P. LI AND C.-C. YANG
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SOME FURTHER RESULTS ON THE UNIQUE RANGE SETS OF MEROMORPHIC FUNCTIONS

PING LI AND CHUNG-CHUN YANG*

Abstract

By improving a generalization of Borel's theorem, the authors have been able to show that there exists a finite set S with 15 elements such that for any two nonconstant meromorphic functions f and g the condition $E_f(S)=E_g(S)$ implies $f\equiv g$. As a special case this also answers an open question posed by Gross [1] about entire functions, and has improved some results obtained recently by Yi [10]. In the last section, the uniqueness polynomials of meromorphic functions which is related to the unique range sets has been studied. A necessary and sufficient condition for a polynomial of degree 4 to be a uniqueness polynomial is obtained.

1. Introduction

Let f be a nonconstant meromorphic function on the complex plane C and S be a subset of distinct elements in C. Define

$$E_f(S)=\bigcup_{a\in S}\{z \mid f(z)-a=0\},$$

here a zero of $f(z)-a$ of multiplicity m appears m times in $E_f(S)$. Usually, the notation $\bar{E}_f(S)$ express the set which contains the same points as $E_f(S)$ but without counting multiplicities. About sixty years ago, R. Nevanlinna [6] proved two general results: (1). If two nonconstant meromorphic functions f and g satisfy $\bar{E}_f(a_i)=\bar{E}_g(a_i)$ $(i=1, \cdots, 5)$ where a_i $(i=1, \cdots, 5)$ are distinct points in $\bar{C}$, then $f\equiv g$. (2) If two nonconstant meromorphic functions f and g satisfy $E_f(a_i)=E_g(a_i)$ $(i=1, \cdots, 4)$ where a_i $(i=1, \cdots, 4)$ are distinct points in $\bar{C}$, then f is a Möbius transformation of g. Actually, above notations S and $E_f(S)$ can be regarded as a range set and a preimage set of f respectively. Recent years, in several papers, for examples [1], [2], [4], [7] and [10], properties of range set and preimage set which can, to some extent, uniquely determine the mero-

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morphic or entire functions were studied. In 1976 Gross [1] proved that there exist three finite sets S_j $(j=1, 2, 3)$ such that for any two nonconstant entire functions f and g if $E_f(S_j)=E_g(S_j)$ $(j=1, 2, 3)$, then $f\equiv g$. In the same paper Gross posed the following problem: can one find two (or possible even one) finite sets S_j $(j=1, 2)$ such that any two entire functions f and g satisfying $E_f(S_j)=E_g(S_j)$ $(j=1, 2)$ must be identical? In 1982, F. Gross and C.C. Yang proved the following result

THEOREM A [2]. *Let* $T=\{z \mid e^z+z=0\}$. *Let* f *and* g *be two nonconstant entire functions. If* $E_f(T)=E_g(T)$, *then* $f\equiv g$.

In [2] the set S such that for any two nonconstant entire functions f and g the condition $E_f(S)=E_g(S)$ implies $f\equiv g$ is called a unique range set of entire functions (URSE in brief). Similar definition (URSM) for meromorphic functions can be defined. Note that the set $T=\{z \mid e^z+z=0\}$ contains infinite number of elements. Recently, Yi [10] exhibited a finite unique range set of entire functions with 15 elements which gave a positive answer to Gross's problem. In [10] Yi asked that whether one can find a URSE with elements less than 15. To answer this problem the authors [8] showed that the set $S=\{z \mid z^9-z^8+1=0\}$ with only 9 elements is a URSE. In this paper, we shall exhibit a URSM with 15 elements and a URSE with 7 elements by improving a generalization of Borel's theorem

THEOREM 1. *Let* $m\geqq 2$, $n>2m+10$ *with* n *and* $n-m$ *having no common factors. Let* a *and* b *be two nonzero constants such that the equation* $z^n+az^{n-m}+b=0$ *has no multiple roots. Let* $S=\{z \mid z^n+az^{n-m}+b=0\}$. *Then for any two nonconstant meromorphic functions* f *and* g, *the condition* $E_f(S)=E_g(S)$ *implies* $f\equiv g$.

THEOREM 2. *Let* $m\geqq 2$, $n>2m+6$ *with* n *and* $n-m$ *having no common factors. Let* a, b *and* S *be as in Theorem* 1. *Then for any two nonconstant meromorphic functions* f *and* g, *the conditions* $E_f(S)=E_g(S)$ *and* $E_f\{\infty\}=E_g\{\infty\}$ *imply* $f\equiv g$.

THEOREM 3. *Let* $m\geqq 1$, $n>2m+4$ *with* n *and* $n-m$ *having no common factors. Let* a, b *and* S *be as in Theorem* 1. *Then for any two nonconstant entire functions* f *and* g, *the condition* $E_f(S)=E_g(S)$ *implies* $f\equiv g$.

In the last section, we define a concept "uniqueness polynomial of meromorphic functions" which is closely related to the unique range set, and it has been used to prove that the cardinality of a unique range set of meromorphic functions is at least 5.

The main tool will be Nevanlinna's value distribution theory of meromorphic functions, and it is assumed that the reader is familiar with its basic notations and results (see Hayman [5]). In the sequel the letter E will be used to denote a set of r values of finite linear measure.

2. Some lemmas

The following lemmas will be needed in the proof of our theorems.

LEMMA 1 [9]. *Let f be a meromorphic function, and*

$$P(f)=a_0f^n+a_1f^{n-1}+\cdots+a_n$$

be a polynomial in f of degree n, where a_0 ($\neq 0$), $a_1, \cdots, a_n$ are finite complex numbers. Then

$$T(r, P(f))=nT(r, f)+S(r, f),$$

here and in the sequel, $S(r, f)$ denotes the quantity $o(T(r, f))$, $r\to\infty$, $r\notin E$.

In the following $N_{n-1}(r, f)$ is a counting function of f which counts a pole according to its multiplicity if the multiplicity is less than or equal to $n-1$ and counts a pole $n-1$ times if its multiplicity is greater than $n-1$.

First we prove a result which is interesting on its own.

LEMMA 2. *Let $f_1, f_2, \cdots, f_n$ be nonconstant meromorphic functions such that $f_1+f_2+\cdots+f_n\equiv 1$. If $f_1, f_2, \cdots, f_n$ are linearly independent, then the following two inequalities hold*

$$T(r, f_1)<\sum_{i=1}^{n} N_{n-1}\left(r, \frac{1}{f_i}\right)+A_n\sum_{i=1}^{n}\overline{N}(r, f_i)+o(T(r)), \tag{1}$$

$$T(r, f_1)<\sum_{i=1}^{n} N_{n-1}\left(r, \frac{1}{f_i}\right)+(n-1)\sum_{i=2}^{n}\overline{N}(r, f_i)+o(T(r)), \tag{2}$$

where

$$A_n=\begin{cases} 1/2, & n=2, \\ \dfrac{2n-3}{3}, & n=3,\ 4,\ 5, \\ \dfrac{2n+1-2\sqrt{2n}}{2}, & n\geqq 6, \end{cases}$$

$T(r)=\max_{1\leq i\leq n}\{T(r, f_i)\}$ and $r\notin E$.

Proof. We give the proof of (1). The proof of (2) is similar. In the case of $n=2$, the inequality (1) can be easily obtained from the second fundamental theorem of Nevanlinna. Now we assume that $n\geqq 3$.

By the proof of a generalization of Borel's theorem (a generalization of Picard's theorem) by Nevanlinna [6] (or [3]), we have

$$T(r, f_1)<\sum_{i=1}^{n} N\left(r, \frac{1}{f_i}\right)-\sum_{i=2}^{n} N(r, f_i)+N(r, D)-N\left(r, \frac{1}{D}\right)+o(T(r)),$$

where D is the Wronskian of $f_1, f_2, \cdots, f_n$, i.e.

(3)
$$D=\begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \cdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Write

$$N(r)=\sum_{i=1}^{n} N\left(r, \frac{1}{f_i}\right)-\sum_{i=2}^{n} N(r, f_i)+N(r, D)-N\left(r, \frac{1}{D}\right)$$

and

$$N^*(r)=\sum_{i=1}^{n} N_{n-1}\left(r, \frac{1}{f_i}\right)+A_n \sum_{i=1}^{n} \bar{N}(r, f_i).$$

Thus clearly Lemma 2 follows immediately from the following inequality

(4) $$N(r) \leqq N^*(r),$$

which is to be shown next.

For a given meromorphic function f and a complex number $b \in \bar{C}$, we define

$$\mu_f^b(z)=\begin{cases} m, & z \text{ is a } b\text{-point of } f \text{ with multiplicity } m \geqq 1, \\ 0, & z \text{ is not a } b\text{-point of } f, \end{cases}$$

$$\bar{\mu}_f^b(z)=\begin{cases} 1, & z \text{ is a } b\text{-point of } f \text{ with multiplicity } m \geqq 1, \\ 0, & z \text{ is not a } b\text{-point of } f, \end{cases}$$

and

$$\nu_f^b(z)=\begin{cases} m, & z \text{ is a } b\text{-point of } f \text{ with multiplicity } m \leqq n-1, \\ n-1, & z \text{ is a } b\text{-point of } f \text{ with multiplicity } m > n-1, \\ 0, & z \text{ is not a } b\text{-point of } f. \end{cases}$$

Let

$$\mu=\sum_{i=1}^{n} \mu_{f_i}^0-\sum_{i=2}^{n} \mu_{f_i}^\infty+\mu_D^\infty-\mu_D^0$$

and

$$\mu^*=\sum_{i=1}^{n} \nu_{f_i}^0+A_n \sum_{i=1}^{n} \bar{\mu}_{f_i}^\infty.$$

Thus inequality (4) follows from $\mu(z) \leqq \mu^*(z)$ for any $z \in C$. To prove this, we consider following two cases for an arbitrary point $z \in C$.

Cass 1. z is not a pole of f_i, $1 \leqq i \leqq n$.

In this case, z is a zero of $f_i^{(n-1)}$ with multiplicity at least $\mu_{f_i}^0(z)-\nu_{f_i}^0(z)$, $1 \leqq i \leqq n$, thus a zero of D with multiplicity at least $\sum_{i=1}^{n}(\mu_{f_i}^0(z)-\nu_{f_i}^0(z))$. This means that

$$\mu_D^0(z) \geqq \sum_{i=1}^{n} (\mu_{f_i}^0(z) - \nu_{f_i}^0(z)).$$

Hence from the definitions of μ and μ^*, we have $\mu(z) \leqq \mu^*(z)$.

Case 2. z is a pole of some f_i.
We consider two subcases:

Subcase 1. z is not a pole of f_1.
Without loss of generality, we suppose that z is a zero of f_i with multiplicity m_i, $i=1, \cdots, k$, $1 \leqq k \leqq n-2$ and a pole of f_i with multiplicity m_i, $i=k+1, \cdots, n$. Since $f_1 + \cdots + f_n \equiv 1$, we have $D=(-1)^{n+1}D_1$, where

$$D_1 = \begin{vmatrix} f_1' & \cdots & f_{n-1}' \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)} & & f_{n-1}^{(n-1)} \end{vmatrix}. \tag{5}$$

Let

$$q = \sum_{i=1}^{k} (m_i - \nu_{f_i}^0(z)) - \sum_{i=k+1}^{n-1} m_i - \frac{(n+k)(n-k-1)}{2}.$$

From (5) it is not difficult to verify that z is a zero of D with multiplicity at least q if $q \geqq 0$ and a pole of D with multiplicity at most $-q$ if $q<0$. Thus we can get

$$\mu(z) \leqq \sum_{i=1}^{k} \nu_{f_i}^0(z) + \frac{(n+k)(n-k-1)}{2} - 1,$$

and

$$\mu^*(z) = \sum_{i=1}^{k} \nu_{f_i}^0(z) + A_n(n-k).$$

It is easy to verify that $(n+k)(n-k-1)/2 - 1 \leqq A_n(n-k)$ for $1 \leqq k \leqq n-2$. Hence $\mu(z) \leqq \mu^*(z)$.

Subcase 2. z is a pole of f_1.
Without loss of generality, we suppose that z is a pole of $f_1, f_{k+1}, \cdots, f_n$ with multiplicity m_i $(i=1, k+1, \cdots, n)$ respectively, and a zero of f_i $(i=2, \cdots, k)$ with multiplicity m_i, $2 \leqq k \leqq n-1$, if any. Since $f_1 + \cdots + f_n \equiv 1$, we have $D=D_2$, where

$$D_2 = \begin{vmatrix} f_2' & \cdots & f_n' \\ \vdots & \ddots & \vdots \\ f_2^{(n-1)} & & f_n^{(n-1)} \end{vmatrix}. \tag{6}$$

Let

$$q = \sum_{i=2}^{k} (m_i - \nu_{f_i}^0(z)) - \sum_{i=k+1}^{n} m_i - \frac{(n+k-1)(n-k)}{2}.$$

From (6) we can see that z is a zero of D with multiplicity at least q if $q \geqq 0$ and a pole of D with multiplicity at most $-q$ if $q<0$. Thus we can get

$$\mu(z) \leqq \sum_{i=2}^{k} \nu_{f_i}^0(z) + \frac{(n+k-1)(n-k)}{2},$$

and

$$\mu^*(z) = \sum_{i=2}^{k} \nu_{f_i}^0(z) + A_n(n-k+1).$$

It is easy to verify that $(n+k-1)(n-k)/2 \leqq A_n(n-k+1)$ for $1 \leqq k \leqq n-1$. Hence $\mu(z) \leqq \mu^*(z)$ which completes the proof of Lemma 2.

3. Proof of Theorem 1

Let $r_1, r_2, \cdots, r_n$ be the roots of the equation $z^n + az^{n-m} + b = 0$. Since $E_f(S) = E_g(S)$ we have from Nevanlinna's second fundamental theorem

$$\begin{aligned}(n-2)T(r, g) &< \sum_{k=1}^{n} \overline{N}\left(r, \frac{1}{g-r_k}\right) + S(r, g) \\ &= \sum_{k=1}^{n} \overline{N}\left(r, \frac{1}{f-r_k}\right) + S(r, g) \\ &\leqq nT(r, f) + S(r, g).\end{aligned}$$

It follows that

$$T(r, g) \leqq \frac{n}{n-2} T(r, f) + S(r, g). \tag{7}$$

Similarly the following inequality holds:

$$T(r, f) \leqq \frac{n}{n-2} T(r, g) + S(r, f). \tag{8}$$

In the sequel, we use $S(r)$ to express either $S(r, f)$ or $S(r, g)$.

Consider now the following meromorphic function

$$\phi = \frac{f^n + af^{n-m} + b}{g^n + ag^{n-m} + b}. \tag{9}$$

The condition $E_f(S) = E_g(S)$ ensures that the zeros of ϕ come from the poles of g, and the poles of ϕ come from the poles of f. This means that the following inequalities hold:

$$\overline{N}\left(r, \frac{1}{\phi}\right) \leqq \overline{N}(r, g) \tag{10}$$

and

$$\overline{N}(r, \phi) \leqq \overline{N}(r, f). \tag{11}$$

Let

$$(12)\qquad f_1=-\frac{1}{b}f^{n-m}(f^m+a),\quad f_2=\frac{1}{b}\phi g^{n-m}(g^m+a),\quad f_3=\phi.$$

Then f_1, f_2 and f_3 are meromorphic functions and f_1 is not a constant. From (9), we have

$$(13)\qquad f_1+f_2+f_3\equiv 1.$$

Now we distinguish following two cases.

Case 1. f_3 is not a constant.

If f_1 and f_2 are linearly dependent, then $f_2=cf_1$, $c\neq -1$. From (13) we have

$$(1+c)f_1+f_3\equiv 1.$$

By using Lemma 1 and Lemma 2 together with the inequalities (7) and (10), we deduce

$$\begin{aligned} nT(r, f)&=T(r, f_1)+S(r)\\ &<\overline{N}\left(r, \frac{1}{f_1}\right)+\overline{N}\left(r, \frac{1}{f_3}\right)+\overline{N}(r, f_1)+S(r)\\ &\leqq\overline{N}\left(r, \frac{1}{f}\right)+\overline{N}\left(r, \frac{1}{f^m+a}\right)+\overline{N}(r, g)+\overline{N}(r, f)+S(r)\\ &\leqq(m+2)T(r, f)+T(r, g)+S(r)\\ &\leqq\left(m+2+\frac{n}{n-2}\right)T(r, f)+S(r)\\ &=\left(m+3+\frac{2}{n-2}\right)T(r, f)+S(r), \end{aligned}$$

which is contradictory to $n>2m+10$. Hence f_1 and f_2 must be linearly independent.

If f_1, f_2 and f_3 are linearly independent and f_2 is not a constant, then by using Lemma 2, we have

$$\begin{aligned} T(r, f_1)<N_2\left(r, \frac{1}{f_1}\right)+N_2\left(r, \frac{1}{f_2}\right)+N_2\left(r, \frac{1}{f_3}\right)\\ +\overline{N}(r, f_1)+\overline{N}(r, f_2)+\overline{N}(r, f_3)+S(r). \end{aligned}$$

From the identities (9) and (12), we can easy to see that the zeros of f_2 can not come from the zeros of ϕ, and the poles of f_2 must come from the poles of f. By above inequality and Lemma 1 together with (10), (11) and (7), we deduce that

$$\begin{aligned} nT(r, f) &< N_2\Big(r, \frac{1}{f^{n-m}}\Big)+N_2\Big(r, \frac{1}{f^m+a}\Big)+N_2\Big(r, \frac{1}{g^{n-m}}\Big)+N_2\Big(r, \frac{1}{g^m+a}\Big) \\ &\quad +2\bar{N}(r, g)+\bar{N}(r, f)+\bar{N}(r, f)+\bar{N}(r, f)+S(r) \\ &\leqq (m+5)T(r, f)+(m+4)T(r, g)+S(r) \\ &\leqq \Big[(m+5)+(m+4)\frac{n}{n-2}\Big]T(r, f)+S(r) \\ &= \Big(2m+9+\frac{2m+8}{n-2}\Big)T(r, f)+S(r). \end{aligned}$$

This contradicts to the assumption $n>2m+10$. It follows that when f_1, f_2 and f_3 are linearly independent, f_2 must be constant and $f_2\neq 1$, i.e. $f_1+f_3=1-f_2$ is a nonzero constant. By Lemma 2,

$$T(r, f_1)<\bar{N}\Big(r, \frac{1}{f_1}\Big)+\bar{N}\Big(r, \frac{1}{f_3}\Big)+\bar{N}(r, f_1)+S(r).$$

This leads to

$$nT(r, f)\leqq\Big(m+3+\frac{2}{n-2}\Big)T(r, f)+S(r)$$

which is a contradiction to $n>2m+10$.

If f_1, f_2 and f_3 are linearly dependent, then there exist three constants c_1, c_2 and c_3, at least one of them is not zero, such that

$$c_1f_1+c_2f_2+c_3f_3\equiv 0. \tag{14}$$

From this and the fact that f_1, f_2 are linearly independent, we must have $c_3\neq 0$. So

$$c_1\frac{f_1}{\phi}+c_2\frac{f_2}{\phi}=-c_3. \tag{15}$$

If $c_1=0$, then $g^{n-m}(g^m+a)$, hence g is a constant. This is impossible.

If $c_2=0$, then

$$\frac{c_1}{b}f^{n-m}(f^m+a)=c_3\phi. \tag{16}$$

Let $s_0=0$, $s_1, \cdots, s_m$ be the distinct roots of the equation: $z^n+az^{n-m}=0$. Then (16) shows that any s_j*—point* of f must be a zero of ϕ and hence a pole of g. But from (9) and (16) one can see that the multiplicity of any zero of ϕ is at least n, so the multiplicity of a s_j*—point* $(j\neq 0)$ of f is at least n and at least m for a s_0*—point* of f. Hence, we have

$$\Theta(s_j, f)=1-\varlimsup_{r\to\infty}\frac{\bar{N}\left(r, \frac{1}{f-s_j}\right)}{T(r, f)}$$

$$\geqq 1-\varlimsup_{r\to\infty}\frac{\bar{N}\left(r, \frac{1}{f-s_j}\right)}{N\left(r, \frac{1}{f-s_j}\right)}\geqq 1-\frac{1}{n}$$

$j=1, 2, \cdots, m$, and

$$\Theta(s_0, f)\geqq 1-\frac{1}{m}.$$

Again by the second fundamental theorem about the deficiencies of meromorphic functions, we have

$$1-\frac{1}{m}+m\left(1-\frac{1}{n}\right)\leqq \sum_{j=0}^{m}\Theta(s_j, f)\leqq 2.$$

This is impossible because $m\geqq 2$, $n>2m+10$.

Thus we have to conclude that $c_1c_2c_3\neq 0$. By Lemma 2 and equation (15)

$$T\left(r, \frac{f_2}{\phi}\right)<\bar{N}\left(r, \frac{\phi}{f_2}\right)+\bar{N}\left(r, \frac{\phi}{f_1}\right)+\bar{N}\left(r, \frac{f_2}{\phi}\right)+S(r)$$

$$<\bar{N}\left(r, \frac{\phi}{f_2}\right)+\bar{N}(r, \phi)+\bar{N}\left(r, \frac{1}{f_1}\right)+\bar{N}\left(r, \frac{f_2}{\phi}\right)+S(r).$$

Hence from Lemma 1 and equations (7), (11) and (12), we have

$$nT(r, g)<\bar{N}\left(r, \frac{1}{g}\right)+\bar{N}\left(r, \frac{1}{g^m+a}\right)+2\bar{N}(r, f)$$

$$+\bar{N}\left(r, \frac{1}{f^m+a}\right)+\bar{N}(r, g)+S(r)$$

$$\leqq (m+2)T(r, g)+(m+2)T(r, f)+S(r)$$

$$\leqq (m+2)\left(1+\frac{n}{n-2}\right)T(r, f)+S(r),$$

a contradiction to $n>2m+10$.

We can rule out Case 1.

Case 2. f_3 is a constant.

In this case, f_2 can not be a constant. And from (9), we have

$$T(r, f)=T(r, g)+S(r). \tag{17}$$

If $f_3\neq 1$, then $f_1+f_2=1-f_3\neq 0$. By Lemma 2

$$T(r, f_1) < \bar{N}\left(r, \frac{1}{f_1}\right) + \bar{N}\left(r, \frac{1}{f_2}\right) + \bar{N}(r, f_1) + S(r).$$

That is

$$nT(r, f) < \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{f^m+a}\right) + \bar{N}\left(r, \frac{1}{g}\right)$$
$$+\bar{N}\left(r, \frac{1}{g^m+a}\right) + 2\bar{N}(r, f) + S(r)$$
$$<(3+m)T(r, f) + (m+1)T(r, g) + S(r)$$
$$=2(m+2)T(r, f) + S(r).$$

This contradicts to the assumption that $n > 2m+10$.

If $f_3 = 1$, then from (9) we get

$$g^m(h^n - 1) = -a(h^{n-m} - 1) \tag{18}$$

where $h = f/g$ is a meromorphic function. Further (18) can be rewritten as

$$g^m(h-u_1)(h-u_2)\cdots(h-u_n) = -a(h^{n-m} - 1) \tag{19}$$

where

$$u_j = e^{i(2j\pi/n)}, \quad j = 1, 2, \cdots, n.$$

Since n and $n-m$ have no common factors, we see that $u_j^{n-m} - 1 \neq 0$, $j = 1, \cdots, n-1$. Hence from (15) the multiplicity of a u_j*-point* of h is at least m. Suppose that h is not a constant, then we have

$$\Theta(u_j, h) = 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}\left(r, \frac{1}{h-u_j}\right)}{T(r, h)}$$
$$\geqq 1 - \overline{\lim_{r\to\infty}} \frac{\bar{N}\left(r, \frac{1}{h-u_j}\right)}{N\left(r, \frac{1}{h-u_j}\right)} \geqq 1 - \frac{1}{m}, \quad j = 1, \cdots, n-1.$$

Thus

$$(n-1)\left(1 - \frac{1}{m}\right) \leqq \sum_{j=1}^{n-1} \Theta(u_j, h) \leqq 2$$

which contradicts to $m \geqq 2$ and $n > 2m+10$. This shows that h must be a constant. Furthermore from (18) we can see h must be equal to 1. Otherwise, we will deduce that g is a constant. Hence $f \equiv g$. This completes the proof of Theorem 1.

Noting that the function ψ in (9) will assume the form e^α with α being an entire function under the assumptions of Theorem 2 and Theorem 3. Furthermore under the assumption of Theorem 3 the inequalities (7) and (8) will be replaced by

$$T(r, g) \leqq \frac{n}{n-1} T(r, f) + S(r, g)$$

and

$$T(r, f) \leqq \frac{n}{n-1} T(r, g) + S(r, f)$$

respectively, we can then prove these two theorems immediately following the same procedure of the proof of Theorem 1.

Example 1. The set $S=\{z \mid z^{15}-z^{13}+1=0\}$ is a URSM with 15 elements.

Example 2. The set $S=\{z \mid z^{7}-z^{6}+1=0\}$ is a URSE with 7 elements.

4. Concluding remarks

We would like to pose the following problems about the unique range set of meromorphic functions and entire functions for further investigations.

PROBLEM 1. Can one find a URSE with less than 7 elements ?. What is the smallest cardinality for a URSE ?

PROBLEM 2. Can one find a URSM with less than 15 elements ?. What is the smallest cardinality for a URSM ?

Now we introduce following notations:

$$U_M=\{S \mid S \text{ is a URSM}\},$$
$$U_E=\{S \mid S \text{ is a URSE}\},$$
$$\lambda_M=\min\{n(S) \mid S \in U_M\},$$
$$\lambda_E=\min\{n(S) \mid S \in U_E\},$$

where $n(S)$ denotes the cardinal number of the set S. Obviously,

$$\lambda_E \leqq \lambda_M .$$

Example 1 and Example 2 show that $\lambda_E \leqq 7$ and $\lambda_M \leqq 15$, respectively. In [8] we have proved that $\lambda_E \geqq 4$. In the following we want to give a lower bound of λ_M. First of all, we introduce two definitions related to unique range set.

DEFINITION 1. Let $S=\{a_1, \cdots, a_n\}$ be a subset in C with finite distinct elements. If S is a URSM (URSE), then any polynomial of degree n which has zeros $a_1, \cdots, a_n$ is called a polynomial of URSM (URSE). We call it a PURSM (PURSE) in brief.

DEFINITION 2. Let P be a polynomial. If the condition $P(f)\equiv P(g)$ implies $f\equiv g$ for any nonconstant meromorphic (entire) functions f and g, then P is called a uniqueness polynomial of meromorphic (entire) functions. We say P is a UPM (UPE) in brief.

Obviously, any nonconstant linear transformation is a trivial UPM. We shall concern the nontrivial UPM and UPE. The following two theorems can easily be obtained from the definitions.

THEOREM 4. *If P is a PURSM (PURSE), then P is a UPM (UPE).*

THEOREM 5. *If P_1 is a UPM (UPE) and P_2 is a polynomial, then $P_1\circ P_2$ is a UPM (UPE) iff P_2 is a UPM (UPE).*

THEOREM 6. *Let P_1 be a PURSM (PURSE) and P_2 a UPM (UPE). If $P_1\circ P_2$ has no multiple zeros, then $P_1\circ P_2$ is a PURSM (PURSE).*

Proof. Let $S=\{a_1, \cdots, a_n\}$ be the zeros of $P_1\circ P_2$ and $S_1=\{b_1, \cdots, b_m\}$ the zeros of P_1. For any two nonconstant meromorphic (entire) functions f and g, if $E_f(S)=E_g(S)$, then

$$(f-a_1)(f-a_2)\cdots(f-a_n)=(g-a_1)(g-a_2)\cdots(g-a_n)h,$$

where h is a meromorphic function whose zeros come from the poles of g and the poles come from the poles of f. From above equation we get

$$(P_2(f)-b_1)\cdots(P_2(f)-b_m)=(P_2(g)-b_1)\cdots(P_2(g)-b_m)h$$

which means that

$$E_{P_2(f)}(S_1)=E_{P_2(g)}(S_1).$$

Since P_1 is a PURSM (PURSE), we get $P_2(f)=P_2(g)$. Hence $f\equiv g$ in terms of the property of UPM (UPE) of P_2.

In general, it is not easy to tell if a polynomial is a UPM (UPE) or not. For the polynomial of degree less than 5, we have

THEOREM 7. *Any polynomial of degree 2 or 3 is not a UPE.*

Proof. By Theorem 5, we only need to prove $P_1(z)=z^2-a$ and $P_2(z)=z^3-az+b$ are not UPE. P_1 is clearly not a UPE. The following two entire functions

$$f(z)=\frac{\omega_2 e^z}{\omega_2-\omega_1}-\frac{a\omega_1 e^{-z}}{\omega_2-\omega_1},$$

and

$$g(z)=\frac{e^{z}}{\omega_2-\omega_1}-\frac{ae^{-z}}{\omega_2-\omega_1},$$

where $\omega_k=e^{i2k\pi/3}$ $(k=1, 2)$ satisfy $f\not\equiv g$ and $P_2(f)\equiv P_2(g)$. Which means that P_2 is not a UPE.

THEOREM 8. *Let* $P(z)=z^4+a_3z^3+a_2z^2+a_1z+a_0$. *Then*

(a) *P is not a UPM.*

(b) *P is a UPE if and only if*

$$\frac{a_3^3}{8}-\frac{a_2a_3}{2}+a_1\neq 0.$$

Proof. By a transformation, we get

$$Q(z)=P\left(z-\frac{a_3}{4}\right)=z^4+az^2+bz+c,$$

where

$$a=a_2-\frac{3a_3^2}{8},\quad b=\frac{a_3^3}{8}-\frac{a_2a_3}{2}+a_1.$$

When $b=0$, P is obviously not a UPM and UPE.

When $b\neq 0$, and $27b^2+8a^3=0$, the following two functions

$$f(z)=\frac{3b}{2a}\cdot\frac{1}{1-e^{2z}}+i\frac{3b}{2a}\left(\frac{e^{z}}{1-e^{2z}}+e^{z}\right)$$

and

$$g(z)=\frac{3b}{2a}\cdot\frac{1}{1-e^{2z}}-i\frac{3b}{2a}\left(\frac{e^{z}}{1-e^{2z}}+e^{z}\right)$$

satisfy $f\not\equiv g$ and $Q(f)\equiv Q(g)$.

When $b\neq 0$, and $27b^2+8a^3\neq 0$, we consider the following functions

$$f=\frac{(-2b)^{1/2}B'+1}{2B},\quad g=\frac{(-2b)^{1/2}B'-1}{2B},$$

where B is the Weierstrass elliptic function which satisfies

$$(B')^2=B^3+\frac{a}{b}B^2+\frac{1}{2b}.$$

One can verify that $Q(f)\equiv Q(g)$ but $f\not\equiv g$. Hence Q and thus P is not a UPM.

Now we prove the part (b) of Theorem 8. If there exist two distinct non-constant entire functions f and g satisfying $Q(f)\equiv Q(g)$, then

$$(f+g)(f^2+g^2)+a(f+g)+b=0.$$

It is clear that $h=f+g$ is not a constant. From the above equation we get

(20) $$(f-g)^2=-\frac{(h-\omega_1)(h-\omega_2)(h-\omega_3)}{h},$$

where ω_i ($i=1, 2, 3$) are the zeros of $z^3+2az+2b$. When $b\neq 0$, one can see from (20) that the entire function h has a exceptional value 0. Furthermore there exists a ω_i say ω_3 such that $\omega_3\neq\omega_1$, $\omega_3\neq\omega_2$. Hence the multiplicity of ω_3*—points* of h is at least 2. This is impossible. This also completes the proof of Theorem 8.

From Theorem 4 and Theorem 8, it follows that

THEOREM 9. $\lambda_M\geqq 5$.

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DEPARTMENT OF MATHEMATICS
HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY
CLEAR WATER BAY, KOWLOON, HONG KONG
e-mail: mayang@uxmail. ust. hk