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MEROMORPHIC FUNCTIONS SHARING THREE VALUES OR SETS CM

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Abstract

In this paper, by using a lemma about common 1-points, the author studied the relationship between two meromorphic functions which share three values or three sets CM, and generalised a result obtained recently by E. Mues [4], and weaked the condition of a theorem in [10].

1. Introduction

Let f and g be two non-constant meromorphic functions in the complex plane $\boldsymbol{C}$ and a be a value in the extended complex plane $\bar{\boldsymbol{C}}$. We say that f and g share the value a CM (IM) provided that $f-a$ and $g-a$ ($1/f$ and $1/g$, resp.) have the same zeros counting multiplicities (ignoring multiplicities) in the case of $a \in \boldsymbol{C}$ ($a=\infty$, resp.). For a positive integer or infinity k let

$$E(a,k,f)=\{z: f(z) \overset{(p)}{=} a, p \le k\},$$

here p is the multiplicity of a-point of f. Then f and g share the value a IM can be expressed as $E(a,\infty,f)=E(a,\infty,g)$. $E(a,1,f)=E(a,1,g)$ means that f and g have the same simple a-points. Let

$$E_f\{a\}=\{z \mid f(z)=a;\ \text{counting multiplicities}\}$$

and

$$\tilde{E}_f\{a\}=\{z \mid f(z)=a;\ \text{ignoring multiplicities}\}.$$

Then f and g share the value a CM (IM) can be expressed as $E_f\{a\}=E_g\{a\}$ ($\tilde{E}_f\{a\}=\tilde{E}_g\{a\}$). A natural generalization of the concept "sharing a value" will be "sharing a set". The following two notations are natural and obvious.

$$E_f(S)=\bigcup_{a\in S} E_f\{a\}, \quad \tilde{E}_f(S)=\bigcup_{a\in S}\tilde{E}_f\{a\},$$

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where S denotes an arbitrary set in $\overline{C}$. For two meromorphic functions f, g and a subset S in $\overline{C}$, we say that f shares S CM (IM) with g provided that

$$E_f(S) = E_g(S) \quad (\tilde{E}_f(S) = \tilde{E}_g(S)).$$

It is well known [5] that if two non-constant meromorphic functions f and g share four values CM, then f is a Möbius transformation of g. In the case that f and g share three values CM, f may not be a Möbius transformation of g, however, f can still be, if some other appropriate condition is added. There have been published many papers to talk about meromorphic functions sharing three values CM (see, [2], [1], [11], [9] and [7]). In 1983, Ueda [8] proved that if f and g share $0, 1, \infty$ CM and if there exists an $a \neq 0, 1, \infty$ such that $E(a, k, f) = E(a, k, g)$ for some $k \geq 2$, then f is a Möbius transformation of g. The bound 2 for k here is best possible. For examples, functions

$$f = \frac{e^{3\gamma} - 1}{e^{\gamma} - 1} \quad \text{and} \quad g = \frac{e^{-3\gamma} - 1}{e^{-\gamma} - 1} \tag{1}$$

share $0, 1, \infty$ CM and satisfy $E(3/4, 1, f) = E(3/4, 1, g)$. Functions

$$f = \frac{e^{\gamma} - 1}{-e^{2\gamma} - 1} \quad \text{and} \quad g = \frac{e^{-\gamma} - 1}{-e^{-2\gamma} - 1} \tag{2}$$

share $0, 1, \infty$ CM and satisfy $E(a, 1, f) = E(a, 1, g)$, where a is one of the solution of

$$\frac{1}{4a^2} = 1 - \frac{1}{a}. \tag{3}$$

Recently, E. Mues [4] proved the following theorem.

THEOREM A. *Let f and g be non-constant meromorphic functions sharing $0, 1, \infty$ CM. Suppose additionally that there exists an $a \neq 0, 1, \infty$ such that $E(a, 1, f) = E(a, 1, g)$. If f is not a Möbius transformation of g, then there exists a Möbius transformation L permuting $\{0, 1, \infty\}$ such that $L \circ f$ and $L \circ g$ have the form* (1) *with $L(a) = 3/4$ or $L \circ f$ and $L \circ g$ have the form* (2) *and $L(a)$ is a solution of* (3).

For functions sharing sets, Gross-Yang [3] asked whether there exist two sets $S_1 = \{a_1, a_2\}$ and $S_2 = \{b_1, b_2\}$ such that for any two non-constant entire functions f and g the conditions $E_f(S_j) = E_g(S_j) (j = 1, 2)$ imply $f \equiv g$ or not. F. Gross and C. C. Yang studied the question for the case $a_1 + a_2 = b_1 + b_2$, and H. X. Yi [10] improved Gross-Yang's result in the following manner.

THEOREM B. *Let f and g be non-constant entire functions with finite order and $S_1 = \{a_1, a_2\}$, $S_2 = \{b_1, b_2\}$ satisfying $a_1 \neq a_2$, $b_1 \neq b_2$, $a_1 + a_2 = b_1 + b_2 = c$, $a_1 a_2 \neq b_1 b_2$. If $E_f(S_j) = E_g(S_j) (j = 1, 2)$, then f and g assume one of the following relations*:

(i) $f \equiv g$,

(ii) $f + g \equiv c$,

(iii) $\left(f - \frac{c}{2}\right)\left(g - \frac{c}{2}\right) \equiv \pm\left(\frac{a_1 - a_2}{2}\right)^2$, *with* $(a_1 - a_2)^2 + (b_1 - b_2)^2 = 0$,

(iv) $(f - a_j)(g - a_k) \equiv (-1)^{j+k}(a_1 - a_2)^2$ $(j, k = 1, 2)$, *with* $3(a_1 - a_2)^2 + (b_1 - b_2)^2 = 0$,

(v) $(f - b_j)(g - b_k) \equiv (-1)^{j+k}(b_1 - b_2)^2 (j, k = 1, 2)$, *with* $(a_1 - a_2)^2 + 3(b_1 - b_2)^2 = 0$.

In this paper, noting that the assumptions of Theorem A leads to $\overline{N}_{1)}(r, 1/(f - a)) = S(r, f)$, we will replace the condition $E(a, 1, f) = E(a, 1, g)$ by $\overline{N}_{1)}(r, 1/(f - a)) = S(r, f)$, and give a generalization of Theorem A. Also we will remove the limitation to the order in Theorem B.

We assume that the readers are familiar with the basic notations and results in value distribution theory. We use $S(r, f)$ to denote the quantity $o(T(r, f))$, $r \to \infty$, $r \notin E$, here and in the sequel, the letter E is a set of $r \in (0, \infty)$ with finite linear measure not necessarily the same at each occurrence. $\overline{N}_{k)}(r, 1/(f - a))$ denotes the reduced counting function of a-points of f with multiplicities less than or equal to k. Similar notation $\overline{N}_{(k}(r, 1/(f - a))$ is also used to denote the reduced counting function of a-points of f with multiplicities more than or equal to k.

2. Lemmas and results

The following lemmas will be used in the proofs of our theorems.

LEMMA 1 (see [6]). *Let f_1 and f_2 be two non-constant meromorphic functions satisfying*

$$\overline{N}(r, f_i) + \overline{N}\left(r, \frac{1}{f_i}\right) = S(r; f_1, f_2), \quad i = 1, 2.$$

If $f_1^s f_2^t - 1$ is not identically zero for arbitrary integers s and t $(|s| + |t| > 0)$, then for any positive number ε, we have

$$N_0(r, 1; f_1, f_2) \le \varepsilon T(r) + S(r; f_1, f_2),$$

where $N_0(r, 1; f_1, f_2)$ denotes the reduced counting function of f_1 and f_2 related to the common 1-points of f_1 and f_2, and $T(r) = T(r, f_1) + T(r, f_2)$, $S(r; f_1, f_2) = o(T(r))$ as $r \to \infty$, $r \notin E$.

LEMMA 2 (see [6]). *Let $f_1, f_2, \ldots, f_n$ be non-constant meromorphic functions satisfying*

$$\overline{N}(r, f_i) + \overline{N}\left(r, \frac{1}{f_i}\right) = S(r), \quad i = 1, 2, \ldots, n,$$

and

$$T(r,f_i) \neq S(r), \quad T\left(r,\frac{f_i}{f_j}\right) \neq S(r), \quad i \neq j,\ i,j = 1,2,\dots,n.$$

Let $a_0, a_1, \dots, a_m$ $(m \leq n)$ *be meromorphic functions satisfying* $T(r,a_i) = S(r)$, $i = 0,1,\dots,m$. *If*

$$\sum_{i=1}^{m} a_i f_i \equiv a_0,$$

then $a_0 \equiv a_1 \equiv \cdots \equiv a_m \equiv 0$, *where* $S(r) = o(T(r))$, *as* $r \to \infty$ *and* $r \notin E$, *and* $T(r) = \sum_{i=1}^{n} T(r,f_i)$.

Lemma 3 (see [4]) and ([12]). *Let f and g be meromorphic functions sharing* $0,1,\infty$ *CM and suppose that f is not a Möbius transformation of g. Then we have*

$$T(r,\ f) \leq 2\overline{N}\left(r,\frac{1}{f-a}\right) + S(r,\ f), \quad \textit{for } a \neq 0,1,\infty.$$

Now we state and prove the main theorems.

Theorem 1. *Let f and g be non-constant meromorphic functions sharing* $0,1,\infty$ *CM. Suppose additionally that f is not a Möbius transformation of g and that there exists an* $a \neq 0,1,\infty$ *such that*

$$T(r,\ f) \leq c\overline{N}_{(2}\left(r,\frac{1}{f-a}\right) + S(r,\ f),$$

here $c > 0$ *is a constant, then there exist a non-constant entire function* γ, *a non-zero constant* λ *and two integers s, t* $(t > 0)$ *which are mutually prime, such that*

$$f = \frac{e^{t\gamma} - 1}{\lambda e^{-s\gamma} - 1}, \quad g = \frac{e^{-t\gamma} - 1}{\frac{1}{\lambda} e^{s\gamma} - 1}, \tag{4}$$

$$\frac{(1-a)^{s+t}}{a^t} = \lambda^t \frac{(1-\theta)^{s+t}}{\theta^t}, \tag{5}$$

with $\theta = -t/s \neq 1, a$.

Proof. Since f and g share 0,1 and ∞ CM, there exist two entire functions α and β such that

$$f = \frac{e^{\alpha} - 1}{e^{\beta} - 1}, \quad g = \frac{e^{-\alpha} - 1}{e^{-\beta} - 1}. \tag{6}$$

Obviously, α, β and $\beta - \alpha$ are not constants, otherwise, f is a Möbius transformation of g. By Nevanlinna second fundamental theorem, it is quite easy to get $T(r,f) \leq 3T(r,g) + S(r,f)$ and $T(r,g) \leq 3T(r,f) + S(r,g)$, and thus

$S(r,f) = S(r,g)$. We use $S(r)$ to express $S(r,f)$ or $S(r,g)$. Formulas in (6) are equivalent to

$$\frac{f}{g} = e^{\alpha-\beta}, \quad \frac{f-1}{g-1} = e^{\alpha}, \tag{7}$$

from which, we deduce that

$$\begin{aligned} T(r,e^{\alpha}) &\le T(r,f) + T(r,g) + O(1) \\ &\le 4T(r,f) + S(r), \end{aligned} \tag{8}$$

and

$$T(r,e^{\beta}) \le 10T(r,f) + S(r). \tag{9}$$

Hence we have $S(r,e^{\alpha}) \le S(r)$ and $S(r,e^{\beta}) \le S(r)$. On the other hand, from (6), we have $S(r) \le S(r;e^{\alpha},e^{\beta})$. Hence

$$S(r) = S(r;e^{\alpha},e^{\beta}). \tag{10}$$

Let z_0 be a multiple a-point of f but not a zero of α', β' and $\beta' - \alpha'$. Then from

$$f - a = \frac{e^{\alpha} - ae^{\beta} + a - 1}{e^{\beta} - 1}, \tag{11}$$

we see that

$$e^{\alpha(z_0)} - ae^{\beta(z_0)} + a - 1 = 0, \tag{12}$$

$$\alpha'(z_0)e^{\alpha(z_0)} - a\beta'(z_0)e^{\beta(z_0)} = 0, \tag{13}$$

which lead to

$$e^{\alpha(z_0)} = \frac{(1-\alpha)\beta'(z_0)}{\beta'(z_0) - \alpha'(z_0)}, \quad e^{\beta(z_0)} = \frac{(1-a)\alpha'(z_0)}{a(\beta'(z_0) - \alpha'(z_0))}. \tag{14}$$

Let

$$F_1 = \frac{(\beta' - \alpha')}{(1-a)\beta'}e^{\alpha}, \quad F_2 = \frac{a(\beta' - \alpha')}{(1-a)\alpha'}e^{\beta}. \tag{15}$$

Then from (10) and (15), we get

$$\begin{aligned} T(r,F_1) &= T(r,e^{\alpha}) + S(r;e^{\alpha},e^{\beta}), \\ T(r,F_2) &= T(r,e^{\beta}) + S(r;e^{\alpha},e^{\beta}), \end{aligned} \tag{16}$$

and thus

$$S(r;F_1,F_2) = S(r;e^{\alpha},e^{\beta}).$$

Since $F_1(z_0) = 1$, $F_2(z_0) = 1$, we have

$$\overline{N}_{(2}\left(r, \frac{1}{f-a}\right) \le N_0(r, 1; F_1, F_2) + S(r; F_1, F_2). \tag{17}$$

From (16), (8), (9), (17) and the assumption, we get

$$T(r, F_1) + T(r, F_2) \le 14cN_0(r, 1; F_1, F_2) + S(r; F_1, F_2). \tag{18}$$

It is obvious that

$$\overline{N}(r, F_i) + \overline{N}\left(r, \frac{1}{F_i}\right) = S(r; F_1, F_2), \quad i = 1, 2.$$

Hence by Lemma 1, we see that there exist two non-zero and mutually prime integers s, t $(t > 0)$ such that $F_1^s F_2^t \equiv 1$, i.e.,

$$e^{s\alpha + t\beta} \equiv \left(\frac{(1-a)\beta'}{\beta' - \alpha'}\right)^s \left(\frac{(1-a)\alpha'}{a(\beta' - \alpha')}\right)^t := \rho, \tag{19}$$

from which, we can see that $\rho'/\rho = s\alpha' + t\beta'$. On the other hand, from the second equality in (19), we have

$$\begin{aligned} \frac{\rho'}{\rho} &= t\frac{\left(\frac{\alpha'}{\beta'}\right)'}{\left(\frac{\alpha'}{\beta'}\right)} - (s+t)\frac{-\left(\frac{\alpha'}{\beta'}\right)'}{\left(1 - \frac{\alpha'}{\beta'}\right)} \\ &= \frac{t + s\frac{\alpha'}{\beta'}}{\frac{\alpha'}{\beta'}\left(1 - \frac{\alpha'}{\beta'}\right)}\left(\frac{\alpha'}{\beta'}\right)'. \end{aligned}$$

If $\alpha'/\beta' \not\equiv -t/s$, then we have

$$\alpha' \equiv \frac{\left(\frac{\alpha'}{\beta'}\right)'}{1 - \frac{\alpha'}{\beta'}},$$

and thus

$$e^{\alpha}\left(1 - \frac{\alpha'}{\beta'}\right) \equiv c_1, \tag{20}$$

where c_1 is a non-zero constant. Above identity can be rewritten as

$$\beta' \equiv \frac{\alpha' e^{\alpha}}{e^{\alpha} - c_1},$$

from which, we get

$$e^{\beta} \equiv c_2(e^{\alpha} - c_1), \tag{21}$$

where c_2 is also a non-zero constant. Formulas (20) and (21) imply $T(r, e^{\alpha}) = S(r)$ and $T(r, e^{\beta}) = S(r)$. And thus from (6), we get $T(r, f) = S(r)$, a contradiction. Hence

$$\frac{\alpha'}{\beta'} \equiv -\frac{t}{s}. \tag{22}$$

If $a = -t/s$, then $a \equiv \alpha'/\beta'$. Hence $e^{\alpha(z_0)} = 1$ and $e^{\beta(z_0)} = 1$. In the case that z_0 is a zero of $e^{\alpha} - ae^{\beta} + a - 1$ with multiplicity not less than 3, we have

$$\alpha''(z_0) + (\alpha'(z_0))^2 - a[\beta''(z_0) + (\beta'(z_0))^2] = 0.$$

This and $a \equiv \alpha'/\beta'$ lead to $a(a-1)\beta'(z_0) = 0$, which contradicts the assumption about z_0. Hence $a \neq -t/s$.

Let $s\alpha + t\beta = c_0$, it is obviously a constant, and let $\gamma = \alpha/t$. Then f and g can be expressed as

$$f = \frac{e^{t\gamma} - 1}{\lambda e^{-s\gamma} - 1}, \quad g = \frac{e^{-t\gamma} - 1}{\frac{1}{\lambda} e^{s\gamma} - 1},$$

where $\lambda = e^{c_0/t}$. From (19) and (22), we have

$$\frac{(1-a)^{s+t}}{a^t} = \lambda^t \frac{(1-\theta)^{s+t}}{\theta^t},$$

where $\theta = -t/s \equiv \alpha'/\beta' \neq 1$, which completes the proof of Theorem 1. □

The following is an example which shows the existence of f and g in Theorem 1.

Example 1. Let γ be a non-constant entire function, $a = (20 + 4\sqrt{2}i)/27$, and

$$f = \frac{e^{4\gamma} - 1}{e^{\gamma} - 1}, \quad g = \frac{e^{-4\gamma} - 1}{e^{-\gamma} - 1}.$$

Since

$$f - a = \left(e^{\gamma} - \frac{-1 + \sqrt{2}i}{3}\right)^2 \left(e^{\gamma} - \frac{-1 - 2\sqrt{2}i}{3}\right)$$

$$g - a = \left(e^{-\gamma} - \frac{-1 + \sqrt{2}i}{3}\right)^2 \left(e^{-\gamma} - \frac{-1 - 2\sqrt{2}i}{3}\right),$$

we have $\overline{N}_{(2}(r, 1/(f-a)) = 1/3T(r, f) + S(r, f)$.

The conditions in the following theorem are stronger slightly than those in Theorem 1, but the conclusion is more clear.

THEOREM 2. *Let f and g be non-constant meromorphic functions sharing $0, 1, \infty$ CM. Suppose additionally that f is not a Möbius transformation of g and that there exists an $a \neq 0, 1, \infty$ such that*

$$\overline{N}_{1)}\left(r, \frac{1}{f-a}\right) = S(r, f),$$

then f and g assume one of the following forms:

(i) $f = \dfrac{e^{3\gamma}-1}{e^{\gamma}-1}$, $g = \dfrac{e^{-3\gamma}-1}{e^{-\gamma}-1}$, *with* $a = \dfrac{3}{4}$;

(ii) $f = \dfrac{e^{\gamma}-1}{e^{3\gamma}-1}$, $g = \dfrac{e^{-\gamma}-1}{e^{-3\gamma}-1}$, *with* $a = \dfrac{4}{3}$;

(iii) $f = \dfrac{e^{3\gamma}-1}{\lambda e^{2\gamma}-1}$, $g = \dfrac{e^{-3\gamma}-1}{\frac{1}{\lambda}e^{-2\gamma}-1}$, *with* $a = -3$ *and* $\lambda^3 = 1$;

(iv) $f = \dfrac{e^{2\gamma}-1}{\lambda e^{3\gamma}-1}$, $g = \dfrac{e^{-2\gamma}-1}{\frac{1}{\lambda}e^{-3\gamma}-1}$, *with* $a = -\dfrac{1}{3}$ *and* $\lambda^2 = 1$;

(v) $f = \dfrac{e^{\gamma}-1}{\lambda e^{2\gamma}-1}$, $g = \dfrac{e^{-\gamma}-1}{\frac{1}{\lambda}e^{-2\gamma}-1}$, *with* $\lambda \neq 1$ *and* $4a(1-a)\lambda = 1$;

(vi) $f = \dfrac{e^{2\gamma}-1}{\lambda e^{\gamma}-1}$, $g = \dfrac{e^{-2\gamma}-1}{\frac{1}{\lambda}e^{-\gamma}-1}$, *with* $\lambda^2 \neq 1$ *and* $a^2\lambda^2 = 4(a-1)$;

(vii) $f = \dfrac{e^{\gamma}-1}{e^{-2\gamma}-1}$, $g = \dfrac{e^{-\gamma}-1}{e^{2\gamma}-1}$, *with* $a = 4$;

(viii) $f = \dfrac{e^{2\gamma}-1}{e^{-\gamma}-1}$, $g = \dfrac{e^{-2\gamma}-1}{e^{\gamma}-1}$, *with* $a = \dfrac{1}{4}$;

(ix) $f = \dfrac{e^{\gamma}-1}{\lambda e^{-\gamma}-1}$, $g = \dfrac{e^{-\gamma}-1}{\frac{1}{\lambda}e^{\gamma}-1}$, *with* $\lambda \neq \dfrac{1-a}{2}$ *and* $(1-a)^2 + 4a\lambda = 0$,

where γ is a non-constant entire function.

Proof. Since $\overline{N}_{1)}r, (1/(f-a)) = S(r, f)$, by Lemma 3, we have $T(r, f) \leq 2\overline{N}_{(2}(r, 1/(f-a)) + S(r, f)$. Hence by Theorem 1, f and g assume the forms in (4) and a, s, t, θ satisfy the equality (5). Since s and t are mutually prime, there exist two non-zero integers p and q such that $ps + qt = 1$. From this and the proof of Theorem 1, we get

$$e^{\gamma} = \lambda^p e^{q\alpha} e^{-p\beta}. \tag{23}$$

At point z_0, we have $e^{\alpha(z_0)} = (1-a)/(1-\theta)$ and $e^{\beta(z_0)} = (1-a)\theta/a(1-\theta)$, and thus

$$e^{\gamma(z_0)} = \lambda^p \frac{a^p(1-a)^{q-p}}{\theta^p(1-\theta)^{q-p}} := u. \tag{24}$$

Now we discuss three cases below.

CASE 1. $t > -s > 0$.

If $t = 2$, then $s = -1$, and thus $p = 1$, $q = 1$ and $\theta = 2$. From (5), we get $a^2\lambda^2 = 4(a-1)$. $a \neq \theta = 2$ and $a^2\lambda^2 = 4(a-1)$ imply $\lambda^2 \neq 1$. Hence f and g assume the form (vi) in Theorem 2.

Now we suppose that $t \geq 3$. Let

$$\varphi := \frac{(e^\gamma - \lambda^p)(e^\gamma - \mu)^2}{e^\alpha - ae^\beta + a - 1}. \tag{25}$$

Since $\overline{N}_{1)}(r, 1/(f-a)) = S(r, f)$, from (11), (23) and (24), we see that $N(r, \varphi) = S(r, f)$. Since $e^\alpha - ae^\beta + a - 1$ is a polynomial in e^γ with degree $t \geq 3$ and non-zero constant term $a - 1$, it is quite easy to see that $m(r, \varphi) = S(r, f)$. Hence we have $T(r, \varphi) = S(r, f)$. Formula (25) can be rewritten as

$$\varphi(e^{t\gamma} - a\lambda e^{-s\gamma} + a - 1) = e^{3\gamma} - (\lambda^p + 2\mu)e^{2\gamma} + (\mu^2 + 2\mu\lambda^p)e^\gamma - \mu^2\lambda^p.$$

From this and Lemma 2, we get $t = 3$ and

$$\varphi \equiv 1, \quad \varphi(a-1) \equiv -\mu^2\lambda^p, \tag{26}$$

and thus

$$-a\lambda e^{-s\gamma} = -(\lambda^p + 2\mu)e^{2\gamma} + (\mu^2 + 2\mu\lambda^p)e^\gamma. \tag{27}$$

By Lemma 2 again, we see that $-s = 2$ or $-s = 1$.

If $-s = 2$, then from (27), we have

$$a\lambda = \lambda^p + 2\mu, \quad u + 2\lambda^p = 0.$$

Since $t = 3$ and $-s = 2$, by the definition of p, q, we have $p = q = 1$. Hence from (26) and the above formulas, we get

$$\lambda^3 = 1, \quad a = -3, \quad \mu = -2\lambda.$$

Therefore

$$f = \frac{e^{3\gamma} - 1}{\lambda e^{2\gamma} - 1}, \quad g = \frac{e^{-3\gamma} - 1}{\frac{1}{\lambda}e^{-2\gamma} - 1},$$

which assume the form (iii) in Theorem 2.

If $-s = 1$, then from (27) and by Lemma 2, we have

$$\lambda^p + 2\mu = 0, \quad -a\lambda = \mu^2 + 2\mu\lambda^p.$$

By the definition of p, q, we have $p = 2$ and $q = 1$. From (26) and the above formulas, we get

$$\mu = -\frac{1}{2}\lambda^2, \quad a = \frac{3}{4}\lambda^3, \quad \frac{1}{4}\lambda^6 + \frac{3}{4}\lambda^3 = 1.$$

Hence

$$f - a = \frac{e^{3\gamma} - \frac{3}{4}\lambda^4 e^\gamma + \frac{3}{4}\lambda^3 - 1}{\lambda e^\gamma - 1} = \frac{e^{3\gamma} - \frac{3}{4}\lambda^4 e^\gamma - \frac{1}{4}\lambda^6}{\lambda e^\gamma - 1} = \frac{(e^\gamma - \lambda^2)(e^\gamma + \frac{1}{2}\lambda^2)^2}{\lambda e^\gamma - 1}.$$

Since $\overline{N}_{1)}(r, 1/(f-a)) = S(r, f)$, the above formulas lead to $\lambda^3 = 1$. Hence $a = 3/4$ and $f = (e^{3\gamma} - 1)/(\lambda e^{\gamma} - 1)$. Select an entire function γ_1 such that $e^{\gamma_1} = \lambda e^{\gamma}$, then

$$f = \frac{e^{3\gamma_1} - 1}{e^{\gamma_1} - 1}, \quad g = \frac{e^{-3\gamma_1} - 1}{e^{-\gamma_1} - 1}, \quad a = \frac{3}{4},$$

which assume the form (i) in Theorem 2.

CASE 2. $-s > t > 0$.

If $-s = 2$, then $t = 1$, and thus from (4), we have

$$f - a = \frac{e^{\gamma} - a\lambda e^{2\gamma} + a - 1}{\lambda e^{2\gamma} - 1}.$$

Since $\overline{N}_{1)}(r, 1/(f-a)) = S(r, f)$, from the above formula, we get

$$1 + 4a\lambda(a-1) = 0$$

and

$$f - a = \frac{-a\lambda\left(e^{\gamma} + \frac{1}{2a\lambda}\right)^2}{\lambda e^{2\gamma} - 1}.$$

If $\lambda = 1$, then $a = 1/2$. Therefore $f - a = -(e^{\gamma} + 1)/2(e^{\gamma} - 1)$, which contradicts $T(r, f) \leq 2\overline{N}_{(2}(r, 1/(f-a)) + S(r, f)$. Hence $\lambda \neq 1$ and f, g assume the form (v) in Theorem 2.

Now we consider the subcase $-s \geq 3$. Similar to Case 1, we can prove the function φ in (25) still satisfies $T(r, \varphi) = S(r, f)$. By Lemma 2, we can get $-s = 3$, and

$$-a\lambda\varphi = 1, \quad \varphi(a-1) = -\mu^2\lambda^p, \tag{28}$$

and thus

$$\varphi e^{t\gamma} = -(\lambda^p + 2\mu)e^{2\gamma} + (\mu^2 + 2\mu\lambda^p)e^{\gamma}. \tag{29}$$

If $t = 2$, then from the above formula and by Lemma 2, we have

$$\varphi = -(\lambda^p + 2\mu), \quad \mu^2 + 2\mu\lambda^p = 0.$$

Since $s = -3$, $t = 2$, by the definition of p, q, we see that $p = q = -1$. Therefore, from the above equalities and (28), we get $a = -1/3$, and $\lambda^2 = 1$. Hence f and g assume the form (iv) in Theorem 2.

If $t = 1$, then $p = -1$, $q = -2$. From (29), we have

$$\varphi = \mu^2 + 2\mu\lambda^p, \quad \lambda^p + 2\mu = 0.$$

This and (28) imply that

$$a = \frac{4}{3}\lambda, \quad \mu = -\frac{1}{2\lambda}, \quad \varphi = -\frac{3}{4\lambda^2}, \quad 4\lambda^2 - 3\lambda - 1 = 0.$$

Then we have

$$f-a=-\frac{4}{3}\lambda\frac{\left(e^{\gamma}-\frac{1}{\lambda}\right)\left(e^{\gamma}+\frac{1}{2\lambda}\right)^{2}}{e^{3\gamma}-\frac{1}{\lambda}}.$$

Since $\overline{N}_{1)}(r,1/(f-a))=S(r,f)$, we can get $\lambda=1$, and thus $a=4/3$. Hence f and g assume the form (ii) in Theorem 2.

CASE 3. $t>0>-s$.

In this case, we express $f-a$ as

$$f-a=-\frac{e^{(s+t)\gamma}+(a-1)e^{s\gamma}-\lambda a}{e^{s\gamma}-\lambda}.$$

If $s+t=2$, then $s=t=1$, and

$$f-a=-\frac{e^{2\gamma}+(a-1)e^{\gamma}-\lambda a}{e^{\gamma}-\lambda}.$$

Since $\overline{N}_{1)}(r,1/(f-a))=S(r,f)$, the numerator in the above formula is a complete square form. Therefore

$$(1-a)^{2}+4a\lambda=0.$$

And then

$$f-a=-\frac{\left(e^{\gamma}+\frac{a-1}{2}\right)^{2}}{e^{\gamma}-\lambda},\quad \lambda\neq\frac{1-a}{2}.$$

Hence f and g assume the form (ix) in Theorem 2.

For the subcase $s+t\geq 3$, let

$$\psi:=\frac{(e^{\gamma}-\lambda^{p})(e^{\gamma}-\mu)^{2}}{e^{(s+t)\gamma}+(a-1)e^{s\gamma}-\lambda a}.$$

Similar to Case 1, we can prove that $T(r,\psi)=S(r,f)$. And then it is not difficult to prove that f and g assume the forms (vii) and (viii) in Theorem 2. The proof of Theorem 2 is then completed. □

THEOREM 3. *Let f and g be non-constant meromorphic functions and $S_1=\{a_1,a_2\}$, $S_2=\{b_1,b_2\}$, $S_3=\{\infty\}$ satisfying $a_1\neq a_2$, $b_1\neq b_2$, $a_1+a_2=b_1+b_2=c$, $a_1a_2\neq b_2b_2$. Let $d=(1/(b_1b_2-a_1a_2))((a_1-a_2)/2)^2$. If $E_f(S_j)=E_g(S_j)$ $(j=1,2,3)$, then f and g assume one of the following relations:*

(i) $f\equiv g$;

(ii) $f+g\equiv c$;

(iii) $f = \frac{c}{2} \pm (b_1 - b_2)\left(e^{\gamma} + \frac{1}{2}\right)$, $g = \frac{c}{2} \pm (b_1 - b_2)\left(e^{-\gamma} + \frac{1}{2}\right)$ *with* $d = \frac{3}{4}$;

(iv) $f = \frac{c}{2} \pm (a_1 - a_2)\left(e^{\gamma} + \frac{1}{2}\right)$, $g = \frac{c}{2} \pm (a_1 - a_2)\left(e^{-\gamma} + \frac{1}{2}\right)$ *with* $d = \frac{1}{4}$;

(v) $f = \frac{c}{2} \pm \frac{a_1 - a_2}{2} e^{\gamma}$ *and* $g = \frac{c}{2} \pm \frac{a_1 - a_2}{2} e^{-\gamma}$ *with* $d = \frac{1}{2}$,

where γ *is a non-constant entire function.*

Proof. Since $E_f(S_j) = E_g(S_j)$ $(j = 1, 2, 3)$, there exist two entire functions α and β such that

$$\frac{(f - a_1)(f - a_2)}{(g - a_1)(g - a_2)} = e^{\alpha}, \quad \frac{(f - b_1)(f - b_2)}{(g - b_1)(g - b_2)} = e^{\beta}, \tag{30}$$

from which we have

$$\frac{(f - g)(f + g - a_1 - a_2)}{(g - a_1)(g - a_2)} = e^{\alpha} - 1, \tag{31}$$

and

$$\frac{(f - g)(f + g - b_1 - b_2)}{(g - b_1)(g - b_2)} = e^{\beta} - 1. \tag{32}$$

Since $a_1 + a_2 = b_1 + b_2 = c$, we see that $f \equiv g$ or $f + g \equiv c$ as long as one of $e^{\alpha} \equiv 1$, $e^{\beta} \equiv 1$ and $e^{\alpha} \equiv e^{\beta}$ hold. In the following, we suppose that $e^{\alpha} \not\equiv 1$, $e^{\beta} \not\equiv 1$ and $e^{\alpha - \beta} \not\equiv 1$.

Since f and g share S_i $(i = 1, 2, 3)$ CM, by the second fundamental theorem, we have

$$\begin{aligned} T(r, f) &< N(r, f) + N\left(r, \frac{1}{f - a_1}\right) + N\left(r, \frac{1}{f - a_2}\right) + S(r, f) \\ &= N(r, g) + N\left(r, \frac{1}{g - a_1}\right) + N\left(r, \frac{1}{g - a_2}\right) + S(r, f) \\ &\leq 3T(r, g) + S(r, f). \end{aligned}$$

Similarly, we have $T(r, g) < 3T(r, f) + S(r, g)$. Hence $S(r, f) = S(r, g)$. Write $S(r) := S(r, f) = S(r, g)$. From (30), we can get

$$T(r, e^{\alpha}) \leq 8T(r, f) + S(r), \quad T(r, e^{\beta}) \leq 8T(r, f) + S(r).$$

Therefore

$$T(r, \alpha) \leq S(r), \quad T(r, \beta) \leq S(r).$$

From (31) and (32), we can easily get

$$\frac{a_1a_2 - b_1b_2}{(g-b_1)(g-b_2)} = \frac{e^{\beta} - e^{\alpha}}{e^{\alpha} - 1}. \tag{33}$$

Therefore the poles of g are multiple zeros of $e^{\beta} - e^{\alpha}$, and thus the zeros of $\beta' - \alpha'$. If $\beta' - \alpha' \equiv 0$, then $e^{\alpha-\beta}$ is a constant different from 1. Hence g have no poles in this case. If $\beta' - \alpha' \not\equiv 0$, then

$$N(r,g) \le N\left(r, \frac{1}{\alpha' - \beta'}\right) \le T(r, \alpha' - \beta') + O(1)$$
$$= S(r).$$

In both cases, we have

$$N(r,g) = S(r). \tag{34}$$

Similarly, we have

$$N(r,f) = S(r). \tag{35}$$

Let

$$F := \frac{(f-a_1)(f-a_2)}{a_1a_2 - b_1b_2}, \quad G := \frac{(g-a_1)(g-a_2)}{a_1a_2 - b_1b_2}. \tag{36}$$

Since

$$F - 1 = \frac{(f-b_1)(f-b_2)}{a_1a_2 - b_1b_2}, \quad G - 1 = \frac{(g-b_1)(g-b_2)}{a_1a_2 - b_1b_2}, \tag{37}$$

we see that F and G share $0, 1, \infty$ CM. Let

$$d := \left(\frac{a_1 - a_2}{2}\right)^2 \frac{1}{b_1b_2 - a_1a_2}. \tag{38}$$

It is quite easy to verify that $d \ne 0, 1, \infty$. From the two equalities in (36), we deduce that

$$F - d = \frac{1}{a_1a_2 - b_1b_2}\left(f - \frac{c}{2}\right)^2, \tag{39}$$

$$G - d = \frac{1}{a_1a_2 - b_1b_2}\left(g - \frac{c}{2}\right)^2. \tag{40}$$

Hence $N_{1)}(r, 1/(F-d)) = N_{1)}(r, 1/(G-d)) = 0$. From (35), (37) and the second fundamental theorem, we get

$$\overline{N}\left(r, \frac{1}{F-1}\right) \ge T(r,f) + S(r).$$

Hence $\overline{N}(r, 1/(F-1)) \neq S(r, F)$. Similarly, $\overline{N}(r, 1/F) \neq S(r, F)$. By Theorem 2, one of the following cases holds:

(A) $F = e^{2\gamma} + e^{\gamma} + 1,\ G = e^{-2\gamma} + e^{-\gamma} + 1,\ d = \dfrac{3}{4}.$

(B) $F = -e^{\gamma}(e^{\gamma} + 1),\ G = -e^{-\gamma}(e^{-\gamma} + 1),\ d = \dfrac{1}{4}.$

(C) F is a Möbius transformation of G,

where γ is a non-constant entire function.

In Case A, we have $F - d = (e^{\gamma} + 1/2)^2$, $G - d = (e^{-\gamma} + 1/2)^2$. Hence

$$\left(f - \frac{c}{2}\right)^2 = (a_1a_2 - b_1b_2)\left(e^{\gamma} + \frac{1}{2}\right)^2, \quad \left(g - \frac{c}{2}\right)^2 = (a_1a_2 - b_1b_2)\left(e^{-\gamma} + \frac{1}{2}\right)^2.$$

Since $d = 3/4$, we have $(a_1 - a_2)^2 = 3(b_1b_2 - a_1a_2)$. Combining this and $a_1 + a_2 = b_1 + b_2$, we get $a_1a_2 - b_1b_2 = (b_1 - b_2)^2$. Hence

$$f = \frac{c}{2} \pm (b_1 - b_2)\left(e^{\gamma} + \frac{1}{2}\right), \quad g = \frac{c}{2} \pm (b_1 - b_2)\left(e^{-\gamma} + \frac{1}{2}\right).$$

Similarly, in Case B, we have

$$F - d = -\left(e^{\gamma} + \frac{1}{2}\right)^2, \quad G - d = -\left(e^{-\gamma} + \frac{1}{2}\right)^2,$$

and

$$d = \frac{1}{4}, \quad a_1a_2 - b_1b_2 = -(a_1 - a_2)^2.$$

Hence

$$f = \frac{c}{2} \pm (a_1 - a_2)\left(e^{\gamma} + \frac{1}{2}\right), \quad g = \frac{c}{2} \pm (a_1 - a_2)\left(e^{-\gamma} + \frac{1}{2}\right).$$

Now we discuss Case C. Since F is a Möbius transformation of G, and F and G share $0, 1, \infty$ CM, furthermore $N(r, F) = S(r, F)$, $N(r, 1/F) \neq S(r, F)$ and $N(r, 1/(F-1)) \neq S(r, F)$, we can see that there exists a $d_1 \neq 0, 1, \infty$ such that d_1, ∞ are exceptional values of F. Since $N_{1)}(r, 1/(F-d)) = 0$, by the second fundamental theorem, we can deduce that $d_1 = d$. Hence $\infty, c/2$ are two exceptional values of f. Similarly, $\infty, c/2$ are two exceptional values of g. We have assumed that $e^{\alpha} \not\equiv 1$, which is equivalent to $F \not\equiv G$. Hence we have

$$(F - d)(G - d) = d^2 \quad \text{and} \quad d = \frac{1}{2},$$

which imply that

$$\left(f - \frac{c}{2}\right)\left(g - \frac{c}{2}\right) = \pm\left(\frac{a_1 - a_2}{2}\right)^2.$$

Hence

$$f = \frac{c}{2} \pm \frac{a_1 - a_2}{2} e^{\gamma}, \quad g = \frac{c}{2} \pm \frac{a_1 - a_2}{2} e^{-\gamma},$$

which completes the proof of Theorem 3. □

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