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THE UNIQUENESS OF MEROMORPHIC FUNCTIONS WITH THEIR DERIVATIVES

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Abstract

In this paper, we deal with the problem of uniqueness of meromorphic functions that share one finite value with their derivatives and obtain some theorems which improve a result given by Rainer Brück.

1. Introduction and main results

By a meromorphic function we shall always mean a function that is meromorphic in the whole complex plane. It is assumed that the reader is familiar with the notations of the Nevanlinna theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, $S(r,f)$ and so on, that can be found, for instance, in [1]. And $N_{1)}(r,1/f)$ denotes the counting function of the simple zeros of f, $\overline{N}_{(2}(r,1/f) = \overline{N}(r,1/f) - N_{1)}(r,1/f)$. Let f and g be meromorphic functions and a be a complex constant, we say that f and g share the value a IM (ignoring multiplicity), if $f-a$ and $g-a$ have the same zeros, they share the value a CM (counting multiplicity), if $f-a$ and $g-a$ have the same zeros with the same multiplicity.

In 1979, E. Mues and N. Steinmetz proved the following theorem in [2].

THEOREM A. *Let f be an entire function which is not constant. If f and f' share two distinct values a, b, then $f' \equiv f$.*

In 1996, Rainer Brück proved the following in [3]

THEOREM B. *Let f be an entire function which is not constant. If f and f' share the value 1 CM, and if $N(r,1/f') = S(r,f)$, then*

$$\frac{f'-1}{f-1} \equiv c \tag{1}$$

for some non-zero constant c.

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In this paper, we prove the following results which are improvements of Theorem B.

THEOREM 1. *Let f be a non-constant meromorphic function. If f and f′ share the value* 1 *CM, and if*

$$\overline{N}(r,f)+N\left(r,\frac{1}{f'}\right)<(\lambda+o(1))T(r,f') \tag{2}$$

for some real constant $\lambda\in(0,1/2)$, *then* f *and* f' *satisfy* (1).

Remark 1. It is clear that if f and f' satisfy the condition of Theorem 1, then $f=Ae^{cz}+1-(1/c)$, where A, c are non-zero constants. And obviously the condition is necessary.

THEOREM 2. *Let f be a non-constant meromorphic function, k be a positive integer. If f and* $f^{(k)}$ *share the value* 1 *CM, and if*

$$2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f'}\right)+N\left(r,\frac{1}{f^{(k)}}\right)<(\lambda+o(1))T(r,f^{(k)}) \tag{3}$$

for some real constant $\lambda\in(0,1)$, *then*

$$\frac{f^{(k)}-1}{f-1}\equiv c \tag{4}$$

for some non-zero constant c.

Remark 2. It is easy to see from (4) that if f and $f^{(k)}$ satisfy the condition of Theorem 2, then $f=Ae^{\mu z}+1-1/c$, where A, c are non-zero constants, and μ is any k-th roots of c. And the condition is necessary.

From Theorem 2, we can obtain the following corollaries:

COROLLARY 1. *Let f be a non-constant meromorphic function, k be a positive integer. If f and* $f^{(k)}$ *share the value* 1 *CM, and if*

$$2\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f'}\right)+N\left(r,\frac{1}{f^{(k)}}\right)<(\lambda+o(1))T(r,f) \tag{5}$$

for some real constant $\lambda\in(0,2/5)$, *then* f *and* $f^{(k)}$ *satisfy* (4).

COROLLARY 2. *Let f be a non-constant meromorphic function, k be a positive integer. If f and* $f^{(k)}$ *share the value* 1 *CM, and if*

$$(k+1)\overline{N}(r,f)+2N\left(r,\frac{1}{f'}\right)<(\lambda+o(1))T(r,f) \tag{6}$$

for some real constant $\lambda\in(0,2/5)$, *then* f *and* $f^{(k)}$ *satisfy* (4).

Corollary 3. *Let f be a non-constant entire function, k be a positive integer. If f and $f^{(k)}$ share the value 1 CM, and if*

$$\bar{N}\left(r,\frac{1}{f'}\right) < (\lambda + o(1))T(r,f)$$

for some real constant $\lambda \in (0,1/4)$, then f and $f^{(k)}$ satisfy (4).

Obviously Theorem B is included in Corollary 3.

Remark 3. Factly, the real constant $\lambda \in (0,2/5)$ in Corollary 1 and Corollary 2 can be stated $\lambda \in (0,(k+1)/(2k+3))$ instead, that is easy to see from the following proving.

2. Proof of main results

The following two lemmas are neened in the following proving.

Lemma 1 (see [4]). *Let f be a non-constant meromorphic function, k be a positive integer, then*

$$N\left(r,\frac{1}{f^{(k)}}\right) < N\left(r,\frac{1}{f}\right) + k\bar{N}(r,f) + S(r,f). \tag{7}$$

Lemma 2. *Let f be a non-constant meromorphic function, k be a positive integer, if f and $f^{(k)}$ share the value 1 CM, then*

$$T(r,f) < \left(2+\frac{1}{k+1}\right)T(r,f^{(k)}) + S(r,f). \tag{8}$$

Especially when f is an entire function, then

$$T(r,f) < 2T(r,f^{(k)}) + S(r,f). \tag{9}$$

Proof. By Milloux inequality (see [1] or [5]) for $f-1$, we have

$$T(r,f) < \bar{N}(r,f) + \bar{N}\left(r,\frac{1}{f-1}\right) + \bar{N}\left(r,\frac{1}{f^{(k)}-1}\right) + S(r,f). \tag{10}$$

Since f and $f^{(k)}$ share the value 1 CM, and

$$T(r,f^{(k)}) > N(r,f^{(k)}) > (k+1)\bar{N}(r,f)$$

so (8) holds. (9) can be got immediately from (10). This lemma is thus proved.

Now we turn to prove the theorems. Theorem 1 is the particular case of Theorem 2, so we need only to prove Theorem 2. Define

$$F = \frac{f^{(k+2)}}{f^{(k+1)}} - \frac{f''}{f'} - 2\frac{f^{(k+1)}}{f^{(k)}-1} + 2\frac{f'}{f-1}.$$

Firstly assume that $F \not\equiv 0$, then $m(r,f) = S(r,f)$. $N_0(r, 1/f^{(k+1)})$ denotes the counting function corresponding the zeros of $f^{(k+1)}$ which are not the zeros of f', $f^{(k)}$ and $f^{(k)} - 1$ with the multiple zeros are counted multiplicity times, $\overline{N}_0(r, 1/f^{(k+1)})$ denotes that case the multiple zeros are only counted one time. Since f and $f^{(k)}$ share the value 1 CM, it is easy to see by calculating that the zeros of $f-1$ are not the poles of F, so we have

$$N(r,F) \le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f'}\right) - \overline{N}_{(2}\left(r,\frac{1}{f-1}\right) + \overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right) + \overline{N}_0\left(r,\frac{1}{f^{(k+1)}}\right).$$

And noticing that $m(r,F) = S(r,f)$ and

$$\overline{N}_{(2}\left(r,\frac{1}{f-1}\right) = \overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right)$$

then

$$\begin{aligned}(11)\qquad T(r,F) \le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f'}\right) + \overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right) + \overline{N}_0\left(r,\frac{1}{f^{(k+1)}}\right) \\ - \overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right) + S(r,f).\end{aligned}$$

By calculating it can be shown that the simple zeros of $f-1$ are the zeros of F. And as f and $f^{(k)}$ share the value 1 CM, we have

$$(12)\qquad N_{1)}\left(r,\frac{1}{f-1}\right) = N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right) \le N\left(r,\frac{1}{F}\right) \le T(r,F) + O(1).$$

Combining (11) and (12) we obtain

$$\begin{aligned}(13)\quad \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) &= N_{1)}\left(r,\frac{1}{f^{(k)}-1}\right) + \overline{N}_{(2}\left(r,\frac{1}{f^{(k)}-1}\right) \\ &\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f'}\right) + \overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right) + \overline{N}_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f).\end{aligned}$$

By using the second fundamental theorem for $f^{(k)}$, we have

$$\begin{aligned}(14)\qquad T(r,f^{(k)}) < \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f^{(k)}}\right) + \overline{N}\left(r,\frac{1}{f^{(k)}-1}\right) \\ - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f^{(k)}).\end{aligned}$$

From Lemma 2 we can get $S(r,f) = S(r,f^{(k)})$. Combining this and (13) and (14) and

$$\overline{N}\left(r,\frac{1}{f^{(k)}}\right) + \overline{N}_{(2}\left(r,\frac{1}{f^{(k)}}\right) \le N\left(r,\frac{1}{f^{(k)}}\right)$$

we get

$$(15) \qquad T(r, f^{(k)}) < 2\bar{N}(r, f) + \bar{N}\left(r, \frac{1}{f'}\right) + N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f).$$

By (15) and (3) we have $T(r, f^{(k)}) = S(r, f)$, and from (8) we conclude $T(r, f) = S(r, f)$ which is a contradiction.

Therefore, we have $F \equiv 0$, and integration yields

$$(16) \qquad C\frac{f^{(k+1)}}{f'} \equiv \left(\frac{f^{(k)} - 1}{f - 1}\right)^2$$

where C is a non-zero constant.

We assume that $f^{(k+1)}/f'$ is not constant. From (16) and that f and $f^{(k)}$ sharing the value 1 CM, it is clear that $f^{(k+1)}/f'$ has no zeros and no poles, so

$$T\left(r, \frac{f^{(k+1)}}{f'}\right) = m\left(r, \frac{f^{(k+1)}}{f'}\right) = S(r, f).$$

If z_0 is a simple zero of $f - 1$, we know by calculating that $(f^{(k)} - 1)/(f - 1)\,|_{z=z_0} = f^{(k+1)}(z_0)/f'(z_0)$, and from (16) thus $f^{(k+1)}(z_0)/f'(z_0) = C$, so z_0 is also the zero of $(f^{(k+1)}/f') - C$, then

$$N_{1)}\left(r, \frac{1}{f - 1}\right) \leq N\left(r, \frac{1}{\frac{f^{(k+1)}}{f'} - C}\right) \leq T\left(r, \frac{f^{(k+1)}}{f'}\right) + O(1) = S(r, f)$$

therefore

$$\bar{N}\left(r, \frac{1}{f^{(k)} - 1}\right) = \bar{N}\left(r, \frac{1}{f - 1}\right) = N_{1)}\left(r, \frac{1}{f - 1}\right) + \bar{N}_{(2}\left(r, \frac{1}{f - 1}\right)$$
$$\leq \bar{N}\left(r, \frac{1}{f'}\right) + S(r, f)$$

this implies that (13) still holds. Similarly we can get the contradition $T(r, f) = S(r, f)$ again. Then $f^{(k+1)}/f'$ is a constant, and so the proof of Theorem 2 is finished.

Corollary 1 can be obtained by Theorem 2 and (8).

By Lemma 1 we get

$$(17) \qquad N\left(r, \frac{1}{f^{(k)}}\right) < N\left(r, \frac{1}{f'}\right) + (k - 1)\bar{N}(r, f) + S(r, f)$$

from this and Corollary 1 we obtained Corollary 2.

Corollary 3 can be obtained by Theorem 2 and (9) and (17).

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References

[1] W. K. Hayman, Meromorphic Function, Clarendon Press, Oxford, 1964.
[2] E. Mues and N. Steinmetz, Meromorphe Funktionen, die mit ihrer Ableitung Werte teilen, Manuscripta Math., **29** (1979), 195–206.
[3] Rainer Brück, On entire functions which share one value CM with their first derivative, Results in Math., **30** (1996), 21–24.
[4] Hong-Xun Yi, Uniqueness of meromorphic functions and a question of C. C. Yang, Complex Variables Theory Appl., **14** (1990), 169–176.
[5] H. Milloux, Extension d'un théorème de M. R. Nevanlinna et applications, Act. Scient. et Ind. No. 888, 1940.

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