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ON THE VALUE DISTRIBUTION OF $ff^{(k)}$

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Abstract

Let f be a transcendental entire function. In this paper we will prove that if f is of finite order, then there exists at most one integer $k \geqq 2$ such that $ff^{(k)}$ may have non-zero and finite Picard exceptional value. We also give a class of entire functions which have no non-zero finite Picard values. If f is a transcendental meromorphic function, we obtain that for non-negative integers $n, n_1, \cdots, n_k$ with $n \geqq 1$, $n_1 + \cdots + n_k \geqq 1$, if $\delta(o, f) > 3/(3n+3n_1+\cdots+3n_k+1)$, then $f^n(f')^{n_1}\cdots(f^k)^{n_k}$ has no finite non-zero Picard values.

I. Introduction and main results

Let f be a transcendental meromorphic function. In 1959, W. K. Hayman [4] proved that if n is an integer satisfying $n \geqq 3$, then $f^n f'$ takes every non-zero complex value a infinitely often. He conjectured [5] that this remains valid for $n=1$ and $n=2$. The case $n=2$ was settled by E. Mues [9] on 1979. The case $n=1$ is still open.

J. Clunie [3] proved that Hayman's conjecture is true when f is entire and $n=1$. W. Hennekemper [7] extended Clunie's result and proved

$$(1) \qquad T(r, f) \leqq \left(4+\frac{1}{k+1}\right)\left\{\bar{N}(r, f)+\bar{N}\left(\frac{1}{(f^{k+1})^{(k)}-c}\right)\right\}+S(r, f)$$

for $k \in N$, $c \in C-\{0\}$, where the argument used here is based on the Nevanlinna theory, its associated standard symbols and notations, see, e.g. [6]. Particularly, $S(r, f)$ will be used to denote any quantity that satisfies $S(r, f)=o\{T(r, f)\}$ as $r \to \infty$ and $r \notin E$ with E being a set of r values of finite linear measure. W. Bergweiler and A. Eremenko [2] proved this for functions of finite order. Recently, Q. Zhang [16] extended Hennekemper's result (1) for $k=1$ and c is replaced by any small function $a(z)$ $(\not\equiv 0)$ of f, i.e. $a(z)$ satisfies $T(r, a)=S(r, f)$. W. Bergweiler [1] proved that if f is a transcendental meromorphic function of finite order and if a is a polynomial which does not vanish identically, then $ff'-a$ has infinitely many zeros.

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Recently a conjecture was raised in [14], it states that for any integer $k \geqq 2$, and nonzero value c, the function $ff^{(k)}-c$ has infinitely many zeros. More generally is it true that for any non-negative integers $n, n_1, \cdots, n_k$ with $n=1$, $n_1+\cdots+n_k \geqq 1$ and for any non-zero value c, the function $f^n(f')^{n_1}\cdots(f^{(k)})^{n_k}-c$ always has infinitely many zeros? C.C. Yang, L. Yang and Y.F. Wang [15] proved that if f is a transcendental entire function and if n, k are non-negative integers with $n \geqq 2$ and $k \geqq 0$, then the only possible Picard value of $f(f^{(k)})^n$ is the value zero. In this note we shall provide some results in towards to a complete solution of the conjecture.

THEOREM 1. *Let f be a transcendental entire function of finite order. Then there exists at most one integer $k \geqq 2$, such that $ff^{(k)}$ has a non-zero and finite exceptional value.*

THEOREM 2. *Take non-negative integers $n, n_1, \cdots, n_k$ with $n \geqq 1$, $n_1+\cdots+n_k \geqq 1$ and define $d=n+n_1+\cdots+n_k$. Let f be a transcendental meromorphic function with $\delta(0, f)>3/(3d+1)$. Then for any non-zero value c, the function $f^n(f')^{n_1}\cdots(f^{(k)})^{n_k}-c$ always has infinitely many zeros.*

THEOREM 3. *Take $c_1, c_2, c_3 \in C$, $k \in N$ with $K_0=2k+1+2c_1 \neq 0$, and define*

$$P(f)=(f')^k+c_1 f(f')^{k-2}f''+c_2 f^2(f')^{k-4}(f'')^2+c_3 f^2(f')^{k-3}f''' \tag{2}$$

where $c_1=c_2=c_3=0$ if $k=1$, $c_2=c_3=0$ if $k=2$ and $c_2=0$ if $k=3$. Write

$$K_1=3k+1+6c_1+6c_3, \quad K_3=\frac{3}{K_0}(k^2+2(k-1)c_1+2c_2), \quad K_2=K_0-K_3 \tag{3}$$

$$A=K_0^2K_1-K_0K_1(k+1)+(K_1-K_0K_2)(k+1)^2 \tag{4}$$

$$B=K_0^2K_1+K_0(3K_0-2K_1)(k+1) \tag{5}$$

Assume that (i) $c_1+c_2+c_3 \neq -1$ and $(1+c_1+c_2+c_3)X^2+(c_1+2c_2+3c_3)X+c_2+2c_3=0$ has no roots of positive integers; (ii) $A \neq 0$ and $B/A \notin N-\{1\}$. If f is a transcendental meromorphic function, then, for $c \in C-\{0\}$,

$$T(r, f) \leqq \left(4+\frac{1}{k+1}\right)\left\{\bar{N}(r, f)+\bar{N}\left(r, \frac{1}{fP(f)-c}\right)\right\}+S(r, f). \tag{6}$$

Remarks. If $k=1$, Theorem 3 yields Clunie's result. If $n \geqq 2$, then the deficient condition in Theorem 2 can be omitted, see e.g. [11]. Furthermore, when $n=1$, $d \geqq 1$, and $N_1(r, 1/f)=S(r, f)$, then Zhang-Li [17] obtained some results similar to that of Theorem 2. Also see [10].

2. Proof of Theorem 1

We first prove the following key point in the verification of Theorem 1.

LEMMA. *Let* $p_i(z)$ $(\not\equiv 0,\ i=1,\ 2)$ *and* $q_i(z)$ $(\not\equiv$*constants*, $i=1,\ 2)$ *be some polynomials, with* $\deg q_1=\deg q_2$, *and* a_1 *and* a_2 *are two constants. If*

$$m\left(r,\ \frac{a_1+p_1e^{q_1}}{a_2+p_2e^{q_2}}\right)=o(1)T(r,\ e^{q_i})\,;\quad i=1,\ 2,$$

then q_1 *and* q_2 *have the same leading coefficient.*

Proof. By definition,

$$m\left(r,\ \frac{a_1+p_1e^{q_1}}{a_2+p_2e^{q_2}}\right)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{a_1+p_1(re^{i\theta})e^{q_1(re^{i\theta})}}{a_2+p_2(re^{i\theta})e^{q_2(re^{i\theta})}}\right|d\theta\,;\quad z=re^{i\theta}$$
$$=I.$$

Let $q_j(z)=b_{j_0}z^q+b_{j_1}z^{q-1}+\cdots+b_{j_{q-1}}z+b_{j_q}$; $q\geqq 1$, $j=1,\ 2$ and $b_{j_0}\neq 0$.

Then clearly on $|z|=r$, for $z\in J_j(\theta)=\{\theta\,|\,\mathrm{Re}\, b_{j_0}e^{iq\theta}>0\}$ $j=1,\ 2$, the value $|a_j+p_j(re^{i\theta})e^{q_j(re^{i\theta})}|$, $(j=1,\ 2)$ is dominated by the term $e^{b_{j_0}z^q}$. Thus if $b_{10}\neq b_{20}$ (either $\arg b_{10}\neq\arg b_{20}$ or $\arg b_{10}=\arg b_{20}$), the integrand in I, when $r>r_0$ will be $\geqq\varepsilon r^q$ for some fixed $\varepsilon>0$ along a piece of arc J_θ^* on $|z|=r$ with meas $J_\theta^*\geqq c_0>0$, c_0 is a fixed positive number depends on q. Hence there exists a positive number ε_0 such that when $r>r_0$

$$m\left(r,\ \frac{a_1+p_1e^{q_1}}{a_2+p_2e^{q_2}}\right)\geqq\varepsilon_0 r^q\neq o(1)T(r,\ e^{q_i})\,;\quad i=1,\ 2,$$

a contradiction. The lemma is thus proved.

Now we proceed to prove the theorem by contradiction.

Assume that there exist two distinct integers k_1 and k_2; $k_1>k_2\geqq 2$ and constants a_1 and a_2 such that both $ff^{(k_1)}-a_1$ and $ff^{(k_2)}-a_2$ have only finitely many zeros. By this and the fact that f is of finite order, we have

$$ff^{(k_1)}-a_1=p_1(z)e^{q_1(z)}\tag{7}$$

and

$$ff^{(k_2)}-a_2=p_2(z)e^{q_2(z)},\tag{8}$$

where p_i $(\not\equiv 0)$ and $q_i(z)$ $(\not\equiv$constant) are polynomials.

It follows that

$$ff^{(k_1)}=a_1+p_1e^{q_1}\tag{9}$$

and

$$ff^{(k_2)}=a_2+p_2e^{q_2}.\tag{10}$$

Hence

$$\frac{f^{(k_1)}}{f^{(k_2)}}=\frac{a_1+p_1e^{q_1}}{a_2+p_2e^{q_2}}.\tag{11}$$

We recall that $k_1>k_2$. By applying the lemma of logarithmic derivative to (11), we have

$$m\left(r, \frac{f^{(k_1)}}{f^{(k_2)}}\right)=S(r, f^{(k_2)})=S(r, f) \tag{12}$$
$$=o(1)T(r, f).$$

On the other hand, it is not difficult to verify that for any entire function f of finite order, $ff^{(k)}$ and f have the same order. Thus we have $\deg q_1=\deg q_2$. Hence it follows from (12) and the lemma we conclude immediately that q_1 and q_2 have the same leading coefficient. Hence

$$e^{q_i(z)}=e^{bz^q}A_i(z)\,;\quad i=1,\,2\,;$$

where $b=b_{10}=b_{20}$, and $A_i(z)$ $(i=1, 2)$ are functions of order no greater than $q-1$. In the following we shall treat two cases (i) $a_1=a_2$ and (ii) $a_1\neq a_2$ separately.

Case (i) $a_1=a_2$. Then from equations (9) and (10), we obtain

$$f(f^{(k_1)}-f^{(k_2)})=e^{bz^q}(p_1A_1-p_2A_2). \tag{13}$$

If $f^{(k_1)}-f^{(k_2)}\equiv 0$ then this yields $f(z)=\sum_{i=1}^{q} c_i e^{\rho_i z}$, where ρ_i $(i=1, 2, \cdots, q)$ are roots of the unity and c_i $(i=1, 2, \cdots, q)$ are constants.

Substituting the form of f into (9) or (10) and then applying lemma on Borel type of identity [12], we can easily derive a contradiction.

Thus $f^{(k_1)}-f^{(k_2)}\not\equiv 0$ and it follows from (13) and the fact that the order of $(p_1A_1-p_2A_2)\leqq q-1$, we have

$$N\left(r, \frac{1}{f}\right)\leqq N\left(r, \frac{1}{p_1A_1-p_2A_2}\right)$$
$$=O(1)r^{q-1}.$$

Hence f assume the form:

$$f(z)=h(z)e^{cz^q}, \tag{14}$$

where $T(r, h)=O(1)r^{q-1}$ and c is a constant $\neq 0$.

Substituting this into either (9) or (10) and applying lemma on Borel type of identity again, we will derive a contradiction.

Now case (ii): $a_1\neq a_2$. Let $a_1/a_2=d$ for some constant d $(\neq 0)$. Then equation (10) is equivalent to

$$dff^{(k_2)}=a_2d+dp_2e^{q_2}=a_1+dp_2e^{q_2}.$$

Combining this and (9), and by following arguments similar to case (i)'s proof, we will arrive at the same contradiction.

This also completes the proof of the theorem.

3. Proof of Theorem 2

We first prove the following key point in the verification of Theorem 2.

THEOREM 4. *Take non-negative integers* n, $n_1, \cdots, n_k$ *with* $n_1+\cdots+n_k\geqq 1$ *and define*

$$d=n+n_1+\cdots+n_k, \quad w=n+2n_1+3n_2+\cdots+(k+1)n_k. \tag{15}$$

Let f *be a transcendental meromorphic function and set*

$$\psi=af^n(f')^{n_1}\cdots(f^{(k)})^{n_k} \quad (a\in C-\{0\}). \tag{16}$$

Then either

$$(3d-2)T(r, f)<(3d+1)N\left(r, \frac{1}{f}\right)+4\bar{N}\left(r, \frac{1}{\psi-1}\right)-2N\left(r, \frac{\psi-1}{\psi'}\right)+S(r, f) \tag{17}$$

or ψ *satisfies the identity*

$$(w-n)\frac{\psi''}{\psi'}-\left(\frac{n}{w}+w+1-n\right)\frac{\psi'}{\psi-1}+2n\frac{f'}{f}=0. \tag{18}$$

Proof. If $N_1(r, f)$ counts simple poles of f and if $\bar{N}_2(r, f)$ counts multiple poles of f, each counted only once irrespective of multiplicity, then

$$\bar{N}(r, f)=N_1(r, f)+\bar{N}_2(r, f), \tag{19}$$

$$N_1(r, f)+2\bar{N}_2(r, f)\leqq N(r, f)\leqq T(r, f). \tag{20}$$

Note that (see Hu [8])

$$dT(r, f)\leqq\bar{N}(r, f)+dN\left(r, \frac{1}{f}\right)+\bar{N}\left(r, \frac{1}{\psi-1}\right)-N\left(r, \frac{\psi-1}{\psi'}\right)+S(r, f). \tag{21}$$

Then (20) and (21) imply

$$(d-1)N_1(r, f)+(2d-1)\bar{N}_2(r, f)\leqq dN\left(r, \frac{1}{f}\right)+\bar{N}\left(r, \frac{1}{\psi-1}\right)-N\left(r, \frac{\psi-1}{\psi'}\right) +S(r, f). \tag{22}$$

Suppose that z_0 is a simple pole of f. Then we may write

$$f(z)=\frac{b}{z-z_0}+\sum_{i=0}^{\infty}b_i(z-z_0)^i \quad (b\neq 0) \tag{23}$$

near $z=z_0$. Consequently

$$\psi(z)=\frac{\lambda}{(z-z_0)^w}\left\{1+\frac{nb_0}{b}(z-z_0)+\cdots\right\} \tag{24}$$

where

$$\lambda = ab^{d}(-1)^{w-d}\prod_{i=1}^{k}(i\,!)^{n_i}. \tag{25}$$

Then we have

$$\frac{\phi'(z)}{\phi(z)-1} = -\frac{w}{z-z_0} + \frac{nb_0}{b} + \cdots \tag{26}$$

$$\frac{\phi''(z)}{\phi'(z)} = -\frac{w+1}{z-z_0} + \frac{n(w-1)b_0}{wb} + \cdots \tag{27}$$

near $z=z_0$. Combining (26) and (27) we obtain, near $z=z_0$,

$$w\frac{\phi''(z)}{\phi'(z)} - (w+1)\frac{\phi'(z)}{\phi(z)-1} = -2\frac{nb_0}{b} + O(z-z_0). \tag{28}$$

Noting that, near $z=z_0$,

$$\frac{f'(z)}{f(z)} = -\frac{1}{z-z_0} + \frac{b_0}{b} + \cdots. \tag{29}$$

Then

$$w\frac{f'(z)}{f(z)} - \frac{\phi'(z)}{\phi(z)-1} = (w-n)\frac{b_0}{b} + O(z-z_0). \tag{30}$$

By (28), (30), it follows that, near $z=z_0$,

$$\begin{aligned} h(z) &:= w(w-n)\frac{\phi''(z)}{\phi'(z)} - \{(w-n)(w+1)+2n\}\frac{\phi'(z)}{\phi(z)-1} + 2nw\frac{f'(z)}{f(z)} \\ &= O(z-z_0). \end{aligned} \tag{31}$$

Now suppose that (18) is not true, i.e., $h\not\equiv 0$. Then $h(z_0)=0$, and thus we have

$$N_1(r, f) \leqq N\left(r, \frac{1}{h}\right). \tag{32}$$

Applying Jensen's formula to h, (32) yields

$$N_1(r, f) \leqq m(r, h) + N(r, h) + O(1) = N(r, h) + S(r, f). \tag{33}$$

Note that $h(z)$ can only have simple poles at zeros and poles of ϕ', $\phi-1$ and f which are not simple poles of f. Thus

$$N(r, h) \leqq \overline{N}_2(r, f) + \varepsilon\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{\phi-1}\right) + \overline{N}\left(r, \frac{\phi-1}{\phi'}\right) \tag{34}$$

where $\varepsilon=1$ if $n\neq 0$, and $\varepsilon=0$ if $n=0$. Therefore by (22), (33) and (34), we have

$$(35)\qquad (3d-2)N_1(r, f)\leqq dN\left(r, \frac{1}{f}\right)+\varepsilon(2d-1)\bar{N}\left(r, \frac{1}{f}\right)+2d\bar{N}\left(r, \frac{1}{\psi-1}\right)$$
$$+2(d-1)N\left(r, \frac{\psi-1}{\psi'}\right)+S(r, f).$$

Multiply (35) by $d/(3d-2)$ and add to (22) to obtain

$$(36)\qquad (3d-2)\bar{N}(r, f)\leqq 2dN\left(r, \frac{1}{f}\right)+\varepsilon d\bar{N}\left(r, \frac{1}{f}\right)+(d+2)\bar{N}\left(r, \frac{1}{\psi-1}\right)$$
$$+(d-2)N\left(r, \frac{\psi-1}{\psi'}\right)+S(r, f).$$

Hence (21) and (36) yield

$$(37)\qquad (3d-2)T(r, f)\leqq 3dN\left(r, \frac{1}{f}\right)+\varepsilon\bar{N}\left(r, \frac{1}{f}\right)+4\bar{N}\left(r, \frac{1}{\psi-1}\right)$$
$$-2N\left(r, \frac{\psi-1}{\psi'}\right)+S(r, f).$$

Now (17) follows from (37). Q.E.D.

Now we proceed to prove Theorem 2 by contradiction.

Assume that there exists $c\in C-\{0\}$ such that $f^n(f')^{n_1}\cdots(f^{(k)})^{n_k}-c$ has only finitely many zeros. Set $a=1/c$ and define ψ by (16). Then

$$(38)\qquad N\left(r, \frac{1}{\psi-1}\right)=O(\log r)=S(r, f).$$

If (17) is true, then

$$(3d-2)T(r, f)<(3d+1)N\left(r, \frac{1}{f}\right)+S(r, f).$$

It follows that

$$(3d-2)\leqq(3d+1)(1-\delta(0, f))$$

i.e., $\delta(0, f)\leqq 3/(3d+1)$, which is a contradiction.

Now suppose that (18) is true. Integrating (18) to obtain

$$(39)\qquad (\psi')^m/(\psi-1)^l=c_1/f^{2n}$$

where $m=w-n$, $l=(n/w)+w+1-n$, $c_1\in C-\{0\}$. Solving (39) to obtain

$$(40)\qquad \frac{1}{(\psi-1)^{w+n}}=\left\{c_3+c_2\int^z\frac{ds}{f^{2n(w-n)}(s)}\right\}^{w(w-n)}.$$

Then (38) and (40) (or (39)) yield

$$(41)\qquad N\left(r, \frac{1}{f}\right)=O\left\{N\left(r, \frac{1}{\psi-1}\right)\right\}=S(r, f).$$

Hence (21), (38) and (41) imply

$$(d-1)T(r, f) \leqq S(r, f)$$

which is impossible.

Thus we complete the proof of Theorem 2.

4. Proof of Theorem 3

Set $b=1/c$ and consider the following differential polynomial

$$\psi = bfP(f) \tag{42}$$

where $P(f)$ is defined by (2). Define

$$h=-\frac{\psi'}{\psi-1}, \quad B_0=-\frac{1}{f}\left\{K_0 f''+\left(h-\frac{h'}{h}\right)f'\right\} \tag{43}$$

and

$$\begin{aligned} F=&(K_1-K_0K_2)h^4+(K_0K_3+K_1-K_0K_1)(h')^2 \\ &+(2K_0^2-K_0K_1+K_0K_2-K_0K_3-2K_1)h^2h' \\ &+(K_0K_1-K_0^2)hh''-K_0K_1h^2B_0. \end{aligned} \tag{44}$$

According to W. Hennekemper [7], we prove the following properties:

(iii) If z_0 is a p-fold zero of f, then z_0 is at least a p-fold zero of F;

(iv) F don't vanish identically;

(v) $T(r, F) \leqq 4\overline{N}(r, f)+4\overline{N}(r, 1/(\psi-1))+S(r, f)$.

Firstly, we prove (iii). Trivially one see

$$\psi'+h\psi-h=0. \tag{45}$$

Differentiating (45) successively we obtain

$$\psi''+h\psi'+h'\psi-h'=0, \tag{46}$$

$$\psi'''+h\psi''+2h'\psi'+h''\psi-h''=0. \tag{47}$$

If $p=1$, then by (42) and (2), we obtain

$$\psi(z_0)=0, \tag{48}$$

$$\psi'(z_0)=b(f'(z_0))^{k+1}, \tag{49}$$

$$\psi''(z_0)=bK_0(f'(z_0))^k f''(z_0), \tag{50}$$

$$\psi'''(z_0)=bK_1(f'(z_0))^k f'''(z_0)+bK_0K_3(f'(z_0))^{k-1}(f''(z_0))^2. \tag{51}$$

By (45), (48) and (49), we have

$$h(z_0)=b(f'(z_0))^{k+1}. \tag{52}$$

By (46), (48)-(50) and (52), then

$$h'(z_0)=(h(z_0))^2+K_0h(z_0)f''(z_0)/f'(z_0) \tag{53}$$

and (47)-(53) yield

$$h''(z_0)=K_1h(z_0)\frac{f'''(z_0)}{f'(z_0)}+(K_2h(z_0)^2+K_3h'(z_0))\frac{f''(z_0)}{f'(z_0)}+2h(z_0)h'(z_0). \tag{54}$$

Note that B_0 is regular at $z=z_0$ according to (52) and (53), and that

$$B_0(z_0)=-K_0\frac{f'''(z_0)}{f'(z_0)}-\left(h(z_0)-\frac{h'(z_0)}{h(z_0)}\right)\frac{f''(z_0)}{f'(z_0)}-h'(z_0)+\frac{h''(z_0)}{h(z_0)}-\left(\frac{h'(z_0)}{h(z_0)}\right)^2. \tag{55}$$

Substituting (52)-(55) into (44), by simple computation we see $F(z_0)=0$.

If $p\geqq 2$, then z_0 is at least $(k+1)(p-1)$-fold zero of h. Hence z_0 is at least zero point of F of order $2(k+1)(p-1)-2(\geqq p)$. Thus we obtain (iii).

Next we prove (iv). If z_0 is a pole of f of order p, writing

$$f(z)=\frac{a}{(z-z_0)^p}+\cdots \quad (a\in C-\{0\}),$$

near $z=z_0$. We have, near $z=z_0$,

$$\phi=a\tilde{a}/(z-z_0)^{k(p+1)+p}+\cdots$$

where

$$\tilde{a}=b(-a)^k p^{k-2}\{(1+c_1+c_2+c_3)p^2+(c_1+2c_2+3c_3)p+c_2+2c_3\}. \tag{56}$$

By the condition (i), we see $\tilde{a}\neq 0$. Hence, near $z=z_0$, we have Laurent expansions

$$h=(k(p+1)+p)/(z-z_0)+\cdots$$
$$B_0=\{p(k(p+1)+p+1)-K_0p(p+1)\}/(z-z_0)^2+\cdots$$
$$F=a'/(z-z_0)^4+\cdots$$

where

$$a'=(k(p+1)+p)^2(p+1)\{A(p+1)-B\}. \tag{57}$$

The condition (ii) implies $a'\neq 0$, thus $F\not\equiv 0$.

Now we proceed to prove (iv) for entire functions f by contradiction. Assume that there exists an entire function f such that $F\equiv 0$. Then we have

$$\begin{aligned} &K_0^2K_1\phi(\phi')^2f''-K_0K_1\phi\phi'\phi''f'+(K_0K_3-K_0K_1+K_1)\phi(\phi'')^2f \\ &+(K_0^2-K_0K_3-K_0K_2)(\phi')^2\phi''f+(K_0K_1-K_0^2)\phi\phi'\phi'''f \\ &-K_0^2K_1(\phi')^2f''+K_0K_1\phi'\phi''f'-(K_0K_3+K_1-K_0K_1)(\phi'')^2f \\ &-(K_0K_1-K_0^2)\phi'\phi'''f\equiv 0. \end{aligned} \tag{58}$$

Note that the sum of the coefficients of ψ is

$$a_0=b(1+c_1+c_2+c_3)\neq 0.$$

Obviously, the sum of the coefficients of $\psi^{(l)}$ is $(k+1)^l a_0$. Hence the sum of the coefficients of former five summands in (58) is

$$\begin{aligned}\sigma_1=&a_0^3\{K_0^2K_1(k+1)^2-K_0K_1(k+1)^3+(K_0K_3-K_0K_1+K_1)(k+1)^4\\&+(K_1^2-K_0K_3-K_0K_2)(k+1)^4+(K_0K_1-K_0^2)(k+1)^4\\=&a_0^3(k+1)^2A\neq 0.\end{aligned}$$

Let σ_2 denote the sum of the coefficients of later four summands in (58). By using the theory of Wiman-Valiron (see [13]), there exists complex sequences $\{\xi_n\}$, $\{\delta_n\}$ and $\{\Delta_n\}$ $(n=1, 2, \cdots)$ such that

$$\lim_{n\to\infty}|\xi_n|=\infty, \quad \lim_{n\to\infty}\delta_n=\lim_{n\to\infty}\Delta_n=0,$$

and for all $n\in N$,

$$0<|f(\xi_n)|=M(|\xi_n|, f), \quad \nu(|\xi_n|)\geqq 1, \tag{60}$$

$$\left(\frac{\nu(|\xi_n|)}{\xi_n}\right)^{3k+4}f^{3k+4}(\xi_n)\{\sigma_1+\delta_n\}+\left(\frac{\nu(|\xi_n|)}{\xi_n}\right)^{2k+4}f^{2k+3}(\xi_n)\{\sigma_2+\Delta_n\}=0 \tag{61}$$

where we apply the formula (8) of Chapter I from [13], into (58) to obtain (61). Thus for all $n\in N$, we have

$$\sigma_1+\delta_n=-(\sigma_2+\Delta_n)\left(\frac{\xi_n}{\nu(|\xi_n|)}\right)^k/f^{k+1}(\xi_n).$$

which implies $\sigma_1=0$ since f is transcendental with

$$\lim_{n\to\infty}\frac{\xi_n^k}{f^{k+1}(\xi_n)}=0.$$

We get a contradiction. This complets the proof of (iv).

By the definition of F, we see

$$m(r, F)=S(r, f). \tag{62}$$

Note that poles of F only come from poles of f and zeros of $\psi-1$; its multiplicities are 4 at most. Hence

$$N(r, F)\leqq 4\bar{N}(r, f)+4\bar{N}\left(r, \frac{1}{\psi-1}\right). \tag{63}$$

Thus (v) follows.

Finally, we prove Theorem 3. By (iii), (iv) and (v), we have

$$N\left(r, \frac{1}{f}\right)\leqq N\left(r, \frac{1}{F}\right)\leqq T(r, F)+O(1)\leqq 4\bar{N}\left(r, \frac{1}{\psi-1}\right)+4\bar{N}(r, f)+S(r, f) \tag{64}$$

Theorem 1 of Hu [8] yields

$$T(r, f) \leqq \frac{1}{k+1}\bar{N}(r, f)+N\left(r, \frac{1}{f}\right)+\frac{1}{k+1}\bar{N}\left(r, \frac{1}{\phi-1}\right)+S(r, f). \tag{65}$$

Then (6) follows from (64) and (65). Q.E.D.

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