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(2, 3) TORUS SEXTICS AND THE ALBANESE IMAGES OF 6-FOLD CYCLIC MULTIPLE PLANES

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Introduction

Let B be an irreducible plane curve of degree n given in affine part, B_a, by the equation $f(x,y)=0$. Consider a k-cyclic extension, K, of $\boldsymbol{C}(\boldsymbol{P}^2)=\boldsymbol{C}(x,y)$, of $\boldsymbol{P}^2$ given by

$$\zeta^k = f(x,y).$$

Let S_k' be the K-normalization of $\boldsymbol{P}^2$; and we denote its smooth model by S_k. S_k is a k-fold cyclic covering of $\boldsymbol{P}^2$ branched along B and possibly along the line L in infinity. S_k is called a cyclic multiple plane by Italian algebraic geometers. There are many results on it ([BdF], [Co], [CC], [DF1], [DF2], [Ku], [L], [Sa], [Z1] and [Z2]). One of the purposes to study cyclic multiple planes is to understand the topology of $\boldsymbol{P}^2\backslash B$; and the irregularity, $q(S_k)$, of S_k (or the first Betti number of S_k) plays a central role for this purpose.

In [Z1] and [Z2], Zariski studied cyclic multiple planes and proved the following:

ZARISKI'S THEOREM. *Assume that singularities of B are only nodes and cusps and B is transversal to L. Then the irregularity of S_k vanishes unless both n and k are divisible by* 6.

In view of Zariski's theorem, 6-fold cyclic multiple planes branched along irreducible sextics are the first possible one with non-vanishing irregularities. This makes study of such cyclic multiple planes worthwhile.

In [Ku], Kulikov studied cyclic multiple planes by using a quasi-torus decomposition of a curve whose definition is as follows:

DEFINITION 0.1. B_a is called a (p,q) quasi-torus curve ($\gcd(p,q)=1$, $p,q>1$) if there exist a positive integer α and polynomials g, h and r with

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$\deg g > 0$, $\deg h > 0$ and $\deg r > 0$ which are pairwise coprime and coprime with $f(x, y)$ such that

$$r^{pq} f^{\alpha} = g^p + h^q.$$

A quasi-torus curve is called a *torus* curve if r in Definition 0.1 is a constant. We simply call this decomposition a (p, q) torus decomposition.

Let $A(S_k)$ be the Albanese variety of S_k and let α_k be the Albanese mapping from S_k to $A(S_k)$.

DEFINITION 0.2. The number

$$a(B_a) = \max_{k \in \boldsymbol{N}} \dim \alpha_k(S_k)$$

is called the Albanese dimension of B_a.

In [Ku], Kulikov studied B_a with $a(B_a) > 0$, and proved the following:

KULIKOV'S THEOREM (Theorem 1, [Ku]). *Suppose* $a(B_a) > 0$. *Then*:
(i) $\dim \alpha_k(S_k) > 0$ *for some* k *and* α_k *gives a quasi-torus decomposition of* f.
(ii) *If* $a(B_a) = 1$, *then* f *possesses a unique quasitorus decomposition.*
(iii) *If* f *possesses different quasi-torus decompositions*:

$$r^{p_1 q_1} f = g_1^{p_1} + h_1^{q_1}, \quad r^{p_2 q_2} f = g_2^{p_1} + h_2^{q_2}$$

such that two pencils determined by

$$\lambda_0 g_1^{p_1} + \lambda_1 h_1^{q_1} = 0, \quad [\lambda_0 : \lambda_1] \in \boldsymbol{P}^1$$

and

$$\lambda_0 g_2^{p_2} + \lambda_1 h_2^{q_2} = 0, \quad [\lambda_0 : \lambda_1] \in \boldsymbol{P}^1$$

are different, then $a(B_a) = 2$.

Kulikov's theorem shows the importance of quasi-torus curves in the study of cyclic multiple planes. In this paper, we study $(2, 3)$ torus sextics and 6-fold cyclic multiple planes from Kulikov's viewpoint. Here a plane sextic B is called a $(2, 3)$ torus curve if its affine part is a $(2, 3)$ torus curve.

Note that the line L in infinity is not contained the branch locus of 6-fold cyclic multiple planes branched along sextics. This means $\dim \alpha_6(S_6)$ is independent of the choice of homogeneous coordinates; and $\dim \alpha_6(S_6)$ is defined for B.

If B is a $(2, 3)$ torus curve given by the affine equation $g^3 + f^2 = 0$, $\deg g = 2$, $\deg f = 3$, then the conic, C, defined by $g = 0$ meets B only at $\mathrm{Sing}(B)$ in a certain special way. We consider a "converse" of this. Along this line, our question may be formulated as follows:

Question 0.3. Let B be an irreducible sextic. Suppose that there exists a conic, C, meeting B only at $\mathrm{Sing}(B)$. In terms of data on how C meets B, find a sufficient condition for B to be a $(2,3)$ torus curve.

One of the results in this article is to give a partial answer to Question 0.3 when B has at most simple singularities:

Theorem 0.4. *Let B be an irreducible sextic with at most simple singularities. Suppose that there exists a conic, C, such that*

(0.4.1) *C meets B only at singularities, and*

(0.4.2) *the type of a singular point in $B \cap C$ is either a_{3k-1} or e_6; and the intersection multiplicity of B and C at an a_{3k-1} (resp. e_6) singularity is $2k$ (resp. 4).*

Then B is a $(2,3)$ torus curve.

In [D], Degtyarev proved Theorem 0.4 for *abundant* sextics. His proof heavily made use of the fact that the degree of the curve is 6; and it seems to be difficult to generalize the statement, for example, to a criterion for a given curve to be a $(2,p)$ (p: odd prime) torus curve. On the other hand, our method is to make use of a certain normal form of a genus 2 curve, $\mathscr{C}$, defined over $\boldsymbol{C}(t)$, the rational function field of one variable, having a 3-torsion in $\mathrm{Pic}^0_{\boldsymbol{C}(t)}(\mathscr{C})$. From this point of view, by considering a normal form of a curve with higher genus, one might be able to generalize the result in Theorem 0.4 to the one for a curve of degree $2p$ to be a $(2,p)$ torus curve.

Now we go on to explain our idea to prove Theorem 0.4. It is based on the following well-known fact on an elliptic curve:

Let $\mathscr{E}$ be an elliptic curve defined over K, $\mathrm{char}(K) \neq 2,3$, given by the equation

$$\mathscr{E} : y^2 = x^3 + ax + b.$$

Suppose that the Mordell-Weil group, $\mathrm{MW}(\mathscr{E})$, of $\mathscr{E}$ over K has a non-trivial 3-torsion element (x_0, y_0). Then the right hand side of the above equation can be rewritten in such a way as

$$x^3 + ax + b = (x - x_0)^3 + (ux + v)^2,$$

where the line $y = ux + v$ is the tangent to $\mathscr{E}$ at (x_0, y_0).

In the case when $K = \boldsymbol{C}(t)$, $a, b \in \boldsymbol{C}[t]$, this decomposition gives rise to a $(2,3)$ torus decomposition of the polynomial $x^3 + ax + b$. We want to make use of this type of argument in finding a $(2,3)$ decomposition of B; and this is the case in [T5]. However B is not always given by such an affine equation as $x^3 + ax + b, a, b \in \boldsymbol{C}[t]$. Hence one can not apply the above fact on $\mathscr{E}$ to general sextics. Instead, we make use of a similar fact for a curve of genus 2 (see Lemma 3.1).

Now we give our strategy to prove Theorem 0.4. Let $f' : S' \to \boldsymbol{P}^2$ be a double covering branched along B, and we denote its canonical resolution by

$\mu : S \to S'$. Choose $x \in \boldsymbol{P}^2 \backslash B$. Then a pencil of lines through x gives rise to a pencil of genus 2 curves with two base point, $x^+, x^- \in (\mu \circ f')^{-1}(x)$ on S. Let $\hat{S} \to S$ be blowing-ups at x^+ and x^-. Then the pencil induces a fibration of genus 2 curves, φ_x, on $\hat{S}$. Let $\hat{S}_\eta$ be the generic fiber of φ_x. Then $\hat{S}_\eta$ is a genus 2 curve over $K = \boldsymbol{C}(\boldsymbol{P}^1)$. Let $\mathrm{Pic}^0_K(\hat{S}_\eta)$ be the degree 0 part of the divisor class group of $\mathscr{C}$ defined over K. We first show that the conic C in Theorem 0.4 gives rise to a 3-torsion in $\mathrm{Pic}^0_K(\hat{S}_\eta)$ (§1 and §2). Hence, by applying Lemma 3.1 to $\hat{S}_\eta$, we eventually obtain a $(2,3)$ torus decomposition of B (§4).

As we have seen in [T5], there are some irreducible plane sextics with the Albanese dimension 2. All of them are, however, either with non-simple singularities or curves with non-zero genus. We use Theorem 0.4 in finding a $(2,3)$ torus sextic such that

(i) all the singularities of B is at most simple,
(ii) the normalization of B is a rational curve, and
(iii) $a(B) = 2$.

Now we state our result.

THEOREM 0.5. *Let B be an irreducible sextic possessing singularities either $4a_2 + a_5 + e_6$ or $6a_2 + e_6$. Then* (i) *there exist irreducible sextics for both cases, and* (ii) $\dim \alpha_6(S_6) = 2$. *In particular, the former satisfies the three conditions as above.*

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Notation and conventions

Throughout this article, the ground field is always the complex number field $\boldsymbol{C}$. Also a *surface* and a *curve* always mean projective ones. For a variety X, We denote the field of rational function of X by $\boldsymbol{C}(X)$.

Let X be a normal variety, and let Y be a smooth variety. Let $\pi : X \to Y$ be a finite morphism from X to Y. We define the branch locus, $\Delta(X/Y)$, of π as follows:

$$\Delta(X/Y) = \{y \in Y \,|\, \sharp(\pi^{-1}(y)) < \deg \pi\}.$$

Let S be a finite double covering of a smooth projective surface Σ. The "*canonical resolution*" of S always means the resolution given by Horikawa in [H].

For singular fibers of an elliptic surface, we use the notation of Kodaira [Ko].

Let D_1, D_2 be divisors.

$D_1 \sim D_2$: linear equivalence of divisors.
$D_1 \approx D_2$: algebraic equivalence of divisors.
$D_1 \approx_{\boldsymbol{Q}} D_2$: $\boldsymbol{Q}$-algebraic equivalence of divisors.

A $(-n)$ curve means a rational curve with self-intersection number $-n$. For simple singularities of a plane curve, we use the same notation as in [P1], while we use the standard one for rational double points.

§1. Preliminaries

Let W be a smooth surface. Let B be a reduced divisor on W such that $B \sim 2L$ for some line bundle on W. Then it is well-known that there exists a normal surface, S', with degree 2 finite morphism $f' : S' \to W$ having the branch locus $\Delta(S'/W) = B$ (*cf.* [H]). Let $\mu : S \to S'$ be the canonical resolution of S' given in [H]. Then we have a commutative diagram:

$$\begin{array}{ccc} S' & \xleftarrow{\mu} & S \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \boldsymbol{P}^2 & \xleftarrow{q} & \Sigma, \end{array}$$

where q is a composition of a finite number of blowing-ups so that the induced morphism f is finite of degree 2. We denote the covering transformation of f by σ.

Let NS(S) and NS(W) be the Néron-Severi group of S and W, respectively. Let T_μ be the subgroup of NS(S) generated by $\pi^*\mathrm{NS}(W)$, where $\pi = f' \circ \mu = q \circ f$, and all irreducible components of the exceptional divisor of μ. T_μ has a decomposition as follows:

LEMMA 1.1. *Let R_v be the subgroup of T_μ generated by irreducible components of the exceptional divisor of $v \in \mathrm{Sing}(S')$. Then*

$$T_\mu = \pi^*\mathrm{NS}(W) \oplus \bigoplus_{v \in \mathrm{Sing}(S')} R_v.$$

This lemma is immediate by the definition of T_μ.

From now on, we always assume that

$(*)$ *$H^2(S, \boldsymbol{Z})$ is torsion free.*

Under the assumption $(*)$, $H^2(S, \boldsymbol{Z})$ becomes a unimodular lattice with respect to the intersection pairing; and NS(S) is a primitive sublattice of it, i.e., $H^2(S, \boldsymbol{Z})/\mathrm{NS}(S)$ is torsion free. T_μ is also a sublattice of $H^2(S, \boldsymbol{Z})$, and the decomposition in Lemma 1.1 is orthogonal with respect to the intersection pairing. T_μ is, however, not primitive in general. Let $T_\mu^\sharp$ be the primitive hull of T_μ. Note that $(\mathrm{NS}(S)/T_\mu)_{\mathrm{tor}} = T_\mu^\sharp/T_\mu$. We next consider when a given divisor D is a member of $T_\mu^\sharp$. Let us start with the following lemma.

LEMMA 1.2. *Let D be a divisor on S and let α be its image in* NS$(S)/T_\mu$. *Then there exists an element, D_α, in* NS$(S) \otimes \boldsymbol{Q}$ *satisfying the conditions as follows*:

(i) $D_\alpha \equiv D \bmod T_\mu \otimes \boldsymbol{Q}$
(ii) $D_\alpha \perp T_\mu$ *with respect to the intersection pairing.*

This is a straight forward modification of Lemma 8.1 in [S2], so, we omit its proof.

We now give a numerical criterion for D to be a member of $T^\sharp$.

LEMMA 1.3. *If* $D_\alpha^2 = 0$, *then* $D \in T_\mu^\sharp$.

Proof. By the Hodge index theorem, $D_\alpha^2 \le 0$; and if the equality holds, then $D_\alpha \approx_{\boldsymbol{Q}} 0$. This implies Lemma 1.3.

We give an explicit formula for D_α when $W = \boldsymbol{P}^2$ for later use.

LEMMA 1.4. *Put* $L = \pi^* l$, *where* l *denotes a line in* $\boldsymbol{P}^2$. *Then we have*

$$D_\alpha = D - \frac{1}{2}(DL)L - \sum_{v \in \mathrm{Sing}(S')} (\Theta_{1,v}, \dots, \Theta_{m_v,v}) A_v^{-1} \begin{pmatrix} \Theta_{1,v} D \\ \vdots \\ \Theta_{m_v,v} D \end{pmatrix},$$

where $m_v = \mathrm{rank}_{\boldsymbol{Z}} R_v$, $A_v =$ *the intersection matrix of the lattice determined by* R_v, *and* $\Theta_{i,v}$ $(i = 1, \dots, m_v)$ *are irreducible components of the exceptional divisor for* v. *In particular, if* $D \in T_\mu \otimes \boldsymbol{Q}$, *we have*

$$D \approx_{\boldsymbol{Q}} \frac{1}{2}(DL)L + \sum_{v \in \mathrm{Sing}(S')} (\Theta_{1,v}, \dots, \Theta_{m_v,v}) A_v^{-1} \begin{pmatrix} \Theta_{1,v} D \\ \vdots \\ \Theta_{m_v,v} D \end{pmatrix}.$$

This is again straightforward by the definition of D_α, so, we omit its proof.

From now on, we restrict ourselves to the case when $W = \boldsymbol{P}^2$ and $\deg B = 2n$. Moreover, we always assume

ASSUMPTION 1.5. B has at most simple singularities.

Under Assumption 1.5, (i) S is the minimal resolution of S' by Lemma 5 in [H], and (ii) S is simply connected by [B1], [B2] and Proposition 1.8 in [Ca]. This implies that $\mathrm{NS}(S)$ is not only torsion free, but also equal to $\mathrm{Pic}(S)$. In particular, there is no difference between linear equivalence and algebraic equivalence.

Let x be an arbitrary point in $\boldsymbol{P}^2 \setminus B$. Let $\hat{\Sigma} \to \Sigma$ be a blowing-up at $q^{-1}(x)$, and let $v : \hat{S} \to S$ be a composition of blowing-ups at two points $\pi^{-1}(x)$. Then $\hat{S}$ satisfies the following:

(i) $\hat{S}$ is a double covering of $\hat{\Sigma}$. We denote its covering morphism and the covering transformation by $\hat{f}$ and $\hat{\sigma}^*$, respectively.

(ii) $\hat{S}$ has a fibration of hyperelliptic curves of genus $\deg B/2 - 1 = n - 1$, $\varphi_x : S \to \boldsymbol{P}^1$, arising from a pencil of lines through x and $\hat{\sigma}$ induces the hyperelliptic involution on a smooth fiber.

(iii) The exceptional divisors of ν give rise to two sections, s^+ and $s^-(= \hat{\sigma}^* s)$, of φ_x.

We define the sublattice, T_{φ_x}, of $\mathrm{Pic}(\hat{S})$ as follows:

$T_{\varphi_x} :=$ the subgroup of $\mathrm{Pic}(\hat{S})$ generated by s^+ and all irreducible components in fibers of $\varphi_x : \hat{S} \to \boldsymbol{P}^1$.

T_{φ_x} has a decomposition as follows:

$$T_{\varphi_x} = \boldsymbol{Z}s^+ \oplus \boldsymbol{Z}F \oplus \bigoplus_{w \in \mathrm{Red}(\varphi_x)} (\oplus_i \boldsymbol{Z}\Theta_{i,w})$$

where $\mathrm{Red}(\varphi_x) = \{w \in \boldsymbol{P}^1 \mid \varphi_x^{-1}(w) \text{ is reducible}\}$, and the $\Theta_{i,w}$'s are irreducible components of $\varphi_x^{-1}(w)$ not meeting s^+. This decomposition is orthogonal with respect to the intersection pairing.

Note that $\nu^* T_\mu$ is not contained in T_{φ_x} since $\nu^* L \sim s^+ + s^- + F$. T_{φ_x}, however, contains $\nu^*(\oplus_{v \in \mathrm{Sing}(S')} R_v)$. In fact, all irreducible components of the exceptional divisors are those of reducible fibers of φ_x not meeting s^+. Let $T_{\varphi_x}^\sharp$ be the primitive hull of T_{φ_x}. Then:

LEMMA 1.6. *Suppose that $T_\mu^\sharp / T_\mu$ has a p-torsion (p: odd prime), and let D be a divisor in $T_\mu^\sharp$ that gives a p-torsion in $T_\mu^\sharp / T_\mu$. Then:*

(i) The intersection number, (DL), is even,

(ii) $D - (DL)/2(s^+ + s^-) \notin T_{\varphi_x}$, and $p(D - (DL)/2(s^+ + s^-)) \in T_{\varphi_x}$.

(iii) $T_{\varphi_x}^\sharp / T_{\varphi_x}$ has a p-torsion.

Proof. Since $T_\mu^\sharp \otimes \boldsymbol{Q} \cong T_\mu \otimes \boldsymbol{Q}$, we have

$$D \sim_{\boldsymbol{Q}} \frac{1}{2}(DL)L + \sum_{v \in \mathrm{Sing}(S')} \left(\sum_i b_{i,v} \Theta_{i,v} \right) \quad a, b_{i,v} \in \boldsymbol{Q}.$$

As $D \notin T_\mu$ and $pD \in T_\mu$, $p(DL)/2$ and all the $pb_{i,v}$'s are in $\boldsymbol{Z}$, and at least one of $1/2(DL)$ and the $b_{i,v}$'s is not in $\boldsymbol{Z}$. As p is odd, (LD) is even. This shows (i). Since $\nu^* L \sim s^+ + s^- + F$, we have

$$\nu^* D - \frac{1}{2}(DL)s^- \sim_{\boldsymbol{Q}} \frac{1}{2}(DL)s^+ + \frac{1}{2}(DL)F + \sum_{v \in \mathrm{Sing}(S')} \left(\sum_i b_{i,v} \nu^* \Theta_{i,v} \right).$$

As DL is even, the left hand side is in $T_{\varphi_x}^\sharp$. Since s^+, F and $\nu^*\Theta_{i,v}$'s are part of basis of T_{φ_x}, the presentation in the right hand side is unique. Hence $v^*D - (DL)/2s^- \notin T_{\varphi_x}$ and $p(v^*D - (DL)/2s^-) \in T_{\varphi_x}$. This implies (ii) and (iii).

Let $\hat{S}_\eta$ be the generic fiber of $\varphi_x : \hat{S} \to \boldsymbol{P}^1$. Then $\hat{S}_\eta$ is a curve of genus $n-1$ over $K = \boldsymbol{C}(\boldsymbol{P}^1)$. Let D be the divisor in Lemma 1.6, and put $D_1 = v^*D|_{\hat{S}_\eta}$, $\infty^+ = s^+|_{\hat{S}_\eta}$, and $\infty^- = s^-|_{\hat{S}_\eta}$. Then we have

PROPOSITION 1.7. *The divisor* $D_1 - (DL)/2(\infty^+ + \infty^-)$ *on* $\hat{S}_\eta$ *is an element in* $\mathrm{Pic}^0_K(\hat{S}_\eta)$ *such that*
(i) $D_1 - (DL)/2(\infty^+ + \infty^-) \not\sim 0$, *and* (ii) $p(D_1 - (DL)/2(\infty^+ + \infty^-)) \sim 0$.

Proof. Since D is a divisor on $\hat{S}$, D_1 is a divisor on $\hat{S}_\eta$ defined over K. Hence $D_1 - (DL)/2(\infty^+ + \infty^-)$ gives an element in $\mathrm{Pic}^0_K(\hat{S}_\eta)$. Suppose that $D_1 - (DL)/2(\infty^+ + \infty^-) \sim 0$. Then there exists g in $\boldsymbol{C}(\hat{S}_\eta)$ such that $(g) = D_1 - (DL)/2(\infty^+ + \infty^-)$ on $\hat{S}_\eta$. If we consider g as an element in $\boldsymbol{C}(\hat{S})$, this equality gives $D - (DL)/2(s^+ + s^-) - (g) = G$, where G is a divisor whose irreducible components are contained in fibers of φ_x. Hence $D - (DL)/2(s^+ + s^-) \sim G \in T_{\varphi_x}$. This contradicts Lemma 1.6 (ii). The second assertion easily follows from our proof of Lemma 1.6.

§2. A 3-torsion of $T^\sharp_\mu/T_\mu$ for a double sextic

We keep the notation as before. In this section, we consider 3-torsions in $T^\sharp_\mu/T_\mu$ in the case when B is a sextic satisfying Assumption 1.5.

Let C a conic satisfying the conditions (0.4.1) and (0.4.2). The purpose of this section is to show that C gives rise to a 3-torsion in $T^\sharp_\mu/T_\mu$. Let $q^{-1}C$ be the proper transform of C. Then it satisfies:
(i) $(q^{-1}C)^2 = -2$,
(ii) $q^{-1}C$ does not meet the branch locus of f, $\Delta(S/\Sigma)$.

Hence $f^*(q^{-1}C)$ has a decomposition in the form of $C' + \sigma^*C'$ for some divisor C' on S with $C'^2 = -2$. For this C', we have the following:

LEMMA 2.1. $C' \notin T_\mu$ *and* $3C' \in T_\mu$.

Proof. Let $\alpha(C')$ be the image in $\mathrm{NS}(S)/T_\mu$. Consider $D_{\alpha(C')}$ obtained in Lemma 1.4. It is in the form of

$$D_{\alpha(C')} = C' - L - \textit{the correction terms}.$$

The correction terms arise from the singularities lying over $C \cap B$. To describe them explicitly, we label irreducible components of the exceptional divisors as follows:

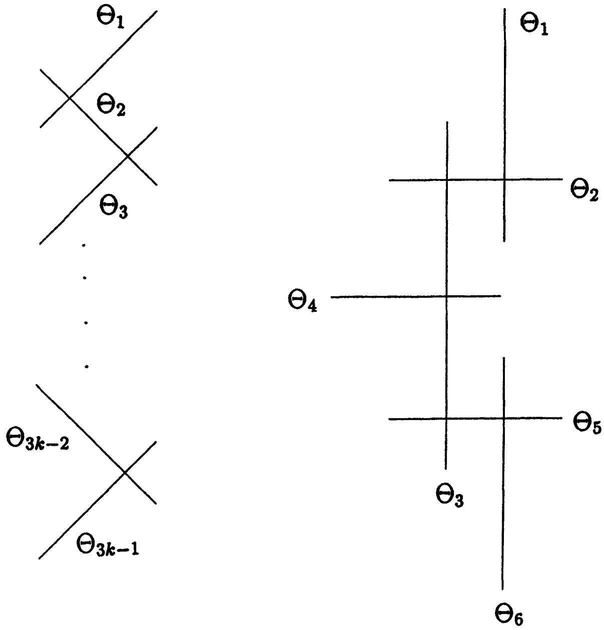

Figure 1

Also, by the conditions (0.4.1) and (0.4.2), we may assume that C' hits Θ_1 at the exceptional divisor of the E_6 singularity lying over an e_6 singularity, and Θ_k at the exceptional divisor of the A_{3k-1} singularity lying over an a_{3k-1} singularity. Then the correction terms are

$$\frac{4}{3}\Theta_1 + \frac{5}{3}\Theta_2 + 2\Theta_3 + \Theta_4 + \frac{4}{3}\Theta_5 + \frac{2}{3}\Theta_6$$

for an E_6 singularity, and

$$\frac{2}{3}\sum_{i=1}^{k} i\Theta_i + \sum_{i=1}^{2k-1}\frac{2k-i}{3}\Theta_{k+i}$$

for an A_{3k-1} singularity.

Using these explicit formulas, we have

Claim. $D^2_{\alpha(C')} = 0$.

Proof of Claim. Suppose that B and C meet at $x_1, \ldots, x_{\lambda_1}, x_{\lambda_1+1}, \ldots, x_{\lambda_1+\lambda_2}$, where

$$x_i : \textit{an } a_{3k_i-1} \textit{ singularity for } 1 \leq i \leq \lambda_1,$$

and

$$x_i : \textit{an } e_6 \textit{ singularity for } \lambda_1 \le i \le \lambda_2.$$

Then, as $BC = 12$, $\sum_{i=1}^{\lambda_1} k_i + 2\lambda_2 = 6$. Hence we have

$$\begin{aligned} D^2_{\alpha(C')} &= -4 + \frac{2}{3}\sum_{i=1}^{\lambda_1} k_i + \frac{4}{3}\lambda_2 \\ &= 0 \end{aligned}$$

Now, by Lemma 1.3, $C' \approx_Q L +$ *the correction terms*. This implies Lemma 2.1.

Summing up, we have

PROPOSITION 2.2. *Let B be a sextic with at most simple singularities. If there exists a conic, C, satisfying* (0.4.1) *and* (0.4.2), *then* $T^\sharp_\mu/T_\mu$ *has a* 3*-torsion.*

§3. A certain canonical form of a curve of genus 2

Let K be a field of characteristic zero, and let $\bar{K}$ be its algebraic closure. Let $\mathscr{C}$ be a curve of genus 2 defined by the affine equation:

$$\mathscr{C} : Y^2 = F(X)$$

where

$$F(X) = f_0X^6 + \cdots + f_6, \quad f_i \in K$$

is of degree 6 and has no multiple factor. Adding up two points at infinity, ∞^+ and ∞^-, we have a complete curve. Put $O = \infty^+ + \infty^-$. Then any effective divisor of degree 2 on $\mathscr{C}$ of form $(x_0, y_0) + (x_0, -y_0)$, $x_0 \in K$ is linearly equivalent to O. Although the following lemma may be well-known to experts, we give a proof for completeness.

LEMMA 3.1. *Let* $(x_1, y_1) + (x_2, y_2)$, *where* $x_1 \neq x_2$, $(x_i, y_i) \neq \infty^+, \infty^-$ $(i = 1, 2)$ *be a divisor on* $\mathscr{C}$ *defined over* K. *Suppose that the divisor* $D = (x_1, y_1) + (x_2, y_2) - O$ *gives rise to a* 3*-torsion of* $\mathrm{Pic}^0_K(\mathscr{C})$, *i.e.*, $D \not\sim 0$ *and* $3D \sim 0$. *Then there exist* $G, H \in K[X]$ *and* $a \in K^\times$ *such that*

(i) $\deg G = 2$, $\deg H = 3$,
(ii) $F(X) = H(X)^2 + aG(X)^3$, *and*
(iii) $G(x_1) = G(x_2) = 0$.

Proof. Since the divisor $(x_1, y_1) + (x_2, y_2)$ is defined over K, there exists a polynomial, $G \in K[X]$, such that $G(x_1) = G(x_2) = 0$. $G(X)$ gives rise to a rational function on $\mathscr{C}$; and $(G(X)) = \sum_{i=1}^2 (x_i, y_i) + (x_i, -y_i) - 2O$. As $3((x_1, y_1) + (x_2, y_2) - O) \sim 0$, we have a rational function $\varphi \in \bar{K}(\mathscr{C})$ on $\mathscr{C}$ such that

$$(\varphi) = 3((x_1, y_1) + (x_2, y_2) - O) \quad \text{i.e., } \varphi \in H^0(\mathscr{C}, \mathcal{O}(3O)).$$

Rational functions 1, X, X^2, X^3, Y form a $\mathrm{Gal}(\bar{K}/K)$-invariant basis of $H^0(\mathscr{C}, \mathcal{O}(3O))$; and (φ) is $\mathrm{Gal}(\bar{K}/K)$-invariant. Hence we may assume

$$\varphi = k_0 + k_1X + k_2X^2 + k_3X^3 + k_4Y, \quad (k_i \in K).$$

Moreover, as $\varphi^\sigma \neq \varphi$ (σ denotes the hyperelliptic involution, $(X, Y) \mapsto (X, -Y)$, $k_4 \neq 0$. Hence, replacing φ by $(1/k_4)\varphi$, we may assume

$$\varphi = Y + h_0X^3 + h_1X^2 + h_2X + h_3, \quad (h_i \in K).$$

Then we have

$$\varphi^\sigma = -Y + h_0X^3 + h_1X^2 + h_2X + h_3$$

and

$$(\varphi^\sigma) = 3((x_1, -y_1) + (x_2, -y_2) - O).$$

Thus we have

$$(\varphi\varphi^\sigma) = (G^3).$$

Hence there exists $a \in K^\times$ such that

$$-\varphi\varphi^\sigma = aG^3.$$

Thus we have

$$Y^2 = F(X) = (h_0X^3 + h_1X^2 + h_2X + h_3)^2 + aG^3$$

on $\mathscr{C}$. Therefore we have $F(X) = (h_0X^3 + h_1X^2 + h_2X + h_3)^2 + aG^3$ as a polynomial.

§4. Proof of Theorem 0.4

The goal of this section is to prove Theorem 0.4. We keep the notation as in §1 and §2. Our proof of Theorem 0.4 is divided into two parts:
Case (I) C is irreducible.
Case (II) C is reducible.

CASE (I). Choose an affine coordinate, (X, Y), of $\boldsymbol{P}^2$ as follows:
(i) B is given by the equation $f(X, Y) = 0$.
(ii) C is given by the equation $Y + X^2 = 0$.
(iii) The point x is the origin $(0, 0)$.
Note that $f(0, 0) \neq 0$ since $x \in \boldsymbol{P}^2 \setminus B$. Let $\mu_x : \hat{\boldsymbol{P}}^2 \to \boldsymbol{P}^2$ be a blowing-up at x. Choose an affine open set U_s of $\hat{\boldsymbol{P}}^2$ in such a way that

$$\mu_x : (s, X) \mapsto (X, Y) = (X, sX).$$

Then the total transforms, $\mu_x^* B$, and $\mu_x^* C$, of B and C are given by the equations:

$$\mu_x^* B : \hat{f}(s, X) = f(X, sX) = f_0(s)X^6 + \cdots + f_6(s) = 0$$

where $f_i(s) \in \boldsymbol{C}[s]$, $\deg f_i = 6 - i$, and

$$\mu_x^* C : X(X + s) = 0.$$

Then the generic fiber, $\hat{S}_\eta$, of φ_x is given by the affine equation

$$Z^2 = \hat{f}(s, X). \tag{4.1}$$

Also, by the construction of $\hat{S}_\eta$ the divisor given by $X(X + s) = 0$ on $\hat{S}_\eta$ is equal to $v^*C' + v^*\sigma^*C'|_{S_\eta}$, where C' is one in Lemma 2.1, and $v^*C'|_{\hat{S}_\eta}$ is an effective divisor of degree 2 on $\hat{S}_\eta$. Hence, by Proposition 1.7, Proposition 2.2, and Lemma 3.1, we have

$$\hat{f}(s, x) = (h_0(s)X^3 + h_1(s)X^2 + h_2(s)X + h_3(s))^2 + a(s)(X(X + s))^3 \tag{4.2}$$

where $h_i(s)$, $(i = 1, 2, 3)$, $a(s) \in \boldsymbol{C}(s)$. Hence, in order to prove Theorem 0.4 in Case (I), it is enough to prove that (i) $a(s)$ is a non-zero constant and (ii) $h_i \in \boldsymbol{C}[s]$, $\deg h_i \leq 3 - i$. Comparing the coefficients of X^i $(0 \leq i \leq 6)$ in (4.2), we have

$$f_0 = a + h_0^2 \tag{4.3.1}$$

$$f_1 = 2h_0h_1 + 3as \tag{4.3.2}$$

$$f_2 = h_1^2 + 2h_0h_2 + 3as^2 \tag{4.3.3}$$

$$f_3 = 2h_1h_2 + 2h_0h_3 + as^3 \tag{4.3.4}$$

$$f_4 = h_2^2 + 2h_1h_3 \tag{4.3.5}$$

$$f_5 = h_2h_3 \tag{4.3.6}$$

$$f_6 = h_3^2. \tag{4.3.7}$$

Since f_6 is a non-zero constant ($f(0, 0) \neq 0$), h_3 is a non-zero constant by (4.3.7). By (4.3.6), $h_2h_3 \in \boldsymbol{C}[s]$ and $\deg h_2h_3 = \deg f_5 \leq 1$. This implies $h_2 \in \boldsymbol{C}[s]$ and $\deg h_2 \leq 1$. Also, by (4.3.5), $h_2^2 + 2h_1h_3 \in \boldsymbol{C}[s]$ and $\deg(h_2^2 + 2h_1h_3) \leq 2$; this means $h_1 \in \boldsymbol{C}[s]$ and $\deg h_1 \leq 2$. Next we put $a = a'/a''$, $h_0 = h_0'/h_0''$, a', a'', h_0', $h_0'' \in \boldsymbol{C}[s]$. Then, by (4.3.1), we may assume $a'' = ch_0''^2$, $c \in \boldsymbol{C}^\times$. If $h_0'' = 0$ has a non-zero root, then $2h_0h_1 + 3as \notin \boldsymbol{C}[s]$. This contradicts (4.3.2). Hence we may assume that $h_0'' = c's^\alpha$ $(c' \in \boldsymbol{C}^\times, \alpha \geq 0)$. Suppose that $\alpha > 0$ or $\deg h_0 > 3$. Then, since $2h_0h_1 + 3as \in \boldsymbol{C}[s]$ by (4.3.2), we have $\alpha = 1$. In this case, $as^3 \in \boldsymbol{C}[s]$. Then we have $h_0h_3 \in \boldsymbol{C}[s]$ by (4.3.4). This is a contradiction as h_0'' is not a constant. Therefore, a, $h_0 \in \boldsymbol{C}[s]$. Now it is enough to show the following claim.

CLAIM. *a is a constant, and* $\deg h_0 \leq 3$.

Proof of Claim. If $\deg a > 0$ or $\deg h_0 > 3$, then we have $\deg h_0 = \deg a + 3$ as $\deg(2h_1h_2 + 2h_0h_3 + as^3) \le 3$ and $\deg h_1h_2 \le 3$. Hence $\deg(h_0^2 + a) = 2\deg a + 6 > 6$. But this contradicts (4.3.1) as $\deg f_0 \le 6$.

CASE (II). Choose an affine coordinate (X, Y) of $\boldsymbol{P}^2$ as follows:
(i) B is given by the equation $f(X, Y) = 0$.
(ii) C is given by the equation $X(X + Y + k) = 0$, k: a non-zero constant.
(iii) x is the origin and any line except $X = 0$ through x meets B more than 3 distinct points.

By the same argument as that in Case (I), we have

$$\hat{f}(s, X) = f_0X^6 + f_1X^5 + f_2X^4 + f_3X^3 + f_4X^2 + f_5X + f_6 \tag{4.4}$$
$$= (h_0X^3 + h_1X^2 + h_2X + h_3)^2 + a(X((1+s)X + k))^3$$

where h_i, $(i = 1, 2, 3)$, $a \in \boldsymbol{C}(s)$. Likewise in Case (I), it is enough to show that a is a constant and $h_i \in \boldsymbol{C}[s]$, $\deg h_i \le 3 - i$. Comparing the coefficients of X^ι in (4.4), we have

$$f_0 = a(1+s)^3 + h_0^2 \tag{4.5.1}$$

$$f_1 = 2h_0h_1 + 3a(s+1)^2 \tag{4.5.2}$$

$$f_2 = h_1^2 + 2h_0h_2 + 3a(1+s) \tag{4.5.3}$$

$$f_3 = 2h_1h_2 + 2h_0h_3 + a \tag{4.5.4}$$

$$f_4 = h_2^2 + 2h_1h_3 \tag{4.5.5}$$

$$f_5 = h_2h_3 \tag{4.5.6}$$

$$f_6 = h_3^2. \tag{4.5.7}$$

By (4.5.7), h_3 is a non-zero constant. Hence, by (4.5.6), $h_2 \in \boldsymbol{C}[s]$ and $\deg h_2 \le 1$. By (4.5.5), $h_1 \in \boldsymbol{C}[s]$ and $\deg h_1 \le 2$. Now put $a = a'/a''$ and $h_0 = h_0'/h_0''$. Then, by (4.5.4), we have $a'' = ch_0''$, $(c \in \boldsymbol{C}^\times)$. But, if $\deg h_0'' > 0$, then $h_0^2 + a(1+s)^3 \notin \boldsymbol{C}[s]$. This contradicts to (4.5.1). Thus $a, h_0 \in \boldsymbol{C}[s]$.

CLAIM. *Both* $\deg a$ *and* $\deg h_0$ *are* ≤ 3.

Proof of Claim. Suppose that $\deg a > 3$ or $\deg h_0 > 3$. Then, by (4.5.4), as $\deg f_3 \le 3$, $\deg h_0 = \deg a$. Hence $\deg(h_0^2 + a(1+s)^3) = 2\deg a > 6$. But this contradicts (4.5.1) as $\deg f_0 \le 6$.

Now Case (II) is immediate from the following claim.

CLAIM. *a is a constant.*

Proof of Claim. Suppose that $\deg a > 0$ and let α be a root of $a = 0$. Then the line $Y - \alpha X = 0$ meets B at less than 4 distinct points. This contradicts our choice of x (see (iii)).

REMARK 4.1. Just Lemma 3.1 is not enough to find a (2, 3) torus decomposition for a given sextic curve. In fact, we have the following example:

$$X^6 - 3X^5 + \frac{5 - 16s}{4}X^4 + (1 + 3s)X^2 + 2s^2X + s^2$$
$$= \left(\frac{1+s}{s}X^3 + \frac{3}{2}X^2 + sX + s\right)^2 - \frac{1+2s}{s^2}X^3(X+s)^3$$

§5. Some examples

In this section, we give some examples of (2, 3) torus sextics. To this purpose, we make use of theory of elliptic surface as we did in [T2], [T3] and [T4]. We first summarize results from theory of elliptic surfaces which we use later. We refer to [Ko], [M], [S1] and [S2] for details.

Let $\psi : \mathscr{E} \to \boldsymbol{P}^1$ be an elliptic surface. We always assume that $\mathscr{E}$ satisfies the following:

ASSUMPTION 5.1. (i) $\mathscr{E}$ has a section, s_0, and (ii) ψ has at least one singular fiber.

By Assumption 5.1 (i), the generic fiber, $\mathscr{E}_\eta$, of ψ becomes an elliptic curve over $\boldsymbol{C}(\boldsymbol{P}^1)$. Hence one can introduce a group structure on $\mathscr{E}_\eta$, $s_0|_\eta$ being the zero. The inverse morphism with respect to the group structure gives an involution on $\mathscr{E}$. We call it *the canonical involution.*

Let $\mathrm{NS}(\mathscr{E})$, $T_{\mathscr{E}}$ and $\mathrm{MW}(\mathscr{E})$ be the Néron-Severi group, the subgroup of $\mathrm{NS}(\mathscr{E})$ generated by s_0 and all irreducible components of fibers, and the Mordell-Weil group, the group of section, of $\mathscr{E}$, respectively. Then under Assumption 5.1 we have the following theorem:

THEOREM 5.2 (Shioda). $\mathrm{MW}(\mathscr{E}) \cong \mathrm{NS}(\mathscr{E})/T_{\mathscr{E}}$. *In particular,* $\mathrm{MW}(\mathscr{E})$ *is finitely generated.*

For a proof, see [S2].

The following fact is useful.

LEMMA 5.3. *Let s be a non-zero torsion section in* $\mathrm{MW}(\mathscr{E})$. *Then s and s_0 are disjoint.*

An important corollary to Lemma 5.3 is the following:

COROLLARY 5.4. (i) *If* MW($\mathscr{E}$) *has a* 3-*torsion, then every singular fiber of* $\mathscr{E}$ *is of type either* IV, IV^*, *or* I_n.

(ii) *If* MW($\mathscr{E}$) *has a p-torsion* ($p \geq 5$), *then* $\mathscr{E}$ *has only* I_n *fibers as its singular fibers.*

For proofs of Lemma 5.3 and Corollary 5.4, see [Mi] VII, 3.

Our method to obtain a sextic is based on the following proposition due to Persson [P2].

PROPOSITION 5.5 (Persson). *Let* $\psi : \mathscr{E} \to \boldsymbol{P}^1$ *be an elliptic* $K3$ *surface with a section* s_0 *having an* I_6 *fiber or* I_2 *and* I_4 *fibers. Then*:

(i) $\mathscr{E}$ *is the canonical resolution of some double covering* $\mathscr{E}' \to \boldsymbol{P}^2$ *branched along a sextic* B *with at most simple singularities.*

(ii) *The elliptic fibration* $\psi : \mathscr{E} \to \boldsymbol{P}^1$ *is the standard fibration centered at a triple point,* x, *of* B. *Namely,* ψ *is induced by a pencil of lines through* x; *and* x *is an* e_6 *singularity for the former, while* x *is a* d_5 *singularity for the latter.*

(iii) *The involution determined by the covering transformation coincides with the canonical involution.*

For a proof, see [P2], p. 282.

Now we have the following:

PROPOSITION 5.6. *Let* $\psi : \mathscr{E} \to \boldsymbol{P}^1$ *be an elliptic* $K3$ *surface as in Proposition* 5.5, *and let* B *be the sextic in Proposition* 5.5 (i). *If* MW($\mathscr{E}$) *has a* 3-*torsion, then* B *is a* (2, 3) *torus curve.*

Proof. Suppose that MW($\mathscr{E}$) has a 3-torsion and let s be the corresponding section. By Lemma 5.4, every singular fiber of $\mathscr{E}$ is of type either IV, IV^*, or I_n. To see at which component s meets at each singular fiber, we label irreducible components of a singular fiber as follows:

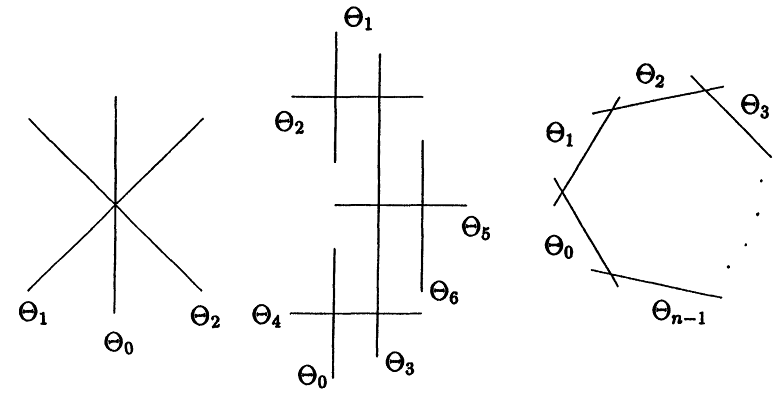

Figure 2

Then, by [M] VII. 3.3, we may assume that s meets Θ_1 at IV and IV^* fibers. Also, by Lemma 3.5 in [T3], if s meets Θ_k at an I_n fiber, then we have $k \equiv 0 \bmod n/3$ if $n \equiv 0 \bmod 3$ and $k = 0$ if $n \not\equiv 0 \bmod 3$. Hence, by considering the process of the canonical resolution $\mathscr{E} \to \mathscr{E}'$ in Proposition 5.5, we can check that the image of s in $\boldsymbol{P}^2$ is a conic satisfying (0.4.1) and (0.4.2). Hence, by Theorem 0.4, B has a $(2, 3)$ torus decomposition.

Now we go on to give rather concrete examples by using Proposition 5.6. We first define the total Milnor number of a plane curve.

Definition 5.7. Let B be a reduced plane curve. For $x \in \mathrm{Sing}(B)$, μ_x denotes its Milnor number. We define the total Milnor number, $\mu(B)$, of B to be $\sum_{x \in \mathrm{Sing}(B)} \mu_x$.

By the definition, $\mu(B)$ is a non-negative integer. For a sextic, B, with at most simple singularities, it is well-known that $\mu(B) \leq 19$ (see [P2], for example). Following to Persson, we define a maximizing sextic as follows:

Definition 5.8 (Persson). Let B be a sextic with at most simple singularities. We call B a maximizing sextic if $\mu(B) = 19$.

For an irreducible maximizing sextic, we have the following:

Proposition 5.9. *Let B be an irreducible maximizing sextic with a triple point. If B has three or more singularities, each of which is of type either e_6 or a_{3k-1} $(k \geq 1)$. Then B has a $(2, 3)$ torus decomposition.*

Proof. Let $\mathscr{E}$ be the canonical resolution of a double covering of $\boldsymbol{P}^2$ branched along B. Let be a triple point of B and let $\psi_x : \mathscr{E} \to \boldsymbol{P}^1$ be the standard fibration centered at x. Then, by Theorem 0.6 in [T2], $\mathrm{MW}(\mathscr{E})$ has a 3-torsion. Hence, by Proposition 5.6, B has a $(2, 3)$ torus decomposition.

Example 5.10. There exist irreducible maximizing sextics, B, for the following 7 cases:

	Singularities of B
1	$3e_6 + a_1$
2	$e_6 + a_5 + 4a_2$
3	$e_6 + a_{11} + a_2$
4	$e_6 + a_8 + a_3 + a_2$
5	$e_6 + a_8 + 2a_2 + a_1$
6	$e_6 + a_5 + a_4 + 2a_2$
7	$d_5 + a_8 + 3a_2$

For the existence of sextics as above, see [P2], [MP2], [T4] and [Y].

Remark 5.11. (i) One can find other examples of $(2,3)$ torus sextics by using elliptic K3 surfaces with 3-torsions. For details, see [T2] and [T4].

(ii) Proposition 5.9 is false if $\mu(B) \leq 18$. In fact, there are two irreducible sextics, B_1 and B_2, such that (i) B_1 and B_2 have the same configuration of singularities, and (ii) B_1 has a $(2,3)$ torus decomposition, while B_2 does not. Such examples give rise to a Zariski pairs. For details on Zariski pairs of degree 6, see [A], [T1], [T2] and [T4].

Now we go on to consider sextics possessing two different $(2,3)$ torus decompositions. By Proposition 5.6, one of possible approaches is to make use of an elliptic K3 surface with $\mathrm{MW}(\mathscr{E})_{\mathrm{tor}} \supset \boldsymbol{Z}/3\boldsymbol{Z} \oplus \boldsymbol{Z}/3\boldsymbol{Z}$. To construct such an elliptic K3 surface, we start with a rational elliptic surface as follows:

Let $g : E(3) \to \boldsymbol{P}^1$ be the elliptic modular surface attached to $\Gamma(3)$, where

$$\Gamma(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\boldsymbol{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\mathrm{mod}\, 3) \right\}.$$

Then it is known that $E(3)$ satisfies the following properties:

(5.12) $E(3)$ has four I_3 fibers.

(5.13) g has 9 sections; and by choosing one of them as the zero, we have $\mathrm{MW}(E(3)) \cong \boldsymbol{Z}/3\boldsymbol{Z} \oplus \boldsymbol{Z}/3\boldsymbol{Z}$.

(5.14) $E(3)$ is obtained from a pencil of cubics given by $\{\lambda_0(X^3 + Y^3 + Z^3) + 3\lambda_1 XYZ\}$, $[\lambda_0 : \lambda_1] \in \boldsymbol{P}^1$, and each base points of the pencil gives rise to a section of $E(3)$.

For these facts, see [I] and [S1] for details.

LEMMA 5.15. *One can label the four singular fibers and their irreducible components in the following way*:

(i) *If* F_i $(i = 1,2,3,4)$ *denote the singular fibers, then* $F_i = \Theta_{i,0} + \Theta_{i,1} + \Theta_{i,2}$ *such that* $s_0\Theta_{i,0} = \Theta_{i,0}\Theta_{i,1} = \Theta_{i,1}\Theta_{i,2} = \Theta_{i,2}\Theta_{i,0} = 1$, $(i = 1,2,3,4)$.

(ii) *There exist* 3-*torsion sections* s_1 *and* s_2 *such that*

$$s_1\Theta_{1,1} = s_1\Theta_{2,1} = s_1\Theta_{3,0} = s_1\Theta_{4,1} = 1,$$

and

$$s_2\Theta_{1,1} = s_2\Theta_{2,2} = s_2\Theta_{3,1} = s_2\Theta_{4,0} = 1.$$

Proof. By (5.14), we can easily check the above fact.

Let $\rho : \boldsymbol{P}^1 \to \boldsymbol{P}^1$ be a morphism of degree 2 with branch points v_1 and v_2. Let $\psi : \mathscr{E} \to \boldsymbol{P}^1$ be the relatively minimal model of the pull-back of $g : E(3) \to \boldsymbol{P}^1$ by ρ.

LEMMA 5.16. *For singular fibers of* ψ, *we have the following*:

	Fibers of $E(3)$ *over* v_1 *and* v_2	*Singular fibers of* $\mathscr{E}$
1	F_1, F_2	I_6, I_6, I_3, I_3, I_3, I_3
2	F_1, *a smooth fiber*	I_6, I_3, I_3, I_3, I_3, I_3, I_3
3	*a smooth fiber, a smooth fiber*	I_3, I_3, I_3, I_3, I_3, I_3, I_3, I_3

Proof. By Table 7.1 in [MP1], our table is immediate.

The sections, s_1 and s_2, in Lemma 5.15 give rise to 3-torsion sections, $\tilde{s}_1$ and $\tilde{s}_2$, of $\mathscr{E}$, respectively. For the first two cases in Lemma 5.16, Figures 3 and 4 as below explain at which component of each singular fiber $\tilde{s}_1$ and $\tilde{s}_2$ meet.

CASE 1.

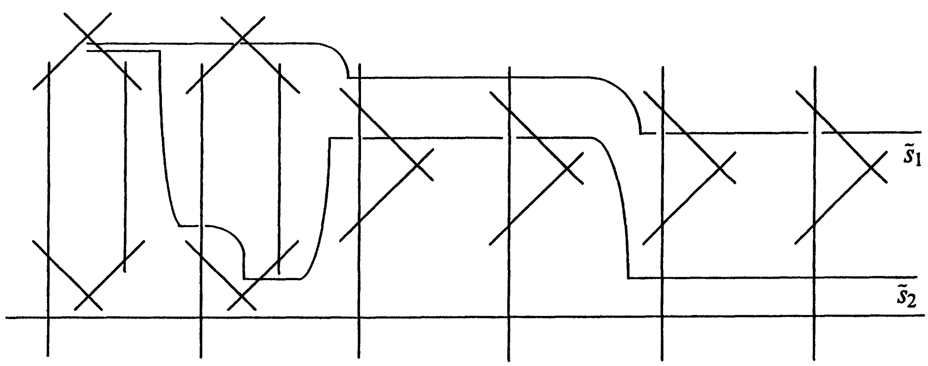

Figure 3

CASE 2.

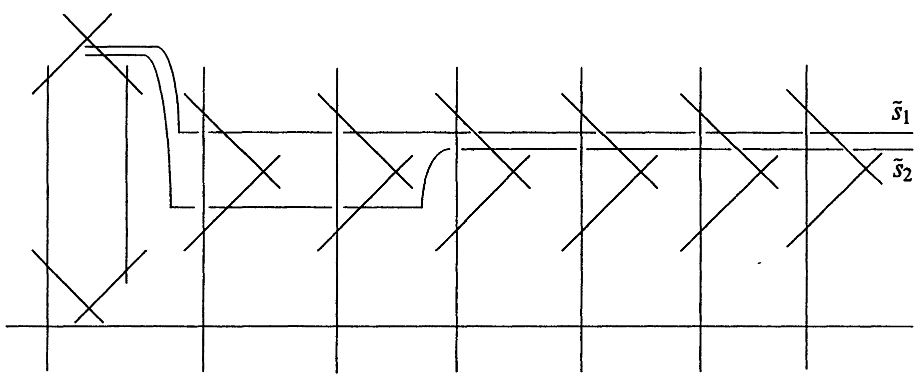

Figure 4

For each case as above, ψ has an I_6 fiber. Hence, by Proposition 5.5, we have

(i) $\mathscr{E}$ is the canonical resolution of a double covering $\mathscr{E}'$ branched along some sextic, B, with at most simple singularities, and

(ii) ψ is the standard fibration centered at an e_6 singularity of B.

Thus, by looking into the process of the resolution $\mathscr{E} \to \mathscr{E}'$, we have the following:

PROPOSITION 5.17. *Let B_1 and B_2 be the sextics as above corresponding to Case* 1 *and Case* 2, *respectively. Then*:

(i) *B_1 has singularities $4a_2 + a_5 + e_6$. Let $C_{i,1}$ be the image of $\tilde{s}_i$. Then both $C_{i,1}$ $(i = 1,2)$ are conics satisfying the conditions* (0.4.1) *and* (0.4.2). *Both $C_{i,1}$ $(i = 1,2)$ meet B_1 at e_6, a_5 and $2a_2$; and the two a_2 points in $C_{1,1} \cap B$ are disjoint from those in $C_{2,1} \cap B$.*

(ii) *B_2 has singularities $6a_2 + e_6$. Let $C_{i,2}$ be the image of $\tilde{s}_i$. Then both $C_{i,2}$ $(i = 1,2)$ are conics satisfying the conditions* (0.4.1) *and* (0.4.2). *Both $C_{i,2}$ $(i = 1,2)$ meets B_2 at $4a_2$ and e_6; and the four a_2 points in $C_{1,2} \cap B$ do not coincide with those in $C_{2,2} \cap B$.*

Proposition 5.17 shows that Theorem 0.5 (i).

§6. Proof of Theorem 0.5 (ii)

We keep the same notation as that in §6. We start with the following lemma.

LEMMA 6.1. *Let B_1 and B_2 be sextics as in Theorem* 0.5, *and let $\varphi_i : \mathscr{E}_i \to \boldsymbol{P}^1$ be the standard fibration centered at the e_6 singularity. Then*:

(i) *The configuration of singular fibers of φ_1 is I_6, I_6, I_3, I_3, I_3, I_3.*

(ii) *The configuration of singular fibers of φ_2 is I_6, I_3, I_3, I_3, I_3, I_3, I_3.*

Proof. For each i, the elliptic fibration φ_i comes from a pencil of lines through the e_6 singularity; and it is easy to see that

(a) a singular fiber arising from the e_6 singularity is of type I_n $(n \geq 6)$, and

(b) a singular fiber arising from an a_2 singularity is of type either I_3 or IV.

Since $\mathrm{rank}_{\boldsymbol{Z}} T_{\mathscr{E}} \leq 20$, and the sum of the topological Euler numbers of singular fibers is 24, the configuration of singular fibers of φ_1 is I_6, I_6, I_3, I_3, I_3, I_3, and the configuration of singular fibers of φ_2 is either I_6, I_3, I_3, I_3, I_3, I_3, I_3 or I_9, I_3, I_3, I_3, I_3, I_3. But the latter case of φ_2 does not occur by [MP2].

LEMMA 6.2. *Let $\psi : \mathscr{E} \to \boldsymbol{P}^1$ be a semistable elliptic $K3$ surface with singular fibers $I_{n_1}, \dots, I_{n_r}$. Let p be a fixed prime. If p divides $r - 1$ or more of n_i's, then $\boldsymbol{Z}/p\boldsymbol{Z} \oplus \boldsymbol{Z}/p\boldsymbol{Z} \subset \mathrm{MW}(\mathscr{E})$.*

This is straightforward generalization of Lemma 9 in [MP3], so, we omit its proof.

By Lemma 6.2, for each of φ_i in Lemma 6.1, $\boldsymbol{Z}/3\boldsymbol{Z} \oplus \boldsymbol{Z}/3\boldsymbol{Z} \subset \mathrm{MW}(\mathscr{E}_i)$. Hence, by [CW], there exist degree 2 morphisms $\rho_i : \boldsymbol{P}^1 \to \boldsymbol{P}^1$ $(i = 1, 2)$ such that $\mathscr{E}_i$ $(i = 1, 2)$ are obtained as relatively minimal model of the pull-back surfaces of $E(3)$ by ρ_i $(i = 1, 2)$, respectively. This means that B_1 and B_2 are obtained in the same way as in Proposition 5.17. This implies Theorem 0.5 (ii).

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