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KODAI MATH. J.
18 (1995), 481—486

STIEFEL-WHITNEY HOMOLOGY CLASSES AND EULER SUBSPACES

Dedicated to Professor Seiya Sasao on his 60th birthday

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1. Introduction and the statement of results

In [5], we gave the construction of integral (or mod 2) Euler spaces of a given homotopy type such that the Stiefel-Whitney homology classes are equal to any given homology elements. In this paper, we give a new construction for mod 2 Euler spaces embedded in a given mod 2 Euler spaces such that the Stiefel-Whitney homology classes are equal to any given homology elements of the given space.

Let X be a locally compact n-dimensional polyhedron. For a point $x \in X$, let $\chi(X, X-x)$ denote the Euler number of the pair $(X, X-x)$. The polyhedron X is called a mod 2 Euler space if for each $x \in X$, $\chi(X, X-x) \equiv 1 \pmod 2$ (cf. [1], [3]). Let K' denote the barycentric subdivision of a triangulation K of a polyhedron X. If X is a mod 2 Euler space, the sum of all k-simplexes in K' is a mod 2 cycle and define an element $s_k(X)$ in $H_k(X, \boldsymbol{Z}_2)$ (cf. [3]). The element $s_k(X)$ is called the k-th Stiefel-Whitney homology classes of X. If X is a smooth manifold, PL-manifold or $\boldsymbol{Z}_2$-homology manifold, the class $s_k(X)$ is known to be the Poincaré dual of the Stiefel-Whitney class $w^{n-k}(X)$ ([2], [3], [4], [10]). Consequently, for such spaces, the Stiefel-Whitney homology classes are homotopy invariant. But in the category of mod 2 Euler spaces, Stiefel-Whitney homology classes are not generally homotopy invariant ([4], [5]). A polyhedron X is called purely n-dimensional if the union of all n-simplexes of a triangulation of X is dense in X. In such case, X is said to be an n-dimensional polyhedron of pure dimension. Our theorem is the following:

THEOREM. *Let X be an n-dimensional* mod 2 *Euler space of pure dimension and let α_i be homology elements of $H_i(X; \boldsymbol{Z}_2)$ for $i=0, 1, \cdots, n-1$. Then for any $k \leqq n-1$, there exist a k-dimensional compact* mod 2 *Euler space Y of pure dimension and a PL-embedding $f: Y \to X$ such that $f_* s_i(Y) = \alpha_i$, for all $i \leqq k$.*

Received April 25, 1994; revised December 22, 1994.

2. Proof of Theorem

To prove Theorem, we need the following lemmas. We devote section 3 and 4 to prove Lemma 2.

LEMMA 1. *Let X be an n-dimensional compact* mod 2 *Euler space of pure dimension. Let α be a homology element of $H_k(X; \boldsymbol{Z}_2)$ where $k \leqq n-1$. Then there exist a k-dimensional compact* mod 2 *Euler space Z of pure dimension and a PL-embedding $g: Z \to X$ such that $g_* s_k(Z) = \alpha$, and for $0 \leqq i \leqq k-1$, $g_*: H_i(Z; \boldsymbol{Z}_2) \to H_i(X; \boldsymbol{Z}_2)$ is a surjection.*

Proof. Let T be a triangulation of X. Let c be a mod 2 k-cycle in T which determines the homology class α. Let T' be the barycentric subdivision of T. Then the cycle c is subdivided to the mod 2 k-cycle c' in T', which is denoted by $c' = \sum_{\lambda \in \Lambda} \sigma_\lambda$. We denote by Λ^c the set all k-simplexes of $T' - \{\sigma_\lambda\}_{\lambda \in \Lambda}$. Let T^k be the k-skelton of T'. Then $|T^k|$ is a mod 2 Euler space (Proposition 2.1 of [4]). For each $\tau \in \Lambda^c$, we choose a P_τ in X such that P_τ is joinable to τ and such that $\mathrm{Int}(P_\tau * \tau) \cap \mathrm{Int}(P_{\tau'} * \tau') = \emptyset$ for different τ, τ' in Λ^c and $(P_\tau * \tau) \cap |T^k| = \tau$. We define Z by $Z = |T^k| \cup (\bigcup_{\tau \in \Lambda^c} P_\tau * \partial\tau)$. Note that $P_\tau * \partial\tau$ is a k-dimensional PL-ball. Since, for $x \in \partial\tau$, $\#\{\tau' \in \Lambda^c \mid \partial\tau' \ni x\} \equiv 0 \pmod 2$, it follows that Z is a k-dimensional compact mod 2 Euler space of pure dimension. Let $g: Z \to X$ be the inclusion. Then by the construction we have $g_* s_k(Z) = \alpha$, and for $0 \leqq i \leqq k-1$, $g_*: H_i(Z; \boldsymbol{Z}_2) \to H_i(X; \boldsymbol{Z}_2)$ are surjections. q.e.d.

LEMMA 2. *Let $g: Z \to X$ be a PL-embedding of compact* mod 2 *Euler spaces of pure dimension. Assume that $k < n$, where* $\dim X = n$, $\dim Z = k$. *For $i = 1, \cdots, k-1$, let β_i be a homology element of $H_i(Z; \boldsymbol{Z}_2)$. Then there exist a k-dimensional compact* mod 2 *Euler space Y, and furthermore a homotopy equivalence $h: Z \to Y$ and a PL-embedding $f: Y \to X$ such that g is homotopic to $f \circ h$, $h_* \beta_i = s_i(Y)$ for $i = 1, \cdots, k-1$, and $h_* s_k(Z) = s_k(Y)$.*

In Lemma 2, the construction of Y and $h: Z \to Y$ is analogous to that in [5]. But we need to construct a PL-embedding $f: Y \to X$.

Proof of Theorem. By Lemma 1, there exist a k-dimensional compact mod 2 Euler space Z of pure dimension and a PL-embedding $g: Z \to X$ such that $g_* s_k(Z) = \alpha_k$, and for $1 \leqq i \leqq k-1$, $g_*: H_i(Z; \boldsymbol{Z}_2) \to H_i(X; \boldsymbol{Z}_2)$ is a surjection. Note that $\chi(Z \vee S^k) \equiv \chi(Z) + 1 \pmod 2$ and $\chi(S^k \vee S^k) \equiv 1 \pmod 2$. Then we may assume that $g_* s_0(Z) = \alpha_0$. Let β_i be the element of $H_i(Z; \boldsymbol{Z}_2)$ such that $g_* \beta_i = \alpha_i$ for $i = 0, 1, \cdots, k-1$. By Lemma 2, there exist a k-dimensional compact mod 2 Euler space Y of pure dimension, and furthermore a homotopy equivalence $h: Z \to Y$ and a PL-embedding $f: Y \to X$ such that g is homotopic to $f \circ h$, $h_* \beta_i = s_i(Y)$ for $i = 0, 1, \cdots, k-1$, and $h_* \beta_k = s_k(Y)$. Then $f_* s_i(Y) = f_* h_* \beta_i = g_* \beta_i = \alpha_i$. q.e.d.

3. An elementary lemma for simplexes

In this section, we consider quotient spaces of the simplex and prove an elementary lemma which is necessary to prove Lemma 2. Let Δ^n and Δ^k be the n-simplex $\langle v_0, v_1, \cdots, v_n\rangle$ and the k-simplex $\langle v_0, v_1, \cdots, v_k\rangle$, respectively, where $k<n$. We will construct a quotient space $\hat{\Delta}^k$ of Δ^k and a PL-embedding $g_\Delta: \hat{\Delta}^k \to \Delta^n$. Let σ^p be the p-simplex $\langle v_0, v_1, \cdots, v_p\rangle$ in Δ^k, where $p<k$. Let τ^p be a p-simplex $\langle v_0, u_1, \cdots, u_p\rangle$ which is linearly embedding in Δ^k such that $\tau^p \cap \partial\Delta^k = v_0$. Let $\alpha: \sigma^p \to \tau^p$ be the linear map such that $\alpha(v_0)=v_0$ and $\alpha(v_i)=u_i$ for $i=1, 2, \cdots, p$. We introduce an equivalence relation $\sim$ on Δ^k as follows: $x\sim y$ if $x=y$ or if $\alpha(x)=y$ for $x\in\sigma^p$. Let $\hat{\Delta}^k$ be the quotient space $\Delta^k/\sim$ and $h_\Delta: \Delta^k \to \hat{\Delta}^k$ the projection. Let $i: \partial\Delta^k \to \Delta^k$ and $j: \Delta^k \to \Delta^n$ be the inclusions. We need the following lemma to prove Lemma 1:

LEMMA 3. *Let Δ^n, Δ^k, σ^p and τ^p be simplexes such that $\sigma^p \prec \Delta^k \prec \Delta^n$ and that τ^p is a linearsubspace of Δ^k as aboves, and let $\hat{\Delta}^k$ be the quotient space $\Delta^k/\sim$. Then there exists a PL-embedding $g_\Delta: \hat{\Delta}^k \to \Delta^n$ such that $g_\Delta \circ h_\Delta \circ i = j \circ i$ and $g_\Delta(\hat{\Delta}^k - h_\Delta \circ i(\partial\Delta^k)) \subset \operatorname{Int} \Delta^n$.*

Proof. Let τ^p be as aboves. Let τ' be any p-simplex $\langle v_0, u_1', \cdots, u_p'\rangle$ which is linear embedding in Δ^k such that $\tau' \cap \partial\Delta^k = v_0$. Then there exists a *PL*- homeomorphism $h: \Delta^k \to \Delta^k$ such that

(1) $h|\partial\Delta^k$ is the identity,

(2) $h(\tau^p)=\tau'$,

(3) $h(u_i)=u_i'$ for $i=1, 2, \cdots, p$, and

(4) $h|\tau^p: \tau^p \to \tau'$ is linear.

By the above, we may prove the lemma for a certain τ^p. First we define τ^p as follows. Let G be the barycenter of Δ^k. Put $u_0=v_0$. We denote by $\operatorname{Int} X$ the interior of X. For $i=1, 2, \cdots, k$, we choose points u_i in $\operatorname{Int}\langle v_i, G\rangle$. We define τ^p by $\tau^p=\langle u_0, u_1, \cdots, u_p\rangle$. Next, to construct $g_\Delta: \hat{\Delta}^k \to \Delta^n$, we construct a subset $\hat{\Delta}'$ of Δ^n which is *PL*-homeomorphic to $\hat{\Delta}^k$. Let G_0 be the barycenter of the simplex $\langle v_{k+1}, v_{k+2}, \cdots, v_n\rangle$. Let G_1 be a point in $\operatorname{Int} G_0 * \Delta^k$, where $X*Y$ is the join of X and Y. Put $a_0'=v_0$. For $i=1, 2, \cdots, k$, let a_i' be a point in $\operatorname{Int}\langle G_1, v_i\rangle$. We denote by A^k the simplex $\langle a_0', a_1', \cdots, a_k'\rangle$. Let G_2 be a point in $\operatorname{Int} G_1 * A^k$. Put $b_0'=v_0$. Put $B^k=\langle b_0', b_1', \cdots, b_k'\rangle$. For $i=1, 2, \cdots, k$, we define b_i' by $b_i'=\langle G_2, v_i\rangle \cap A^k$. For $i=0, 1, \cdots, p$, put $u_i'=v_i$. For $i=p+1, p+2, \cdots, k$, let u_i' be a point in $\operatorname{Int}\langle b_i', v_i\rangle$. We denote by C^k the simplex $\langle u_0', u_1', \cdots, u_k'\rangle$. We define $\hat{\Delta}'$ by $\hat{\Delta}'=(G_1*\partial\Delta^k - \operatorname{Int}(G_1*\partial A^k)) \cup (A^k - \operatorname{Int} B^k) \cup (G_2*\partial C^k - \operatorname{Int}(G_2*\partial B^k)) \cup C^k$. Put $a_0=b_0=v_0$. For $i=1, 2, \cdots, k$, we choose different points a_i, b_i in $\operatorname{Int}\langle v_i, u_i\rangle$ such that $\langle b_0, b_1, \cdots, b_k\rangle \prec \langle a_0, a_1, \cdots, a_k\rangle$. Put $A=\langle a_0, a_1, \cdots, a_k\rangle$, $B=\langle b_0, b_1, \cdots, b_k\rangle$, $C=\langle u_0, u_1, u_2, \cdots, u_k\rangle$. Then we have the decomposition of Δ^k as follows:

$$\Delta^k=(\Delta^k - \operatorname{Int} A) \cup (A - \operatorname{Int} B) \cup (B - \operatorname{Int} C) \cup C.$$

Considering the construction of $\hat{\Delta}'$ and the decomposition of Δ^k, we can construct a PL-homeomorphism $g'_\Delta : \hat{\Delta}^k \to \hat{\Delta}'$ such that, for $i=0, 1, \cdots, k$,

(1) $g'_\Delta(h_\Delta(a_i))=a'_i$

(2) $g'_\Delta(h_\Delta(b_i))=b'_i$

(3) $g'_\Delta(h_\Delta(u_i))=u'_i$

(4) $g'_\Delta \circ h_\Delta \circ i = h_\Delta \circ i$.

We define $g_\Delta : \hat{\Delta}^k \to \Delta^n$ by $g_\Delta = j' \circ g'_\Delta$, where $j' : \hat{\Delta}' \subset \Delta^n$ is the inclusion. Then $g_\Delta \circ h_\Delta \circ i = j \circ i$ and $g_\Delta(\hat{\Delta}^k - h_\Delta \circ i(\partial \Delta^k)) \subset \text{Int}\, \Delta^n$. q. e. d.

4. Proof of Lemma 2

We need the following lemma at the induction step in the proof of Lemma 2.

LEMMA 4. *Let $g: Z \to X$ be a PL-embedding of compact* mod 2 *Euler spaces of pure dimension. Let α be a homology element of $H_p(Z ; \boldsymbol{Z}_2)$. Suppose that $0<p<k<n$, where* $\dim X=n$ *and* $\dim Z=k$. *Then there exist compact k-dimensional* mod 2 *Euler space Y of pure dimension and futhermore a homotopy equivalence $h: Z \to Y$ and a PL-embedding $f: Y \to X$ such that $h_*(\alpha)=s_p(Y)$, $h_*(s_i(Z))=s_i(Y)$ for $p<i \leqq k$ and $f \circ h$ is homotopic to g.*

The following lemma (cf. Lemma 2.4 in [5]) is immediately induced from the definition of Stiefel-Whitney homology classes and homology groups. So we omit the proof.

LEMMA 5. *Let $h: K \to L$ be a surjective simplicial map, where $Z=|K|$ and $Y=|L|$ are compact* mod 2 *Euler spaces with same pure dimension. Let $c=\sum_{\lambda \in \Lambda} \sigma_\lambda$ be a* mod 2 *p-cycle, where $\{\sigma_\lambda\}_{\lambda \in \Lambda}$ is a set of p-simplexes in K. Suppose that $\# h^{-1}(y) \equiv 0 \pmod 2$ for $y \in h(\bigcup_{\lambda \in \Lambda} \text{Int}\, \sigma_\lambda)$ and $\# h^{-1}(y) \equiv 1 \pmod 2$ for $y \in h(Z - \bigcup_{\lambda \in \Lambda} \sigma_\lambda)$. Then $h_*(s_i(Z))=s_i(Y)$ for $i>p$ and $h_*(s_p(Z)-[c])=s_p(Y)$, where $[c]$ is a homology class in $H_p(Z ; Z_2)$ defined by the chain c.*

Proof of Lemma 4. Let T be a triangulation of X and let K be the subcomplex which is a triangulation of $g(Z)$. We may suppose that $\text{Link}(\sigma ; T) \cap \text{Link}(\sigma', T)=\emptyset$ for different k-simplexes σ and σ' in K. Let c be a mod 2 cycle which is a sum of p-simplexes of K, such that $[c]=s_p(Z)-\alpha$ in $H_p(Z ; Z_2)$. Let T' and K' be the barycentric subdivision of T and K, respectively. Let $c'=\sum_{\lambda \in \Lambda} \sigma^p_\lambda$ be the barycentric subdivision of c. For each $\lambda \in \Lambda$, choose a k-simplex Δ^k_λ in K' and an n-simplex Δ^n_λ in $T'-K'$ such that $\sigma^p_\lambda < \Delta^k_\lambda < \Delta^n_\lambda$. Let v_λ be a vertex of σ^p_λ. Choose p-simplex τ^p_λ linearly embedded in Δ^k_λ such that $\tau^p_\lambda \cap \partial \Delta^k_\lambda = v_\lambda$. As in Lemma 3, let $\hat{\Delta}^k_\lambda$ be the quotient space of Δ^k_λ and $h_\lambda : \Delta^k_\lambda \to \hat{\Delta}^k_\lambda$ the projection. Furthermore let $g_\lambda : \hat{\Delta}^k_\lambda \to \Delta^n_\lambda$ be the PL-embedding as in Lemma 3. Put $Y=(Z-\bigcup_{\lambda \in \Lambda} \Delta^k_\lambda) \cup (\bigcup_{\lambda \in \Lambda} \hat{\Delta}^k_\lambda)$. Then Y is a mod 2 Euler space of pure dimension. Define $h: Z \to Y$ by $h|\Delta^k_\lambda = h_\lambda$ for $\lambda \in \Lambda$ and $h|(Z-\bigcup_{\lambda \in \Lambda} \Delta^k_\lambda)$ is the identity. By the construction, h is a homotopy equivalence. Furthermore we have that $\#h^{-1}(y)\equiv 0 \pmod 2$ for $y\in h(\bigcup_{\lambda\in\Lambda}\mathrm{Int}\,\sigma_\lambda^p)$ and $\#h^{-1}(y)\equiv 1 \pmod 2$ for $y\in h(Z-\bigcup_{\lambda\in\Lambda}\sigma_\lambda^p)$. By Lemma 5, we have $h_*(s_i(Z))=s_i(Y)$ for $p<i\leqq k$ and $h_*(\alpha)=h_*(s_p(Z)-[c'])=s_p(Y)$. Define a PL-map $f:Y\to X$ by $f|\hat{\Delta}_\lambda^k=g_\lambda$ for $\lambda\in\Lambda$ and $f|(Z-\bigcup_{\lambda\in\Lambda}\Delta_\lambda^k)=g|(Z-\bigcup_{\lambda\in\Lambda}\Delta_\lambda^k)$. By the construction, we have $h\circ f$ is homotopic to g. q.e.d.

Proof of Lemma 2. For $i=1, 2, \cdots, k-1$, let β_{k-i} be a homology element of $H_{k-i}(Z\,;\boldsymbol{Z}_2)$. By the induction on i, we prove Lemma 2. By Lemma 4, there exist a k-dimensional compact mod 2 Euler space Y_1, and a homotopy equivalence $h_1:Z\to Y_1$ and PL-embedding $g_1:Y_1\to X$ such that $g_1\circ h_1$ is homotopic to g, $h_{1*}(s_k(Z))=s_k(Y_1)$ and $h_{1*}(\beta_{k-1})=s_{k-1}(Y_1)$. Next we assume that $i<k-1$ and that there exist a k-dimensional compact mod 2 Euler space Y_i of pure dimension, and furthermore a homotopy equivalence $h_i:Z\to Y_i$ and PL-embedding $g_i:Y_i\to X$ such that $g_i\circ h_i$ is homotopic to g, $h_{i*}(s_k(Z))=s_k(Y_i)$ and $h_{i*}(\beta_j)=s_j(Y_i)$ for $k-i\leqq j<k$. By Lemma 4, there exist a k-dimensional compact mod 2 Euler space Y_{i+1} of pure dimension and furthermore a homotopy equivalence $h'_{i+1}:Y_i\to Y_{i+1}$ and PL-embedding $g_{i+1}:Y_{i+1}\to X$ such that $g_{i+1}\circ h'_{i+1}$ is homotopic to g_i, $h'_{i+1*}(s_j(Y_i))=s_j(Y_{i+1})$ for $k-i\leqq j\leqq k$ and $h'_{i+1*}(h_{i*}(\beta_{k-i-1}))=s_{k-i-1}(Y_{i+1})$. Put $h_{i+1}=h'_{i+1}\circ h_i:Y\to Y_{i+1}$. Then $g_{i+1}\circ h_{i+1}$ is homotopic to g and $h_{i+1*}(\beta_j)=s_j(Y_{i+1})$ for $k-i-1\leqq j\leqq k$. Then, for $i=1, 2, \cdots, k-1$, there exist k-dimensional compact mod 2 Euler spaces Y_i of pure dimension, and homotopy equivalences $h_i:Z\to Y_i$ and *PL*-embeddings $g_i:Y_i\to X$ such that $g_i\circ h_i$ are homotopic to g, $h_{i*}(s_k(Z))=s_k(Y_i)$ and $h_{i*}(\beta_j)=s_j(Y_i)$ for $k-i\leqq j<k$. Put $Y=Y_{k-1}$, $f=g_{k-1}$: $Y\to X$ and $h=h_{k-1}:Z\to Y$. Then $f\circ h=g$, $h_*(\beta_j)=s_j(Y)$ for $1\leqq j\leqq k-1$ and $h_*(s_k(Z))=s_k(Y)$. q.e.d.

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