,2)$. Let $f \not\equiv 0$ be an entire solution of the differential equation*

$$f'' + a_1(z)f' + a_0(z)f = b(z). \tag{1.9}$$

Then

(i) *If $n_0 \ge 2n_1$, then $\bar{\lambda}(f) = \sigma(f) = (n_0 + 2)/2$; if $n_0 < n_1 - 1$, then $\bar{\lambda}(f) = \sigma(f) = n_1 + 1$; if $n_1 - 1 < n_0 < 2n_1$, then $\bar{\lambda}(f) = \sigma(f) = n_1 + 1$ or $\bar{\lambda}(f) = \sigma(f) = n_0 - n_1 + 1$, with at most one exceptional polynomial solution f_0 for three cases above.*

(ii) *If $n_0 = n_1 - 1$, then every transcendental entire solution f satisfies $\bar{\lambda}(f) = \sigma(f) = n_1 + 1$ (or 0).*

Remarks. If the corresponding homogeneous equation of (1.9) has a polynomial solution $p(z)$, then (1.9) may have a family of polynomial solutions $\{cp(z)+f_0\}$ (f_0 is a polynomial solution of (1.9), c is a constant). If the corresponding homogeneous equation of (1.9) has no polynomial solution, then (1.9) has at most one polynomial solution.

Example of a family of polynomial solutions in case (ii). The equation

$$f'' - (z^3+z^2+z+1)f' + (z^2+1)f = z^2+1$$

has a family of polynomial solutions $\{c(z+1)+1, c \text{ is a constant}\}$.

THEOREM 3. *Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$, and let A_0, A_1 be entire functions, with $\sigma(A_1) \le \sigma(A_0) = \sigma < +\infty$ such that for real constant $C(>0)$ and for any given $\varepsilon > 0$,*

$$|A_1(z)| \le \exp\{o(1)|z|^{\sigma-\varepsilon}\} \tag{1.10}$$

and

$$|A_0(z)| \ge \exp\{(1+o(1))C|z|^{\sigma-\varepsilon}\} \tag{1.11}$$

as $z \to \infty$ for $z \in H$. Then every entire solution $f \not\equiv 0$ of the equation

$$f'' + A_1 f' + A_0 f = 0 \tag{1.12}$$

satisfies $\sigma(f) = +\infty$ and $\sigma_2(f) = \sigma(A_0)$.

THEOREM 4. *Let H, A_1 and A_0 satisfy the hypothesis of Theorem 3, and let $F \not\equiv 0$ be an entire function with $\sigma(F) < +\infty$. Then every entire solution $f(z)$ of the equation*

$$f'' + A_1 f' + A_0 f = F \tag{1.13}$$

satisfies $\bar{\lambda}_2(f) = \sigma_2(f) = \sigma$, with at most one exceptional f_0 satisfying $\sigma(f_0) < \sigma$.

THEOREM 5. *Let $A_1, A_0 \not\equiv 0$ be entire functions such that $\sigma(A_0) < \sigma(A_1) < 1/2$ (or A_1 is transcendental, $\sigma(A_1) = 0$, A_0 is a polynomial). Then every entire solution $f \not\equiv 0$ of the equation (1.12) satisfies $\sigma(f) = \infty$, $\sigma_2(f) = \sigma(A_1)$.*

NOTE. A relatively simplier case when $\sigma(A_0) > \sigma(A_1)$, was discussed by L. Kinnunen in [13] and the conclusion: $\sigma(f) = \infty$ was derived earlier in [7].

THEOREM 6. *Let A_1, A_0 satisfy the hypothesis of Theorem 5, and let $F \not\equiv 0$ be an entire function. Conside a solution f of the equation (1.13), we have*

(i) If $\sigma(F) < \sigma(A_1)$ (or F is a polynomial when A_1 is transcendental, $\sigma(A_1) = 0$, A_0 is a polynomial), then every solution $f(z)$ of (1.13) satisfies $\bar{\lambda}_2(f) = \sigma_2(f) = \sigma(A_1)$.

(ii) If $\sigma(A_1) \le \sigma(F) < \infty$, then every entire solution $f(z)$ of (1.13) satisfies $\bar{\lambda}_2(f) = \sigma_2(f) = \sigma(A_1)$, with at most one exceptional solution f_0 satisfying $\sigma(f_0) < \sigma(A_1)$.

Set $g(z) = f(z) - z$. Then clearly $\bar{\lambda}_2(f - z) = \bar{\lambda}_2(g)$, $\sigma_2(g) = \sigma_2(f)$. By Theorems 1-6, we can get the following corollaries.

COROLLARY 1. *Under the hypothesis of Theorem* 1, *if* $n_0 \neq n_1 - 1$, *or* $n_0 = n_1 - 1$ *and* $a_1 + za_0 \not\equiv 0$, *then every transcendental entire solution* $f(z)$ *of* (1.8) *satisfies* $\bar{\lambda}(f - z) = \sigma(f)$.

COROLLARY 2. *Under the hypothesis of Theorem* 2, *if* $b - a_1 - za_0 \not\equiv 0$, *then every transcendental entire solution* $f(z)$ *of* (1.9) *satisfies* $\bar{\lambda}(f - z) = \sigma(f)$.

COROLLARY 3. *In Theorem* 3, *every entire solution* $f(z)$ *of* (1.12) *satisfies* $\bar{\lambda}_2(f - z) = \sigma_2(f)$.

COROLLARY 4. *In Theorem* 4, *if* $F - A_1 - zA_0 \not\equiv 0$, *then every entire solution* $f(z)$ *with* $\bar{\lambda}_2(f) = \sigma_2(f) = \sigma$ of (1.13) *satisfies* $\bar{\lambda}_2(f - z) = \sigma_2(f)$.

COROLLARY 5. *In Theorem* 5, *every entire solution* $f(z)$ *of* (1.12) *satisfies* $\bar{\lambda}_2(f - z) = \sigma_2(f)$.

COROLLARY 6. *In Theorem* 6, *in case* (i), *every entire solution* $f(z)$ *of* (1.13) *satisfies* $\bar{\lambda}_2(f - z) = \sigma_2(f)$; *in case* (ii), *if* $F - A_1 - zA_0 \not\equiv 0$, *then every entire solution* $f(z)$ *with* $\bar{\lambda}_2(f) = \sigma_2(f) = \sigma$ *of* (1.13) *satisfies* $\bar{\lambda}_2(f - z) = \sigma_2(f)$.

2. Preliminary lemmas

We need following lemmas for the proofs of our theorems.

LEMMA 1 ([10, Theorems 1.9 and 1.10, or 11, Satz 4.3 and 4.4]). *Let* $g(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a entire function*, $\mu(r)$ *be the maximum term, i.e.* $\mu(r) = \max\{|a_n|r^n; n = 0, 1, \dots\}$, $\nu(r)$ (*we simplify* $\nu_g(r)$ *by* $\nu(r)$ *if no confusion may arise*) *be the central index, i.e.* $\nu(r) = \max\{m, \mu(r) = |a_m|r^m\}$. *Then*

(i)
$$\log \mu(r) = \log |a_0| + \int_0^r \frac{\nu(t)}{t}\, dt,$$

here we assume that $a_0 \neq 0$.

(ii) *For* $r < R$,

$$M(r, g) < \mu(r)\left\{\nu(R) + \frac{R}{R - r}\right\}.$$

LEMMA 2. *Let* $g(z)$ *be an entire function of infinite order with the hyper-order* $\sigma_2(g) = \sigma$, *and let* $\nu(r)$ *be the central index of* g. *Then*

$$\overline{\lim_{r \to \infty}} \frac{\log \log \nu(r)}{\log r} = \sigma.$$

Proof. Set $g(z) = \sum_{n=0}^{\infty} a_n z^n$. Without loss of generality, we may assume $|a_0| \neq 0$. By (i) of Lemma 1, the maximum term $\mu(r)$ of g satisfies

$$\log \mu(2r) = \log |a_0| + \int_0^{2r} \frac{\nu(t)}{t}\, dt \geq \log|a_0| + \nu(r) \log 2. \tag{2.1}$$

By Cauchy's inequality, we have

$$\mu(2r) \leq M(2r, g). \tag{2.2}$$

This and (2.1) yield

$$\nu(r) \log 2 \leq \log M(2r, g) + C, \tag{2.3}$$

where $C(>0)$ is a constant. By this and (1.2), we have

$$\overline{\lim_{r\to\infty}} \frac{\log\log \nu(r)}{\log r} \leq \overline{\lim_{r\to\infty}} \frac{\log\log\log M(r,g)}{\log r} = \sigma_2(g) = \sigma. \tag{2.4}$$

On the other hand, by (ii) of Lemma 1, we have

$$M(r,g) < \mu(r)\{\nu(2r) + 2\} = |a_{\nu(r)}| r^{\nu(r)} \{\nu(2r) + 2\}. \tag{2.5}$$

Hence, we get from (2.5)

$$\begin{aligned} \log M(r,g) &\leq \nu(r) \log r + \log \nu(2r) + C_1, \\ \log\log M(r,g) &\leq \log \nu(r) + \log\log \nu(2r) + \log\log r + C_2 \\ &\leq \log \nu(2r) \left[1 + \frac{\log\log \nu(2r)}{\log \nu(2r)}\right] + \log\log r + C_3, \end{aligned} \tag{2.6}$$

where $C_j(>0)$ $(j = 1, 2, 3)$ are constants. By (2.6) and (1.2), we get

$$\sigma_2(g) = \overline{\lim_{r\to\infty}} \frac{\log\log\log M(r,g)}{\log r} \leq \overline{\lim_{r\to\infty}} \frac{\log\log \nu(2r)}{\log 2r} = \overline{\lim_{r\to\infty}} \frac{\log\log \nu(r)}{\log r}. \tag{2.7}$$

Lemma 2 follows from this and (2.4).

LEMMA 3 ([7, Theorem 6]). *Let A_0, A_1 satisfy the hypothesis of Theorem 5. Then every entire solution $f \not\equiv 0$ of (1.12) satisfies $\sigma(f) = \infty$.*

LEMMA 4 ([6]). *Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exists a set $E \subset (1, +\infty)$ of finite logarithmic measure and a constant $B > 0$ that depends only α and (m, n) (m, n positive integers with $m < n$) such that for all z satisfying $|z| = r \notin [0, 1] \cup E$, we have*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| \leq B\left(\frac{T(\alpha r, f)}{r} (\log^{\alpha} r) \log T(\alpha r, f)\right)^{n-m}.$$

LEMMA 5 ([2]). *Let $f(z)$ be an entire function of order $\sigma(f) = \sigma < 1/2$ and denote $A(r) = \inf\{\log|f(z)|; |z| = r\}$, $B(r) = \sup\{\log|f(z)|; |z| = r\}$. If $\sigma < \alpha <$*

1, *then*

$$\underline{\log dens}\{r : A(r) > (\cos \pi\alpha)B(r)\} \geq 1 - \frac{\sigma}{\alpha},$$

where

$$\underline{\log dens}(E) = \varliminf_{r\to\infty} \left(\int_1^r (\chi_E(t)/t)\, dt \right) \Big/ \log r,$$

and

$$\overline{\log dens}(E) = \varlimsup_{r\to\infty} \left(\int_1^r (\chi_E(t)/t)\, dt \right) \Big/ \log r.$$

Lemma 6 ([3]). *Let $f(z)$ be an entire function with $\mu(f) = \mu < 1/2$ and $\mu < \sigma(f) = \sigma$. If $\mu \leq \delta < \min(\sigma, 1/2)$ and $\delta < \alpha < 1/2$, then*

$$\overline{\log dens}\{r : A(r) > (\cos \pi\alpha)B(r) > r^{\delta}\} > C(\sigma, \delta, \alpha),$$

where $C(\sigma, \delta, \alpha)$ is a positive constant depending only on σ, δ, and α.

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. Assume that $f(z)$ is a transcendental entire solution of (1.8). First of all, from Wiman-Valiron theory (see [9, or 14]), we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)), \quad (j = 1, 2), \tag{3.1}$$

where $|z| = r$, $|f(z)| = M(r, f)$, $r \notin E_1$ which has a finite logarithmic measure. Substituting (3.1) into (1.8), we obtain

$$\left(\frac{\nu_f(r)}{z}\right)^2 (1 + o(1)) + d_1 z^{n_1} \frac{\nu_f(r)}{z}(1 + o(1)) + d_0 z^{n_0}(1 + o(1)) = 0, \tag{3.2}$$

where $a_j = d_j z^{n_j}(1 + o(1))$, d_j are constants $(j = 1, 2)$. Since any solution of an algebraic equation is continuous function of the coefficients, therefore $\nu_f(r)$ is asymptotically equal to the solution of the equation

$$(\nu_f(r))^2 z^{-2} + d_1 z^{n_1 - 1} \nu_f(r) + d_0 z^{n_0} = 0. \tag{3.3}$$

From the argument used in [14, pp. 106–108], for sufficiently large r, we have

$$\nu_f(r) \sim \alpha r^{\sigma}, \quad r \notin E_1, \tag{3.4}$$

where $\alpha(>0)$ is constant and σ is rational number. By (3.4), it is easy to see that the degrees (in z) of three terms of (3.3) are respectively,

$$2(\sigma - 1), \; n_1 + (\sigma - 1), \; n_0. \tag{3.5}$$

Then by Wiman-Valiron theory, (3.4) and (3.5), we can easily conclude (i). In (ii), if $n_0 = n_1 - 1$, it is easy to see that (1.8) possibly has polynomial solutions.

Now we discuss polynomial solutions of equation (1.8), if $f_1(z)$ and $f_2(z)$ are linearly independent polynomial solutions, then by Able's identity, the Wronskian of f_1, f_2 satisfies

$$\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \exp\left\{-\int_0^z a_1(s)\,ds\right\}.$$

This is a contradiction. Therefore, any two polynomial solutions are linear dependent, hence all polynomial solutions $f(z)$ have the form $f_c(z) = cp(z)$, where p is a polynomial and c is an arbitrary constant.

Proof of Theorem 2. Assume that $f(z)$ is a transcendental entire solution of (1.9). We adopt the argument as used in the proof of Theorem 1, and notice that when z satisfies $|f(z)| = M(r,f)$ and $|z| \to \infty$, $|b(z)/f(z)| \to 0$, we can prove that if $n_0 \geq 2n_1$, then $\sigma(f) = (n_0+2)/2$; if $n_0 < n_1 - 1$, then $\sigma(f) = n_1 + 1$; if $n_1 - 1 \leq n_0 < 2n_1$, then $\sigma(f) = n_1 + 1$ or $\sigma(f) = n_0 - n_1 + 1$. We know that when $n_0 \geq 2n_1$, or $n_0 < n_1 - 1$, or $n_1 - 1 < n_0 < 2n_1$, every solution $f \not\equiv 0$ of the corresponding homogeneous equation of (1.9) is transcendental, so that the equation (1.9) has at most one exceptional polynomial solution, in fact if $f_1, f_2 (\not\equiv f_1)$ are polynomial solutions of (1.9), then $f_1 - f_2 \not\equiv 0$ is a polynomial solution of the corresponding homogeneous equation of (1.9), this is a contradiction. When $n_0 = n_1 - 1$, if the corresponding homogeneous equation of (1.9) has no polynomial solution, then (1.9) has clearly at most one exceptional polynomial solution, if the corresponding homogeneous equation of (1.9) has a polynomial solution $p(z)$, then (1.9) may have a family of polynomial solutions $\{cp(z) + f_0\}$ (f_0 is polynomial solution of (1.9), c is a constant).

Now we prove $\overline{\lambda}(f) = \sigma(f)$ for a transcendental solution f of (1.9). Since $b(z)$ is a polynomial which has only finitely many zeros, it follows that if z_0 is a zero of $f(z)$ and $|z_0|$ is sufficiently large, then the order of zero at z_0 is less than or equal to 2 from (1.9). Hence

$$N\left(r,\frac{1}{f}\right) \leq 2\overline{N}\left(r,\frac{1}{f}\right) + O(1). \tag{3.6}$$

By (1.9), we have

$$\frac{1}{f} = \frac{1}{b}\left(\frac{f''}{f} + a_1\frac{f'}{f} + a_0\right). \tag{3.7}$$

Hence

$$m\left(r,\frac{1}{f}\right) \leq m\left(r,\frac{f'}{f}\right) + m\left(r,\frac{f''}{f}\right) + O(\log r). \tag{3.8}$$

By $\sigma(f) < \infty$ and $m(r, f^{(j)}/f) = O(\log r)$ $(j = 1, 2)$, we get, from (3.6) and (3.8),

$$T(r, f) = T\left(r, \frac{1}{f}\right) + O(1) \leq 2\overline{N}\left(r, \frac{1}{f}\right) + d(\log r), \tag{3.9}$$

where $d(>0)$ is a constant. By (3.9), we have $\sigma(f) \leq \bar{\lambda}(f)$, hence $\bar{\lambda}(f) = \sigma(f)$.

4. Proofs of Theorems 3 and 4

Proof of Theorem 3. Assume that $f(z) \not\equiv 0$ is an entire solution of the equation (1.12). Then by the elementary theory of differential equations, it is easy to see from (1.10) and (1.11) that $\sigma(f) = \infty$.

Now we prove that $\sigma_2(f) = \sigma(A_0) = \sigma$. By Theorem B, we have $\sigma_2(f) \geq \sigma - \varepsilon$, and since ε is arbitrary, we get $\sigma_2(f) \geq \sigma(A_0) = \sigma$.

On the other hand, from Wiman-Valiron theory, there is a set $E_2 \subset (1, +\infty)$ with logarithmic measure $lm\, E_2 < \infty$, we can choose z satisfying $|z| = r \notin [0, 1] \cup E_2$ and $|f(z)| = M(r, f)$, such that (3.1) holds. For any given $\varepsilon > 0$, when r is sufficiently large, we have

$$|A_j(z)| \leq \exp\{r^{\sigma+\varepsilon}\}, \quad (j = 1, 2). \tag{4.1}$$

Substituting (3.1) and (4.1) into (1.12), we obtain

$$\left(\frac{\nu_f(r)}{|z|}\right)^2 |1 + o(1)| \leq \exp\{r^{\sigma+\varepsilon}\} \frac{\nu_f(r)}{|z|} |1 + o(1)| + \exp\{r^{\sigma+\varepsilon}\}, \tag{4.2}$$

where z satisfies $|z| = r \notin [0, 1] \cup E_2$ and $|f(z)| = M(r, f)$. By (4.2), we get

$$\overline{\lim_{r \to \infty}} \frac{\log \log \nu_f(r)}{\log r} \leq \sigma + \varepsilon. \tag{4.3}$$

Since ε is arbitrary, by (4.3) and Lemma 2 we have $\sigma_2(f) \leq \sigma$. This and the fact that $\sigma_2(f) \geq \sigma$ yield $\sigma_2(f) = \sigma$.

Proof of Theorem 4. We assume that f is a solution of (1.13) and f_1, f_2 are two entire solutions of the corresponding homogeneous equation (1.12). Then by Theorem 3, we have $\sigma_2(f_j) = \sigma(A_0)$ $(j = 1, 2)$. Since f can be expressed in the form

$$f(z) = B_1(z) f_1(z) + B_2(z) f_2(z), \tag{4.4}$$

where $B_1(z), B_2(z)$ are some entire functions satisfying

$$B_1' = -f_2 F/(f_1 f_2' - f_2 f_1'), \quad B_2' = -f_1 F/(f_1 f_2' - f_2 f_1').$$

By (4.4) and $\sigma(F) < +\infty$, it is easy to see that $\sigma_2(f) \leq \sigma(A_0)$.

By using the similar arguments as used in the proof of Theorem 2, we can conclude that all solutions f of (1.13) satisfy $\sigma_2(f) = \sigma$, with at most one exceptional f_0 such that $\sigma_2(f) < \sigma$.

By (1.13), it is easy to see that if f has a zero at z_0 with order greater than 2, then it must be a zero of F. Hence

$$N\left(r,\frac{1}{f}\right) \le 3\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{F}\right). \tag{4.5}$$

Now (1.13) can rewritten as

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f''}{f} + A_1\frac{f'}{f} + A_0\right). \tag{4.6}$$

By (4.6), we have

$$m\left(r,\frac{1}{f}\right) \le m\left(r,\frac{f'}{f}\right) + m\left(r,\frac{f''}{f}\right) + m(r,A_0) + m(r,A_1) + m\left(r,\frac{1}{F}\right). \tag{4.7}$$

By (4.5) and (4.7), we get for $|z| = r$ outside a set E_3 of finite linear measure,

$$\begin{aligned} T(r,f) &= T\left(r,\frac{1}{f}\right) + O(1) \\ &\le 3\overline{N}\left(r,\frac{1}{f}\right) + T(r,A_1) + T(r,A_0) + T(r,F) + d(\log\{rT(r,f)\}), \end{aligned} \tag{4.8}$$

where $d(>0)$ is a constant. For sufficiently large r, we have

$$d\log T(r,f) \le \frac{1}{2}T(r,f), \tag{4.9}$$

$$T(r,A_1) + T(r,A_0) \le 2r^{\sigma+\varepsilon}, \tag{4.10}$$

$$T(r,F) \le r^{\sigma(F)+\varepsilon}. \tag{4.11}$$

Thus, by (4.8)–(4.11), we have

$$T(r,f) \le 6\overline{N}\left(r,\frac{1}{f}\right) + 4r^{\sigma+\varepsilon} + 2r^{\sigma(F)+\varepsilon}, \quad (|z| = r \notin E_3). \tag{4.12}$$

Hence for any f with $\sigma_2(f) = \sigma$, by (4.12), we have $\sigma_2(f) \le \bar{\lambda}_2(f)$. Therefore, $\bar{\lambda}_2(f) = \sigma_2(f) = \sigma$.

5. Proofs of Theorems 5 and 6

Proof of Theorem 5. Assume that $f(z)$ is a nonzero entire solution of (1.12). By $A_0 \not\equiv 0$, it is easy to see that (1.12) does not have a polynomial solution. Hence every nonzero solution of (1.12) is transcendental. By Lemma 3, we have $\sigma(f) = \infty$.

If A_1 is transcendental, $\sigma(A_1) = 0$, A_0 is a polynomial, then we have $\sigma_2(f) \ge \sigma(A_1)$ obviously; if $\sigma(A_0) < \sigma(A_1) < 1/2$, then by Theorem A, we have $\sigma_2(f) \ge \sigma(A_1)$. Similarly, as in the proof of Theorem 3, we have $\sigma_2(f) \le \sigma(A_1)$. Therefore, $\sigma_2(f) = \sigma(A_1)$.

Proof of Theorem 6. Assume that $f(z)$ is an entire solution of (1.13). For case (i), we assume $\sigma(A_1) > 0$ (when $\sigma(A_1) = 0$, Theorem 6 holds clearly), by (1.13) we get

$$A_1 = \frac{F}{f'} - A_0\frac{f}{f'} - \frac{f''}{f'} = \frac{F}{f}\frac{f}{f'} - A_0\frac{f}{f'} - \frac{f''}{f'}. \tag{5.1}$$

By Lemma 4, we see that there exists a set $E_4 \subset (1, +\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_4$, we have

$$\left|\frac{f''(z)}{f'(z)}\right| \le dr^c[T(2r,f)]^2, \tag{5.2}$$

where $d(>0)$ and $c(>0)$ are some constants. Now set $b = \max\{\sigma(A_0), \sigma(F)\}$, and choose α, β such that

$$b < \alpha < \beta < \sigma(A_1). \tag{5.3}$$

Then for sufficiently large r, we have

$$|A_0(z)| \le \exp\{r^\alpha\}, |F(z)| \le \exp\{r^\alpha\}. \tag{5.4}$$

By Lemma 5 (if $\mu(A_1) = \sigma(A_1)$) or Lemma 6 (if $\mu(A_1) < \sigma(A_1)$) there exists a subset $H \subset (1, +\infty)$ with logarithmic measure $lmH = \infty$ suth that for all z satisfying $|z| = r \in H$, we have

$$|A_1(z)| > \exp\{r^\beta\}. \tag{5.5}$$

Since $M(r,f) > 1$ for sufficiently large r, we have by (5.4)

$$\frac{|F(z)|}{M(r,f)} \le \exp\{r^\alpha\}. \tag{5.6}$$

On the other hand, by Wiman-Valiron theory, there is a set $E_5 \subset (1, +\infty)$ of finite logarithmic measure such that (3.1) holds for some point z satisfying $|z| = r \notin [0,1] \cup E_5$ and $|f(z)| = M(r,f)$. By (3.1), we get

$$\left|\frac{f'(z)}{f(z)}\right| \ge \frac{1}{2}\left|\frac{\nu_f(r)}{z}\right| > \frac{1}{2r}$$

or

$$\left|\frac{f(z)}{f'(z)}\right| \le 2r. \tag{5.7}$$

Now by (5.1)–(5.7), we get

$$\exp\{r^\beta\} \le Mr^c[T(2r,f)]^2 2\exp\{r^\alpha\}2r \tag{5.8}$$

for $|z| = r \in H \backslash ([0,1] \cup E_4 \cup E_5)$ and $|f(z)| = M(r,f)$. From this and since β is arbitrary, we get $\sigma_2(f) \geq \sigma(A_1)$. Using the similar argument as used in the proof of Theorem 3, we have $\sigma_2(f) \leq \sigma(A_1)$. Thus, $\sigma_2(f) = \sigma(A_1)$.

Similarly by argument as used in the proof of Theorem 4, we can get $\overline{\lambda}_2(f) = \sigma_2(f) = \sigma(A_1)$.

Finally case (ii) can also be obtained by using argument similar to that in the proof of Theorem 4.

Concluding remark: For some general and related results of n-th order linear differential equations with coefficients of infinite order, we refer the reader to [13].

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